[18 POINTS] Find the area between the curves $y = x^3 - 4x^2$ and $y = x - 4$.

INTERSECT $x = -1, 1, 4$

$$\int_{-1}^{1} (x^3 - 4x^2 - (x-4))\,dx + \int_{1}^{4} (x - 4 - (x^3 - 4x^2))\,dx$$

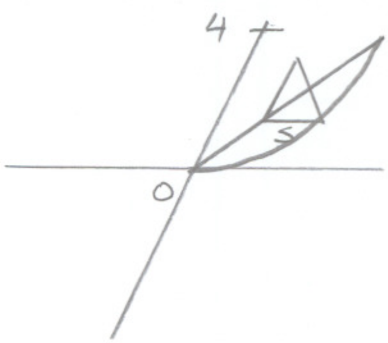
$$= \int_{-1}^{1} (x^3 - 4x^2 - x + 4)\,dx + \int_{1}^{4} (x - 4 - x^3 + 4x^2)\,dx$$

$$= \left(\tfrac{1}{4}x^4 - \tfrac{4}{3}x^3 - \tfrac{1}{2}x^2 + 4x\right)\Big|_{-1}^{1} + \left(\tfrac{1}{2}x^2 - 4x - \tfrac{1}{4}x^4 + \tfrac{4}{3}x^3\right)\Big|_{1}^{4}$$

$$= \left(\frac{29}{12} - -\frac{35}{12}\right) + \left(\frac{40}{3} - -\frac{29}{12}\right)$$

$$= \frac{253}{12}$$

[24 POINTS] The base of a solid is the region bounded by $y = x^2$ and $y = 2x$. Cross sections perpendicular to the y-axis are equilateral triangles. Find the volume of the solid.



$$A = \frac{\sqrt{3}}{4}s^2 = \frac{\sqrt{3}}{4}\left(\sqrt{y} - \tfrac{1}{2}y\right)^2$$

$$s = \sqrt{y} - \tfrac{1}{2}y$$

$$\int_0^4 \frac{\sqrt{3}}{4}\left(\sqrt{y} - \tfrac{1}{2}y\right)^2 dy$$

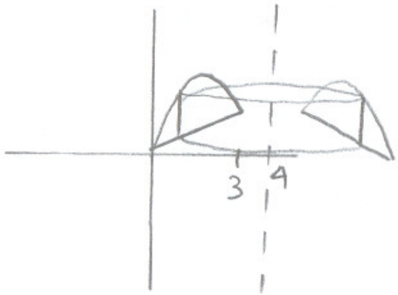
$$= \frac{\sqrt{3}}{4}\int_0^4 \left(y - y^{\frac{3}{2}} + \tfrac{1}{4}y^2\right) dy$$

$$= \frac{\sqrt{3}}{4}\left(\tfrac{1}{2}y^2 - \tfrac{2}{5}y^{\frac{5}{2}} + \tfrac{1}{12}y^3\right)\Big|_0^4$$

$$= \frac{\sqrt{3}}{4} \cdot \frac{8}{15}$$

$$= \frac{2\sqrt{3}}{15}$$

[24 POINTS] The region bounded by $y = 4x - x^2$ and $y = x$ is revolved around $x = 4$. Find the volume of the resulting solid.



$$\int_0^3 2\pi (4-x)(4x-x^2-x)\,dx$$

$$= 2\pi \int_0^3 (4-x)(3x-x^2)\,dx$$

$$= 2\pi \int_0^3 (12x - 4x^2 - 3x^2 + x^3)\,dx$$

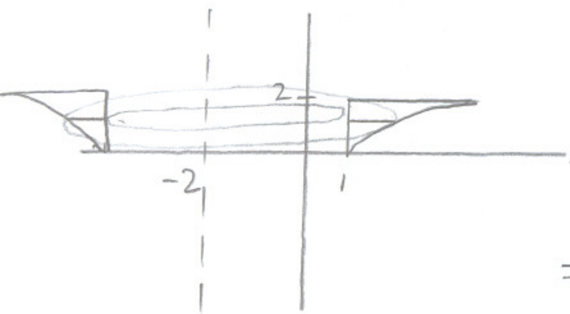
$$= 2\pi \int_0^3 (12x - 7x^2 + x^3)\,dx$$

$$= 2\pi \left(6x^2 - \frac{7}{3}x^3 + \frac{1}{4}x^4\right)\Big|_0^3$$

$$= 2\pi \left(\frac{45}{4}\right)$$

$$= \frac{45\pi}{2}$$

[24 POINTS] The region bounded by $y = \ln x$, $x = 1$ and $y = 2$ is revolved around $x = -2$. Find the volume of the resulting solid.



$$\int_0^2 \pi \left((e^y+2)^2 - (3)^2\right)dy$$

$$= \pi \int_0^2 (e^{2y} + 4e^y + 4 - 9)\,dy$$

$$= \pi \int_0^2 (e^{2y} + 4e^y - 5)\,dy$$

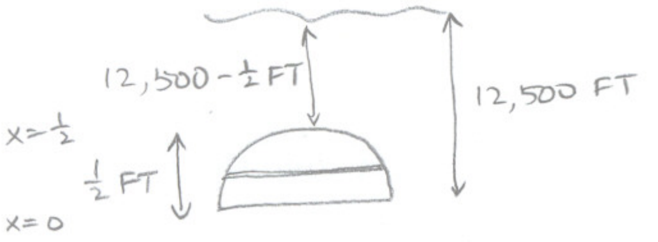
$$= \pi \left(\frac{1}{2}e^{2y} + 4e^y - 5y\right)\Big|_0^2$$

$$= \pi \left(\frac{1}{2}e^4 + 4e^2 - 10 - \left(\frac{1}{2} + 4\right)\right)$$

$$= \pi \left(\frac{1}{2}e^4 + 4e^2 - \frac{29}{2}\right)$$

$$= \frac{1}{2}\pi (e^4 + 8e^2 - 29)$$

[30 POINTS] In 1985, the wreck of the Titanic was discovered southeast of Newfoundland, Canada, at a depth of 12,536 feet. Suppose the ship had a semi-circular porthole (window) of radius 6 inches, located at a depth of 12,500 feet. How much hydrostatic force would the window have had to withstand if the porthole were standing vertically, with the straight side down, circular side up ? Assume the straight side of the porthole was located at a depth of exactly 12,500 feet.



ON 1 SLICE

$P = \delta h$

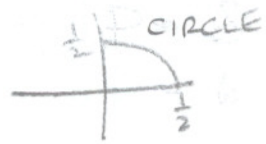
$= 62.4(12{,}500 - x)$

$F = PA$

$= 62.4(12{,}500 - x) 2\sqrt{\frac{1}{4} - x^2}\,\Delta x$

$$\int_0^{\frac{1}{2}} 124.8(12500 - x)\sqrt{\tfrac{1}{4} - x^2}\,dx$$

$$= (124.8)(12500)\underbrace{\int_0^{\frac{1}{2}} \sqrt{\tfrac{1}{4} - x^2}\,dx} - 124.8\int_0^{\frac{1}{2}} x\sqrt{\tfrac{1}{4} - x^2}\,dx$$



$\int = \frac{1}{4}\pi\left(\frac{1}{2}\right)^2$

$= \frac{\pi}{16}$

$u = \frac{1}{4} - x^2$

$\frac{du}{dx} = -2x$

$\frac{du}{-2x} = dx$

$\cancel{x}\sqrt{\frac{1}{4} - x^2}\,\frac{du}{-2\cancel{x}} = x\sqrt{\frac{1}{4} - x^2}\,dx$

$-\frac{1}{2}\int_{\frac{1}{4}}^{0} u^{\frac{1}{2}}\,du = -\frac{1}{2}\cdot\frac{2}{3}u^{\frac{3}{2}}\Big|_{\frac{1}{4}}^{0}$

$= -\frac{1}{3}u^{\frac{3}{2}}\Big|_{\frac{1}{4}}^{0}$

$= \frac{1}{3}\left(\frac{1}{4}\right)^{\frac{3}{2}}$

$= \frac{1}{24}$

$$= (124.8)(12500)\frac{\pi}{16} - 124.8\left(\frac{1}{24}\right)$$

$$= 97500\pi - 5.2$$

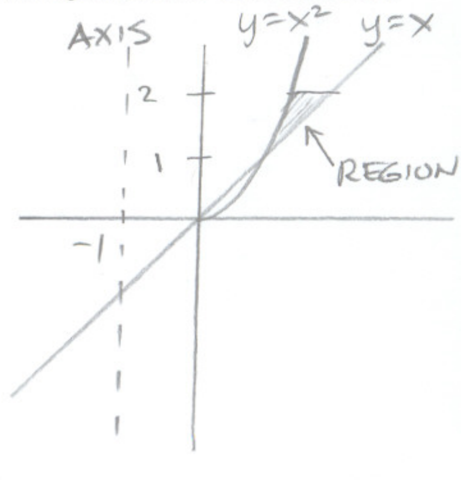
[6 POINTS] The integral $\int_1^2 \pi\left((y+1)^2 - \left(\sqrt{y}+1\right)^2\right)dy$ represents the volume of a solid. Sketch the region and axis of revolution that produce the solid.

WASHER : HORIZONTAL CUT

VERTICAL AXIS $x = -1$

$x = y$ AND $x = \sqrt{y}$

$y = x^2$



[24 POINTS] Abomino's Pizza promises that they will deliver your pizza within half an hour or your pizza is free. The actual delivery time of a randomly selected pizza is a continuous random variable, X, with values between 0 and 35 minutes. The probability density function for X is $f(x) = kx^2$ for some constant k.

Your answers for the following questions should be in decimal (or percent, where applicable).
You may use fnInt, but YOU MUST WRITE DOWN THE INTEGRALS YOU ARE USING.

(a) Find the value of k.

$$\int_0^{35} kx^2\,dx = 1 \implies k\int_0^{35} x^2\,dx = 1 \implies k\left(\tfrac{1}{3}x^3\right)\Big|_0^{35} = 1$$

$$k\left(\frac{35^3}{3}\right) = 1 \implies k = \frac{3}{35^3} = \frac{3}{42875} \quad \text{OR} \quad 7\times 10^{-5}$$

(b) Find the probability that your pizza is free.

$$P(30 < X \leq 35) = \int_{30}^{35} \frac{3}{42875}x^2\,dx = 37.026\%$$

(c) Find the mean delivery time for a pizza.

$$\int_0^{35} x\left(\frac{3}{42875}x^2\right)dx = \frac{3}{4\times 42875}x^4\Big|_0^{35} = \frac{3}{4\times 35^3}\,35^4$$

$$= \frac{105}{4} = 26.25 \text{ MINUTES}$$

(d) Find the median delivery time for a pizza.

$$\int_0^{M} \frac{3}{42875}x^2\,dx = \frac{1}{2}$$

$$\frac{1}{42875}x^3\Big|_0^{M} = \frac{1}{2}$$

$$\frac{1}{42875}M^3 = \frac{1}{2}$$

$$M^3 = \frac{42875}{2}$$

$$M = \sqrt[3]{\frac{42875}{2}}$$

$$= 27.78 \text{ MINUTES}$$

☻ BONUS POINTS ☻ BONUS POINTS ☻ BONUS POINTS ☻ BONUS POINTS ☻ BONUS POINTS ☻ BONUS POINTS ☻

[6 BONUS POINTS] A hemispherical sink is filled with water. The water is to be removed by pumping it over the top of the sink. What depth of water has been removed when half the work has been done ?

[4 BONUS POINTS] *(CONTINUED FROM PREVIOUS QUESTION)* What volume of water has been removed ?