Pile Splitting Problem – Example – Math 22, Fall 2009

Here's the explanation of the case I showed in class: suppose that if a pile of size k is split into two piles of size r and s, then the product (r+s)(r)(s) is computed. At the end, when all piles are size 1, then those products are summed. If you try this a number of times, you will discover that when you start with n objects, you always end with final sum n(n+1)(n–1)/3. Discovering this takes gathering data and looking for an overall pattern, you cannot do it without making this effort!!

So statement n, labeled P(n) is: the final sum is n(n+1)(n–1)/3.

Proof:
Basic Step: Start with k=2 (the theorem does not really make sense for k=1, unless we assume the sum is 0). Split it into two piles of size 1. The product (and final sum) is (1+1)(1)(1) = 2, which is also = 2(2+1)(2–1)/3.

Induction step: Assume the theorem has been proved for statements P(2), P(3),…, P(k–1). Now use those to prove that P(k) is true. If you can do this, you have proved P(n) is true for all n > 1, by the strong principle of induction.

For P(k), suppose we split the pile of size k into two piles of size j and k–j, for j any number from 1 to k–1. The product at this step is then $[j+(k-j)]j(k-j)$. What happens to the piles of sizes j and k–j? Well, by the strong principle of induction we are assuming that they will give final sums of products of $\frac{j^3-j}{3}$ for the pile of size j, and $\frac{(k-j)^3-(k-j)}{3}$ for the pile of size k–j. Now we add all three of these terms to get the final sum for the case when we start with k objects:

$$[j+(k-j)]j(k-j) + \frac{j^3-j}{3} + \frac{(k-j)^3-(k-j)}{3}$$

Now we do some algebraic manipulations, as in class, and see that this actually reduces to:

$$= (k-j)[kj+\frac{(k-j)^2-1}{3}]+\frac{j^3-j}{3}$$

which further reduces, after more algebra, to

$$= (k-j)[\frac{3kj}{3}+\frac{k^2-2kj+j^2-1}{3}]+\frac{j^3-j}{3}$$

which, with a little more algebra, gives

$$= (k-j)[\frac{k^2+kj+j^2-1}{3}]+\frac{j^3-j}{3}$$

$$= \frac{k^3-j^3-(k-j)}{3}+\frac{j^3-j}{3}$$

$$= \frac{k^3-k}{3}$$

But this is just the result we were looking for in the case P(k). So by the principle of strong induction, since P(2) is true, and since also P(2), P(3),…,P(k–1) together imply P(k), the theorem must be true for all k>1.