HYPERBOLIC FUNCTIONS

The following worksheet is a self-study method for you to learn about the hyperbolic functions, which are algebraically similar to, yet subtly different from, trigonometric functions. In order to complete the worksheet, you need to refer back to topics from trigonometry, precalculus and differential calculus. In particular, you may need to refresh yourself on the following:

Trigonometry:

quotient, reciprocal, Pythagorean, sum/difference of angles, double angle identities
(you only need to know the identities, not how to prove them)

Precalculus:

determining if a function has odd/even symmetry algebraically and graphically
determining if a function is one-to-one graphically
relationship between the graph/domain/range of a function and its inverse
using function composition to determine if two functions are inverses of each other
solving for the inverse of a function algebraically

Calculus:

finding vertical/horizontal asymptotes of a function algebraically
using L'Hopital's Rule for appropriate limits
proofs of the derivatives of inverse trigonometric functions

The solutions of the questions on the worksheet depend on some skills you should already have, as well as some you will develop this quarter. You should already be able to handle all questions except #14 and parts of #8. You will be able to handle those two remaining questions after we cover the chapter on advanced integration techniques.

Skills Required	**Questions On Worksheet**
Algebra	1, 2, 3a-d, 4, 9, 10, 11
Limits & Differentiation	3e, 5, 6, 8ac, 12, 13
Integration	7, 8bde, 14

This worksheet must be completed before Thanksgiving, and the material on it will appear on the final exam.

$\sinh x = \dfrac{e^x - e^{-x}}{2}$

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$\cosh x = \dfrac{e^x + e^{-x}}{2}$

[1] Rewrite in terms of exponential functions using the definitions above, simplify, then rewrite in terms of hyperbolic functions, if possible.

[a] $\sinh(-x)$
[b] $\cosh(-x)$
[c] $\cosh^2 x + \sinh^2 x$
[d] $\cosh^2 x - \sinh^2 x$
[e] $2\sinh x \cosh x$
[f] $e^x \sinh x$
[g] $\dfrac{\cosh x}{e^x}$
[h] $\sinh(\ln x)$
[i] $\cosh(2\ln x)$

[2] Based on the similarities to the properties of trigonometric functions noted in [1], conjecture and prove formulae for the following in terms of hyperbolic functions.

[a] $\sinh(x+y)$
[b] $\sinh(x-y)$
[c] $\cosh(x+y)$
[d] $\cosh(x-y)$

[3] Define $\tanh x = \dfrac{\sinh x}{\cosh x}$.

[a] Rewrite $\tanh x$ in terms of exponential functions.
[b] Simplify $\tanh(-x)$ in two ways
  [i] by first rewriting in terms of exponential functions, simplifying, then rewriting in terms of hyperbolic functions
  [ii] by using your answers in [1][a] and [1][b]
[c] Define the three remaining hyperbolic functions in a parallel fashion in terms of other hyperbolic functions.
[d] Rewrite the three remaining hyperbolic functions in terms of exponential functions.
[e] Find the limits as $x \to \pm\infty$ of all hyperbolic functions.

Once you get used to the identities, it is much easier to manipulate the hyperbolic functions without rewriting them in terms of exponential functions.

[4] You should have discovered a hyperbolic parallel to the Pythagorean Identity in [1][d].

[a] Rewrite the identity in [1][d] in two ways
  [i] by solving for $\sinh^2 x$
  [ii] by solving for $\cosh^2 x$
[b] Rewrite the identity in [1][c] in two ways
  [i] by substituting using your answer from [4][a][i]
  [ii] by substituting using your answer from [4][a][ii]
[c] Find the hyperbolic parallels to the other Pythagorean Identities by dividing both sides of [1][d]
  [i] by $\sinh^2 x$
  [ii] by $\cosh^2 x$

CALCULUS OF HYPERBOLIC FUNCTIONS

[5] Using the original exponential definitions of $\sinh x$ and $\cosh x$, find the derivatives of $\sinh x$ and $\cosh x$, and rewrite in terms of hyperbolic functions.

[6] Using the hyperbolic definitions from [3], the quotient rule for derivatives, the derivatives from [5], and the various identities from [4], find the derivatives of the other four hyperbolic functions in terms of hyperbolic functions.

[7] Rewrite your derivatives from [5] and [6] using integral notation. Do not use negatives in your integrands.
(eg. if the derivative of $\frac{1}{x}$ is $-\frac{1}{x^2}$, write $\int \frac{1}{x^2}\,dx = -\frac{1}{x} + C$.)

[8] You now have an arsenal of basic identities, and differentiation and integration rules for the hyperbolic functions. Use those rules, along with the product, quotient and chain rules for derivatives, and substitution and integration by parts, to evaluate the following. Simplify your answers.

[a] Find the derivatives of the following.
[i] $\sinh 8x$
[ii] $\cosh 4x^2$
[iii] $x \tanh^2 x$
[iv] $e^{-2x} \sinh 3x$
[v] $\dfrac{\sinh x}{1 - \cosh x}$
[vi] $\tanh(\ln x)$
[vii] $\sin^{-1}(\tanh x)$

[b] Find the anti-derivatives of the following. HINT: If you get stuck, consider what technique you would use if you replaced the hyperbolic functions with their trigonometric parallels.
[i] $\sinh 8x$
[ii] $\dfrac{\cosh \sqrt{x}}{\sqrt{x}}$
[iii] $x^2 \sinh 2x$
[iv] $e^{-2x} \cosh 3x$
[v] $\dfrac{\cosh x}{1 - \sinh x}$
[vi] $\sinh(\ln x)$
[vii] $\sinh^3 x \cosh^4 x$

[c] Redo [8][a][iv] and [8][a][vi] by rewriting the functions using the exponential definitions, simplifying, differentiating, then simplifying again. Your final answers should not involve hyperbolic functions.

[d] Redo [8][b][iv] and [8][b][vi] by rewriting the functions using the exponential definitions, simplifying, integrating, then simplifying again. Your final answers should not involve hyperbolic functions. Would you consider this technique for [8][b][vii] ?

[e] Find $\int \cosh(2\ln x)dx$ and $\int \tanh(\ln x)dx$. (HINT: See [8][d].)

INVERSE HYPERBOLIC FUNCTIONS

[9] Use your calculator's graphing capabilities to sketch $y = \sinh x$, $y = \cosh x$ and $y = \tanh x$.

[a] Do the graphs confirm the symmetry and limits you found in [1][a] and [b], and [3][b] and [d] ? State the domain and range of each function, and identify all intercepts, and horizontal and vertical asymptotes.

[b] Recall that a function has an inverse function if and only if the function is one-to-one. Which of the three functions is one-to-one ?

[c] A function which is not one-to-one, can be "made" one-to-one by restricting its domain (eg. the trigonometric functions). How would you restrict the domain of the non one-to-one function(s) in [9][b] to "make" it (them) one-to-one ?

[10] Solve $\sinh x = 1$ and $\cosh x = 1$ by using the exponential definitions and an algebraic substitution $z = e^x$.

[11] $y = \sinh^{-1} x$ if and only if $x = \sinh y$.

[a] Define $\cosh^{-1} x$ and $\tanh^{-1} x$. State the domain and range of all three inverse hyperbolic functions, and identify all intercepts, and horizontal and vertical asymptotes.

[b] Prove that $\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$ in two ways

[i] by simplifying $\sinh\left(\ln\left(x + \sqrt{x^2 + 1}\right)\right)$ using the exponential definition

[ii] by solving $x = \sinh y$ for y using the exponential definition and an algebraic substitution $z = e^y$

[c] Find formulae for $\cosh^{-1} x$ and $\tanh^{-1} x$. (HINT: See [11][b][ii].)

[12] Find the derivatives of all three inverse hyperbolic functions in two ways

[a] by using the formulae in [11][b] and [c]

[b] by using implicit differentiation on $y = \sinh^{-1} x$, $y = \cosh^{-1} x$ and $y = \tanh^{-1} x$ (NOTE: This technique is similar to how you found the derivatives of the inverse trigonometric functions.)

[13] Find the derivatives of the following. Simplify your answers.

[a] $\sinh^{-1} 8x$

[b] $\cosh^{-1} 4x^2$

[c] $x\left(\tanh^{-1} x\right)^2$

[d] $e^{-2x} \sinh^{-1} 3x$

[e] $\cosh^{-1}(\csc x)$

[f] $\tanh^{-1}(\cos x)$

[14] Find $\int \sinh^{-1} x \, dx$, $\int \cosh^{-1} x \, dx$ and $\int \tanh^{-1} x \, dx$. HINT: The technique is similar to how you found the integrals of the inverse trigonometric functions.