THE LANGUAGE OF THEOREMS

Many theorems in math are worded using "if … then" sentences.

For example, the Mean Value Theorem says

> "If f is continuous on $[a, b]$ and differentiable on (a, b),
> then there exists a $c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$."

To understand theorems, you have to understand

what is meant by "the conditions and conclusion of a theorem"
what it means to "satisfy the conditions of a theorem" or for "a theorem to apply"
what it means to "satisfy the conclusion of a theorem"
what it means to "contradict (or not contradict) an if/then statement"

The “if” part of a theorem is called the **conditions**, **assumptions** or **hypotheses** of the theorem.

The **conditions** of the Mean Value Theorem are

f is continuous on $[a, b]$
f is differentiable on (a, b)

A situation **satisfies the conditions** of a theorem if, in that situation, all the conditions are true.
Another way of saying “a situation **satisfies the conditions** of a theorem” is “**the theorem applies** to the situation”.

In order to show that a situation **satisfies the conditions** of a theorem
(or **the theorem applies** to the situation**),**
you must show that all the conditions are true.

In order to show that a situation **does not satisfy the conditions** of a theorem
(or **the theorem does not apply** to the situation**),**
you must show that one of the conditions is false.

[A] $f(x)=x^2$ on $[-1, 2]$ **satisfies the conditions** of the Mean Value Theorem
or the Mean Value Theorem **applies** to $f(x)=x^2$ on $[-1, 2]$

because $f(x)=x^2$ is continuous on $[-1, 2]$ since it is a polynomial function
and $f(x)=x^2$ is differentiable on $(-1, 2)$ since it is a polynomial function

[B] $f(x)=\sqrt[3]{x}$ on $[-1, 8]$ **does not satisfy the conditions** of the Mean Value Theorem
or the Mean Value Theorem **does not apply** to $f(x)=\sqrt[3]{x}$ on $[-1, 8]$

because $f(x)=\sqrt[3]{x}$ is not differentiable on $(-1, 8)$ since it is not differentiable at $x=0$

NOTE: It is true that $f(x)=\sqrt[3]{x}$ is continuous on $[-1, 8]$, but that isn't important, because the second condition is not satisfied, so even though the first condition is satisfied, the condition**s** are not satisfied

[C] $f(x)=\dfrac{1}{x^2}$ on $[-1, 2]$ **does not satisfy the conditions** of the Mean Value Theorem
or the Mean Value Theorem **does not apply** to $f(x)=\dfrac{1}{x^2}$ on $[-1, 2]$

because $f(x)=\dfrac{1}{x^2}$ is not continuous on $[-1, 2]$ since it is not continuous at $x=0$

NOTE: It is also true that $f(x)=\dfrac{1}{x^2}$ is not differentiable on $(-1, 2)$, but that isn't important, because the first condition is already not satisfied, so the condition**s** are not satisfied

[D] $f(x)=\dfrac{1}{x}$ on $[-1, 1]$ **does not satisfy the conditions** of the Mean Value Theorem
or the Mean Value Theorem **does not apply** to $f(x)=\dfrac{1}{x}$ on $[-1, 1]$

because $f(x)=\dfrac{1}{x}$ is not continuous on $[-1, 1]$ since it is not continuous at $x=0$

NOTE: It is also true that $f(x)=\dfrac{1}{x}$ is not differentiable on $(-1, 1)$, but that isn't important, because the first condition is already not satisfied, so the condition**s** are not satisfied

The "then" part of a theorem is called the **conclusion** of the theorem.

The **conclusion** of the Mean Value Theorem is

there exists a $c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

A situation **satisfies the conclusion** of a theorem if, in that situation, the conclusion is true.

In order to show that a situation **satisfies the conclusion** of a theorem,
you must show that the conclusion is true.

In order to show that a situation **does not satisfy the conclusion** of a theorem,
you must show that the conclusion is false.

NOTE: **If you are only trying to determine whether a situation satisfies the conclusion of a theorem, you do NOT consider whether the conditions are satisfied.**

[A] $f(x) = x^2$ on $[-1, 2]$ **satisfies the conclusion** of the Mean Value Theorem

because $\dfrac{f(2) - f(-1)}{2 - (-1)} = \dfrac{4 - 1}{2 - (-1)} = 1 = f'(\frac{1}{2})$ and $\frac{1}{2} \in (-1, 2)$

[B] $f(x) = \sqrt[3]{x}$ on $[-1, 8]$ **satisfies the conclusion** of the Mean Value Theorem

because $\dfrac{f(8) - f(-1)}{8 - (-1)} = \dfrac{2 - (-1)}{8 - (-1)} = \dfrac{1}{3} = f'(1)$ and $1 \in (-1, 8)$

NOTE: It is also true that $f'(-1) = \dfrac{1}{3}$ and $-1 \notin (-1, 8)$, but that doesn't change the fact the conclusion is true for $c = 1 \in (-1, 8)$

NOTE: It is also true that $f(x) = \sqrt[3]{x}$ on $[-1, 8]$ **does not satisfy the conditions** of the Mean Value Theorem (the Mean Value Theorem **does not apply**), but that doesn't change the fact that the conclusion is true

[C] $f(x) = \dfrac{1}{x^2}$ on $[-1, 2]$ **does not satisfy the conclusion** of the Mean Value Theorem

because $\dfrac{f(2) - f(-1)}{2 - (-1)} = \dfrac{\frac{1}{4} - 1}{2 - (-1)} = -\dfrac{1}{4} = f'(c)$ only when $c = 2$, but $2 \notin (-1, 2)$

[D] $f(x) = \dfrac{1}{x}$ on $[-1, 1]$ **does not satisfy the conclusion** of the Mean Value Theorem

because $\dfrac{f(1) - f(-1)}{1 - (-1)} = \dfrac{1 - (-1)}{1 - (-1)} = 1 \neq f'(c)$ for any value of c anywhere

A situation **contradicts an if/then statement** if the situation **satisfies the conditions** of the statement, but **does not satisfy the conclusion** of the statement.

In order to show that a situation **contradicts an if/then statement**,
you must show that the situation **satisfies the conditions (the statement applies),**
but **does not satisfy the conclusion** of the statement.
That is, you must show that all the conditions are true, but the conclusion is false.

In order to show that a situation **does not contradict an if/then statement**,
either you must show that the situation **does not satisfy the conditions (the statement does not apply)**
or the situation **satisfies the conditions (the statement applies) and the conclusion** of the statement.
That is, either you must show that one of the conditions is false,
or you must show that the conditions and the conclusion are true.

[A] $f(x)=x^2$ on $[-1, 2]$ **does not contradict** the Mean Value Theorem

because $f(x)=x^2$ on $[-1, 2]$ **satisfies the conditions** of the Mean Value Theorem
(the Mean Value Theorem **applies** to $f(x)=x^2$ on $[-1, 2]$)
and $f(x)=x^2$ on $[-1, 2]$ **satisfies the conclusion** of the Mean Value Theorem

[B] $f(x)=\sqrt[3]{x}$ on $[-1, 8]$ **does not contradict** the Mean Value Theorem

because $f(x)=\sqrt[3]{x}$ on $[-1, 8]$ **does not satisfy the conditions** of the Mean Value Theorem
(the Mean Value Theorem **does not apply** to $f(x)=\sqrt[3]{x}$ on $[-1, 8]$)

NOTE: It is true that $f(x)=\sqrt[3]{x}$ on $[-1, 8]$ **satisfies the conclusion** of the Mean Value Theorem, but that does not contradict the Mean Value Theorem since the theorem only says what must happen in situations which satisfy the conditions, not what must happen in situations which do not satisfy the conditions

[C] $f(x)=\dfrac{1}{x^2}$ on $[-1, 2]$ **does not contradict** the Mean Value Theorem

because $f(x)=\dfrac{1}{x^2}$ on $[-1, 2]$ **does not satisfy the conditions** of the Mean Value Theorem

(the Mean Value Theorem **does not apply** to $f(x)=\dfrac{1}{x^2}$ on $[-1, 2]$)

NOTE: It is true that $f(x)=\dfrac{1}{x^2}$ on $[-1, 2]$ **does not satisfy the conclusion** of the Mean Value Theorem, but that does not contradict the Mean Value Theorem since the theorem only says what must happen in situations which satisfy the conditions, not what must happen in situations which do not satisfy the conditions

[D] $f(x)=\dfrac{1}{x}$ on $[-1, 1]$ **does not contradict** the Mean Value Theorem

because $f(x)=\dfrac{1}{x}$ on $[-1, 1]$ **does not satisfy the conditions** of the Mean Value Theorem

(the Mean Value Theorem **does not apply** to $f(x)=\dfrac{1}{x}$ on $[-1, 1]$)

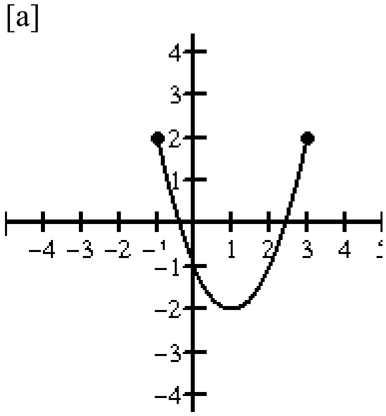
NOTE: It is true that $f(x)=\dfrac{1}{x}$ on $[-1, 1]$ **does not satisfy the conclusion** of the Mean Value Theorem, but that does not contradict the Mean Value Theorem since the theorem only says what must happen in situations which satisfy the conditions, not what must happen in situations which do not satisfy the conditions

You may have noticed that none of the examples contradict the Mean Value Theorem. That's because, in math, an if/then statement is not called a theorem if it is possible to contradict it. If you can contradict an if/then statement, in math, we simply say the statement is false.

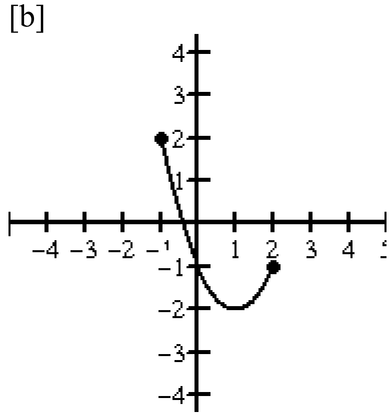
Now, some problems for you to try, based on Rolle's Theorem:

[1] What are the conditions of Rolle's Theorem ? HINT: There are 3 of them.
[2] What is the conclusion of Rolle's Theorem ?
[3] Which of the functions below do NOT satisfy the conditions of Rolle's Theorem on the domain shown ? Name all conditions which are NOT satisfied. NOTE: [d], [g] have cusps.
[4] Which of these functions below do NOT satisfy the conclusion of Rolle's Theorem on the domain shown ?
[5] Why does each of the functions below NOT contradict Rolle's Theorem ? Answer in terms of conditions and conclusions. Don't just answer "because it's a theorem".
[6] Is it possible to find a function and a domain which contradict Rolle's Theorem ? Why or why not ?

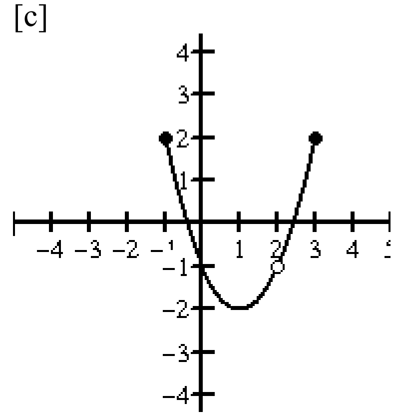
[a]



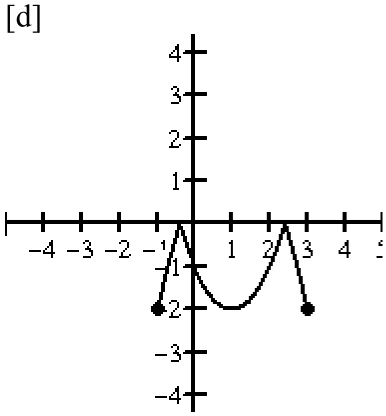
[b]



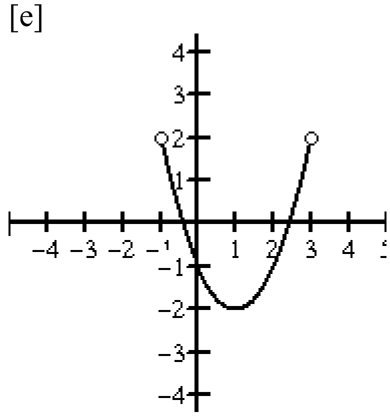
[c]



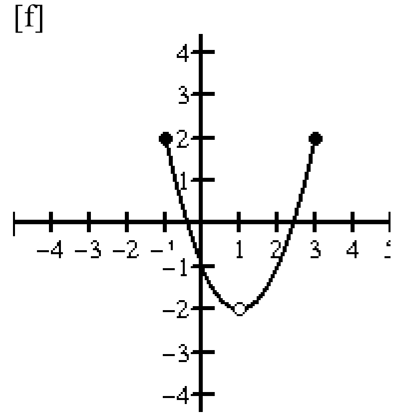
[d]



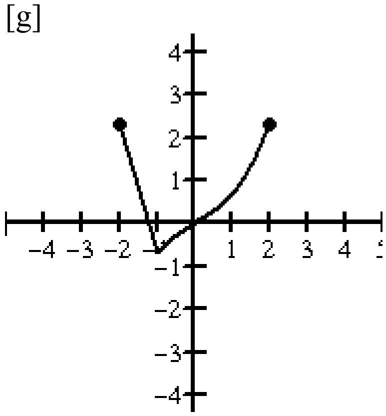
[e]



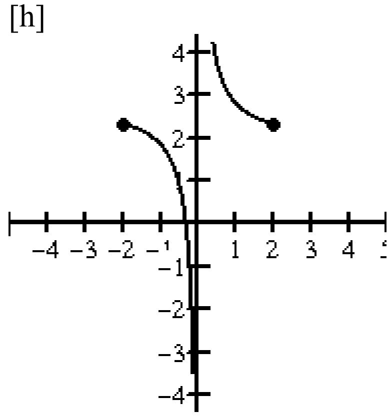
[f]



[g]



[h]



[j]