Math 22, Fall 2006

Some problems on Predicates and Quantifiers

$(\forall x)\ P(x)$ is a statement that says that for all values x the statement $P(x)$ is true.
$(\exists x)\ P(x)$ is a statement that says that for some value x the statement $P(x)$ is true.

$\forall x$ is called the "universal quantifier."
$\exists x$ is called the "existential quantifier."
$P(x)$ is called the "predicate."

$(\forall x)\ P(x)$ is true when $P(x)$ is true for every x, and it is false if there is one x for which $P(x)$ is false.
$(\exists x)\ P(x)$ is true if there is an x for which $P(x)$ is true, and it is false if $P(x)$ is false for every x.

We can think of $(\forall x)\ P(x)$, intuitively, as $P(a) \wedge P(b) \wedge P(c) \wedge \ldots$, where we run over all elements of the universe.
We can think of $(\exists x)\ P(x)$, intuitively, as $P(a) \vee P(b) \vee P(c) \vee \ldots$, where we run over all elements of the universe.

De Morgan's Laws:
$\sim (\forall x)\ P(x) = (\exists x) \sim P(x)$
$\sim (\exists x)\ P(x) = (\forall x) \sim P(x)$

(1) From Lewis Carroll:
"All lions are fierce."
"Some lions do not drink coffee."
Some fierce creatures do not drink coffee."

Use $P(x)$, $Q(x)$, and $R(x)$ for "x is a lion," "x is fierce," and "x drinks coffee,"respectively. Express the statements above using predicates and quantifiers, assuming the universe is the set of all creatures.

(2) Why can the second statement in problem (1) not be written
$(\exists x)\ (P(x) \rightarrow \sim R(x))$? (Think about what might make this statement true.)

(3) Do you think the concluding statement in problem (1) is valid (true when the premises, or preceeding statements, are true)? Why or why not?

(4) Also from Lewis Carroll:
"No professors are ignorant."
"All ignorant people are vain."
"No professors are vain."

Let P(x), Q(x), and R(x) be the statements "x is a professor," "x is ignorant," and "x is vain, respectively. Express the statements above using predicates and quantifiers, assuming the universe is the set of all people.

(5) Do you think the concluding statement in (4) is true, assuming the previous two are true (that is, the entire argument is "valid")? Why or why not?

(6) Decide whether the following statements are true or false, and explain. The universe is the set of integers.

(a) $(\exists n)\ 7n \equiv 8 \pmod 9$

(b) $(\exists n)\ 14n \equiv 9 \pmod{18}$

(c) $(\exists n)\ n^2 \equiv 5 \pmod 9$

(d) $\sim (\forall n)\ [n \equiv 1 \pmod 2 \rightarrow n^2 \equiv 1 \pmod 8]$

(e) $(\forall n)\ (\exists k)\ nk \equiv 1 \pmod 5$

(f) $(\exists n)\ n^2 \equiv n \pmod 3$

(g) $(\exists n)\ (\exists k)\ (\exists m)\ [nk \equiv 0 \pmod m] \wedge \sim [n \equiv 0 \pmod m] \wedge \sim [k \equiv 0 \pmod m]$

(h) $[(\forall m)\ [(m>2) \rightarrow [(\exists n)\ (\exists k)\ [n^2 + k^2 \equiv 2 \pmod m] \wedge \sim [n \equiv k \pmod m]]$