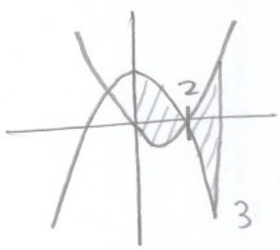
Math 1B
Midterm 3
Mon Jun 16, 2008

What day of the month is your birthday ? __ __
What are the last 2 digits of your address ? __ __
What are the last 2 digits of your zip code ? __ __

CALCULATORS ARE NOT ALLOWED ON THIS SECTION OF THE MIDTERM

[20 POINTS] Find the area of the region between $y = 4 - x^2$ and $y = x^2 - 2x$ on [0, 3].



$4 - x^2 = x^2 - 2x$

$0 = 2x^2 - 2x - 4$

$0 = 2(x^2 - x - 2)$

$0 = 2(x-2)(x+1)$

$x = 2, -1$

$$\int_0^2 ((4-x^2)-(x^2-2x))\,dx + \int_2^3 ((x^2-2x)-(4-x^2))\,dx$$

$$= \int_0^2 (-2x^2+2x+4)\,dx + \int_2^3 (2x^2-2x-4)\,dx$$

$$= \left(-\tfrac{2}{3}x^3 + x^2 + 4x\right)\Big|_0^2 + \left(\tfrac{2}{3}x^3 - x^2 - 4x\right)\Big|_2^3$$

$$= -\tfrac{2}{3}\cdot 8 + 4 + 8 + \tfrac{2}{3}(27-8) - (9-4) - 4(3-2)$$

$$= -\tfrac{16}{3} + 12 + \tfrac{38}{3} - 5 - 4$$

$$= \tfrac{22}{3} + 3$$

$$= \tfrac{31}{3}$$

[15 POINTS] Find the length of the curve $y = \int_1^x \sqrt{(t+2)(t+4)}\,dt$ on [0, 4].

$$y' = (x+2)(x+4) = x^2 + 6x + 8$$

$$\int_0^4 \sqrt{1 + [y']^2}\,dx = \int_0^4 \sqrt{x^2+6x+9}\,dx$$

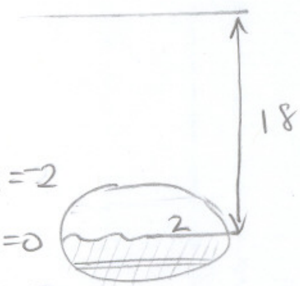
$$= \int_0^4 (x+3)\,dx$$

$$= \tfrac{1}{2}x^2 + 3x \Big|_0^4$$

$$= 8 + 12$$

$$= 20$$

[20 POINTS] A spherical tank of radius 2 feet containing water is buried underground, so that its center is 18 feet below ground level. Find the work done in pumping the water to ground level if the tank is half full.



$d=16 \quad x=-2$

$d=18 \quad x=0$

$d=20 \quad x=2$

$d = x+18$

$V = \pi r^2 \Delta x$

$= \pi\left(\sqrt{2^2-x^2}\right)^2 \Delta x$

$= \pi(4-x^2)\Delta x$

$$W = \int_0^2 \delta\pi(4-x^2)(x+18)\,dx$$

$$= \delta\pi\int_0^2 (72+4x-18x^2-x^3)\,dx$$

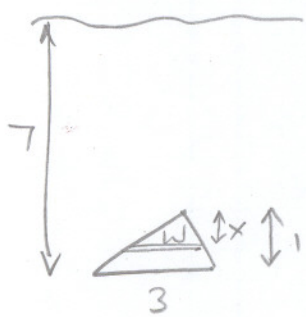
$$= \delta\pi\left(72x+2x^2-6x^3-\tfrac{1}{4}x^4\right)\Big|_0^2$$

$$= \delta\pi(144+8-48-4)$$

$$= 100\,\delta\pi$$

$$= 6240\pi \text{ ft-lb}$$

[20 POINTS] Find the hydrostatic force on the window of an aquarium if the window is a triangle of height 1 foot and base 3 feet with the base down and 7 feet below the surface of the water.



$x=0 \quad h=6$

$x=1 \quad h=7$

$h = x+6$

$\frac{w}{x} = \frac{3}{1}$

$w = 3x$

$$F = \int_0^1 \delta(x+6)(3x)\,dx$$

$$= \delta\int_0^1 (3x^2+18x)\,dx$$

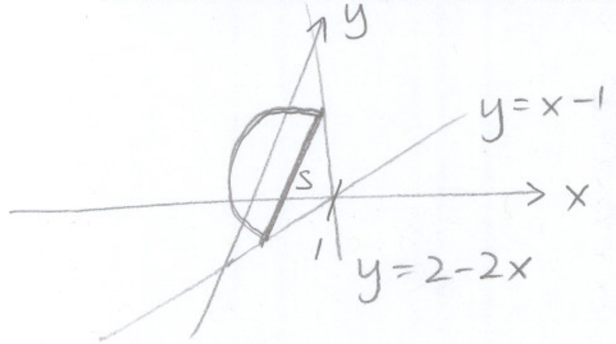
$$= \delta(x^3+9x^2)\Big|_0^1$$

$$= \delta(1+9-0)$$

$$= 10\,\delta$$

$$= 624 \text{ lb}$$

[20 POINTS] The region bounded by $x = 0$, $y = x - 1$ and $y = 2 - 2x$ is the base of a solid whose cross sections perpendicular to the x-axis are semicircles. Find the volume of the solid.



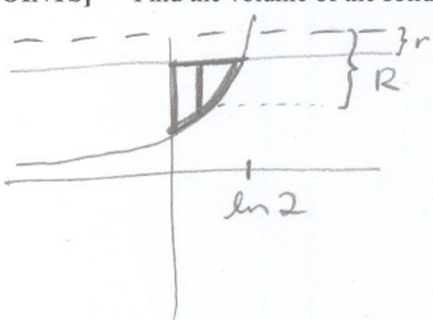
$$A(x) = \frac{\pi}{8} s^2$$

$$s = (2-2x) - (x-1)$$
$$= 3 - 3x$$

$$A(x) = \frac{\pi}{8}(3-3x)^2$$

$$\int_0^1 \frac{\pi}{8}(3-3x)^2\,dx$$
$$= \frac{\pi}{8}\int_0^1 (9 - 18x + 9x^2)\,dx$$
$$= \frac{\pi}{8}\left(9x - 9x^2 + 3x^3\right)\Big|_0^1$$
$$= \frac{\pi}{8}(9 - 9 + 3 - 0)$$
$$= \frac{3\pi}{8}$$

[20 POINTS] Find the volume of the solid if the region bounded by $y = e^x$, $y = 2$ and $x = 0$ is revolved around the line $y = 3$.

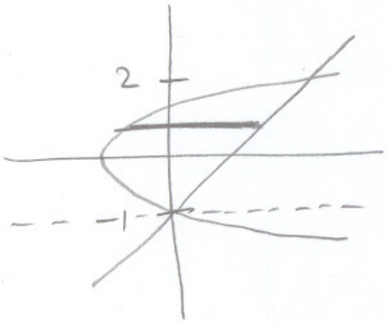


$$\pi\int_0^{\ln 2}\left((3-e^x)^2 - (3-2)^2\right)dx$$
$$= \pi\int_0^{\ln 2}(9 - 6e^x + e^{2x} - 1)\,dx$$
$$= \pi\int_0^{\ln 2}(e^{2x} - 6e^x + 8)\,dx$$
$$= \pi\left(\tfrac{1}{2}e^{2x} - 6e^x + 8x\right)\Big|_0^{\ln 2}$$
$$= \pi\left(\tfrac{1}{2}(4) - 6(2) + 8\ln 2 - \left(\tfrac{1}{2} - 6\right)\right)$$
$$= \pi\left(2 - 12 + 8\ln 2 + \tfrac{11}{2}\right) = \pi\left(8\ln 2 - \tfrac{9}{2}\right) = \frac{\pi}{2}(16\ln 2 - 9)$$

OR $2\pi\int_1^2 (3-y)\ln y\,dy$

$$= 2\pi\left((3y - \tfrac{1}{2}y^2)\ln y + \tfrac{1}{4}y^2 - 3y\right)\Big|_1^2$$
$$= 2\pi\left(4\ln 2 + 1 - 6 - \tfrac{1}{4} + 3\right)$$
$$= \frac{\pi}{2}(16\ln 2 - 9)$$

[25 POINTS] Find the volume of the solid if the region bounded by $x = y^2 - 1$ and $y = x - 1$ is revolved around the line $y = -1$.



$$x = y + 1$$
$$y^2 - 1 = y + 1$$
$$y^2 - y - 2 = 0$$
$$(y+1)(y-2) = 0$$
$$y = -1, 2$$

$$2\pi \int_{-1}^{2} (y - -1)((y+1) - (y^2 - 1))\,dy$$
$$= 2\pi \int_{-1}^{2} (y+1)(-y^2 + y + 2)\,dy$$
$$= 2\pi \int_{-1}^{2} (-y^3 + y^2 + 2y - y^2 + y + 2)\,dy$$
$$= 2\pi \int_{-1}^{2} (-y^3 + 3y + 2)\,dy$$
$$= 2\pi \left(-\tfrac{1}{4}y^4 + \tfrac{3}{2}y^2 + 2y\right)\Big|_{-1}^{2}$$
$$= 2\pi \left(-\tfrac{1}{4}(16-1) + \tfrac{3}{2}(4-1) + 2(2 - -1)\right)$$
$$= 2\pi \left(-\tfrac{15}{4} + \tfrac{9}{2} + 6\right)$$
$$= 2\pi \left(\frac{-15 + 18 + 24}{4}\right)$$
$$= \frac{27\pi}{2}$$

**END OF NO-CALCULATOR SECTION:
YOU MUST HAND IN THIS SECTION TO
BEGIN USING YOUR CALCULATOR**

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YOU MAY USE YOUR CALCULATOR ON THIS SECTION OF THE MIDTERM

[10 POINTS] Find the area of the surface generated by revolving $y = \dfrac{1}{x^2}$ on [1, 4] around the x-axis.

$$\int_1^4 2\pi \left(\frac{1}{x^2}\right) \sqrt{1 + \left(\frac{-2}{x^3}\right)^2}\, dx \approx 6.041245974$$

☺ BONUS POINTS ☺

[10 BONUS POINTS] From our examples in class, we know that the hydrostatic forces on the top and bottom halves of a circular window of an aquarium are not equal.

[a] Find a formula for the difference between the forces in terms of the radius of the window (r) and the distance from the surface of the water to the center of the window (d). Assume all measurements are in feet. **[8 POINTS]**

[b] You should have noticed that the difference between the forces does not depend on how far below the surface of the water the window is. Find the smallest radius such that the difference between the forces is a whole number. (Assume that the density of water is EXACTLY 62.4 lb/ft^3.)

THANK YOU TO ANUJ FOR THE INSPIRATION FOR THIS BONUS QUESTION, ALTHOUGH HE PROBABLY DIDN'T EXPECT IT TO BE USED HERE.