$\sinh x = \dfrac{e^x - e^{-x}}{2}$

HYPERBOLIC FUNCTIONS

$\cosh x = \dfrac{e^x + e^{-x}}{2}$

[1] Rewrite in terms of exponential functions, simplify, then rewrite in terms of hyperbolic functions, where possible.

[a] $\sinh(-x)$
[b] $\cosh(-x)$
[c] $\cosh^2 x + \sinh^2 x$
[d] $\cosh^2 x - \sinh^2 x$
[e] $2\sinh x \cosh x$
[f] $e^x \sinh x$
[g] $\dfrac{\cosh x}{e^x}$
[h] $\sinh(\ln x)$
[i] $\cosh(2\ln x)$

[2] Based on the similarities to the properties of trigonometric functions noted in [1], conjecture and prove formulae for the following.

[a] $\sinh(x+y)$
[b] $\sinh(x-y)$
[c] $\cosh(x+y)$
[d] $\cosh(x-y)$

[3] Define $\tanh x = \dfrac{\sinh x}{\cosh x}$.

[a] Rewrite $\tanh x$ in terms of exponential functions.
[b] Simplify $\tanh(-x)$ in 2 ways
  [i] by first rewriting in terms of exponential functions, simplifying, then rewriting in terms of hyperbolic functions
  [ii] by using your answers in [1][a] and [1][b]
[c] Define the 3 remaining hyperbolic functions in a parallel fashion in terms of other hyperbolic functions.
[d] Rewrite the 3 remaining hyperbolic functions in terms of exponential functions.
[e] Find the limits as $x \to \pm\infty$ of all hyperbolic functions.

Once you get used to the identities, it is much easier to manipulate the hyperbolic functions without rewriting them in terms of exponential functions.

[4] You should have discovered a hyperbolic parallel to the Pythagorean Identity in [1][d].

[a] Rewrite the identity in [1][d] in 2 ways
  [i] by solving for $\sinh^2 x$
  [ii] by solving for $\cosh^2 x$
[b] Rewrite the identity in [1][c] in 2 ways
  [i] by substituting using your answer from [4][a][i]
  [ii] by substituting using your answer from [4][a][ii]
[c] Find the hyperbolic parallels to the other Pythagorean Identities by dividing both sides of [1][d]
  [i] by $\sinh^2 x$
  [ii] by $\cosh^2 x$

CALCULUS OF HYPERBOLIC FUNCTIONS

[5] Using the original exponential definitions of $\sinh x$ and $\cosh x$, find the derivatives of $\sinh x$ and $\cosh x$, and rewrite in terms of hyperbolic functions.

[6] Using the hyperbolic definitions from [3], the quotient rule for derivatives, the derivatives from [5], and the various identities from [4], find the derivatives of the other 4 hyperbolic functions in terms of hyperbolic functions.

[7] Rewrite your derivatives from [5] and [6] using integral notation. Do not use negatives in your integrands.
(eg. if the derivative of $\frac{1}{x}$ is $-\frac{1}{x^2}$, write $\int \frac{1}{x^2}\,dx = -\frac{1}{x} + C$.)

[8] You now have an arsenal of basic identities, differentiation and integration rules for the hyperbolic functions. Use those rules, along with the product, quotient and chain rules for derivatives, and substitution and integration by parts, to evaluate the following. Simplify your answers.

[a] Find the derivatives of the following.
- [i] $\sinh 8x$
- [ii] $\cosh 4x^2$
- [iii] $x \tanh^2 x$
- [iv] $e^{-2x} \sinh 3x$
- [v] $\dfrac{\sinh x}{1-\cosh x}$
- [vi] $\tanh(\ln x)$
- [vii] $\sin^{-1}(\tanh x)$

[b] Find the anti-derivatives of the following. HINT: If you get stuck, consider what technique you would use if you replaced the hyperbolic functions with their trigonometric parallels.
- [i] $\sinh 8x$
- [ii] $\dfrac{\cosh\sqrt{x}}{\sqrt{x}}$
- [iii] $x^2 \sinh 2x$
- [iv] $e^{-2x} \cosh 3x$
- [v] $\dfrac{\cosh x}{1-\sinh x}$
- [vi] $\sinh(\ln x)$
- [vii] $\sinh^3 x \cosh^4 x$

[c] Redo [8][a][iv] and [8][a][vi] by rewriting the functions using the exponential definitions, simplifying, differentiating, then simplifying again. Your final answers should not involve hyperbolic functions.

[d] Redo [8][b][iv] and [8][b][vi] by rewriting the functions using the exponential definitions, simplifying, integrating, then simplifying again. Your final answers should not involve hyperbolic functions. Would you consider this technique for [8][b][vii] ?

[e] Find $\int \cosh(2\ln x)\,dx$ and $\int \tanh(\ln x)\,dx$. (HINT: See [8][d].)

INVERSE HYPERBOLIC FUNCTIONS

[9] Use your calculator to sketch $y = \sinh x$, $y = \cosh x$ and $y = \tanh x$.

[a] Do the graphs confirm the symmetry and limits you found in [1][a] and [b], and [3][b] and [d] ? State the domain and range of each function, and identify all intercepts, and horizontal and vertical asymptotes.

[b] Recall that a function has an inverse function if and only if the function is one-to-one. Which of the 3 functions has an inverse ?

[c] A function which is not one-to-one, can be "made" one-to-one by restricting its domain (eg. the trigonometric functions). How would you restrict the domain of the non one-to-one function(s) in [9][b] to "make" it (them) one-to-one ?

[10] Solve $\sinh x = 1$ and $\cosh x = 1$ by using the exponential definitions and an algebraic substitution $z = e^x$.

[11] $y = \sinh^{-1} x$ if and only if $x = \sinh y$.

[a] Define $\cosh^{-1} x$ and $\tanh^{-1} x$. State the domain and range of all 3 inverse hyperbolic functions, and identify all intercepts, and horizontal and vertical asymptotes.

[b] Prove that $\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$ in 2 ways

[i] by simplifying $\sinh\left(\ln\left(x + \sqrt{x^2 + 1}\right)\right)$ using the exponential definition

[ii] by solving $x = \sinh y$ for y using the exponential definition and an algebraic substitution $z = e^x$

[c] Find formulae for $\cosh^{-1} x$ and $\tanh^{-1} x$. (HINT: See [11][b][ii].)

[12] Find the derivatives of all 3 inverse hyperbolic functions in 2 ways

[a] by using the formulae in [11][b] and [c]

[b] by using implicit differentiation on $y = \sinh^{-1} x$, $y = \cosh^{-1} x$ and $y = \tanh^{-1} x$ (NOTE: This technique is similar to how you found the derivatives of the inverse trigonometric functions. Consult your textbook's chapters on the derivatives of the inverse trigonometric functions.)

[13] Find the derivatives of the following. Simplify your answers.

[a] $\sinh^{-1} 8x$

[b] $\cosh^{-1} 4x^2$

[c] $x\left(\tanh^{-1} x\right)^2$

[d] $e^{-2x} \sinh^{-1} 3x$

[e] $\cosh^{-1}(\csc x)$

[f] $\tanh^{-1}(\cos x)$