Math 1A (9:30am – 10:20am)
Midterm 2
Tue Feb 23, 2010

SCORE: ___ / 140 POINTS

What month is your birthday ? __ __
What are the first 2 digits of your address ? __ __
What are the last 2 digits of your zip code ? __ __
What are the last 2 digits of your social security number ? __ __
[IF YOU DO NOT HAVE A SOCIAL SECURITY NUMBER, USE YOUR STUDENT ID NUMBER]

★ NO CALCULATORS ALLOWED
★ YOU MUST SHOW PROPER CALCULUS-LEVEL WORK AND LOGIC

State the squeeze theorem.

SCORE: ✳ / 0 POINTS
IF WRONG, –7 POINTS

IF $f(x) \le g(x) \le h(x)$ FOR ALL $x \in (b, c)$ (EXCEPT POSSIBLY AT $x = a$) AND $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$ THEN $\lim_{x\to a} g(x) = L$

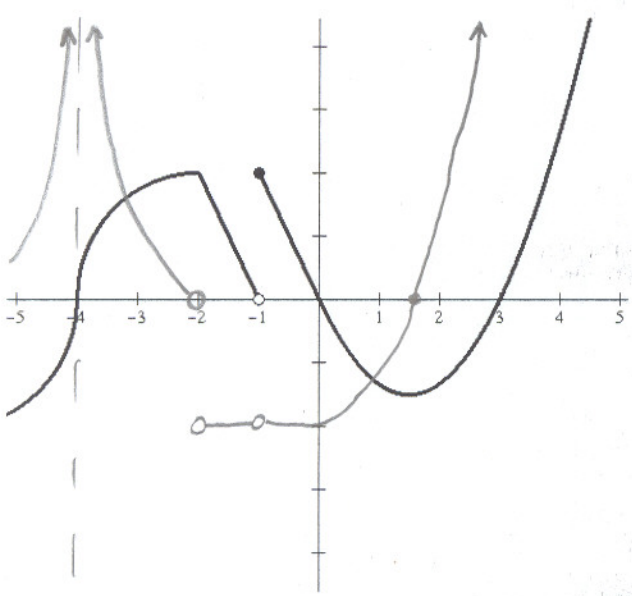
State the definition of a horizontal asymptote.

SCORE: ✳ / 0 POINTS
IF WRONG, –7 POINTS

A FUNCTION f HAS A HORIZONTAL ASYMPTOTE AT $y = b$ IF $\lim_{x\to\infty} f(x) = b$ OR $\lim_{x\to -\infty} f(x) = b$

The graph of $f(x)$ is shown below.

SCORE: ✳ / 16 POINTS



[a] Find all x-coordinates where $f'(x)$ is undefined, and explain **very briefly** why.

$x = -4$ VERTICAL T.L.

$x = -2$ CUSP

$x = -1$ DISCONT.

[b] Sketch a graph of $f'(x)$ on the same axes above.

State the definition of e. (You may give either the definition from lecture, or the definition in section 3.6.) **SCORE: ___ / 8 POINTS**

E IS THE NUMBER SUCH THAT $\lim_{h\to 0} \frac{e^h-1}{h}$

OR $e = \lim_{x\to 0}(1+x)^{\frac{1}{x}}$ OR $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$

A baby's weight (w) at birth depends on her mother's age (a) when the baby was conceived. If $w = f(a)$, where w is measured in ounces, and a is measured in years, give the practical meaning, including units, for the statement $f'(36) = -2$. **SCORE: ___ / 8 POINTS**

IF A WOMAN CONCEIVES A BABY WHEN SHE IS 36 YEARS OLD, THE BABY'S BIRTH WEIGHT WOULD DECREASE BY 2 OUNCES FOR EACH YEAR THE WOMAN DELAYED GETTING PREGNANT.

Using the definition of the derivative, prove the derivative of $f(x) = \csc x$. **SCORE: _____ / 14 POINTS**

You may use the two trigonometric limits proved in class, without reproving them. You MUST NOT use any differentiation shortcuts.

$$f'(x) = \lim_{h\to 0}\frac{\csc(x+h)-\csc x}{h}$$

$$= \lim_{h\to 0}\frac{\frac{1}{\sin(x+h)}-\frac{1}{\sin x}}{h}$$

$$= \lim_{h\to 0}\frac{\sin x - \sin(x+h)}{h\sin x\sin(x+h)}$$

$$= \lim_{h\to 0}\frac{\sin x - \sin x\cos h - \cos x\sin h}{h\sin x\sin(x+h)}$$

$$= \lim_{h\to 0}\left(\frac{\sin x}{\sin x\sin(x+h)}\cdot\frac{1-\cos h}{h} - \frac{\cos x}{\sin x\sin(x+h)}\cdot\frac{\sin h}{h}\right)$$

$$= 0 - 1\cdot\frac{\cos x}{\sin x}\frac{1}{\sin x}$$

$$= -\cot x\csc x$$

Prove the derivative of $f(x) = \cos^{-1} x$. **SCORE: _____ / 14 POINTS**

You may use the derivatives of any trigonometric function without reproving them.

$y = \cos^{-1} x$

$x = \cos y$ AND $y \in [0, \pi]$

$1 = -\sin y \cdot \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{-\sin y}$

$= -\frac{1}{\sqrt{1-x^2}}$

$\sin y \geq 0$

SO, $\sin y = \sqrt{1-\cos^2 y} = \sqrt{1-x^2}$

Show that $xy = a$ and $x^2 - y^2 = b$ are orthogonal trajectories (where a and b are constants). **SCORE: * / 14 POINTS**

<u>HINT: Do NOT try to solve for y.</u>

$$xy = a \qquad\qquad x^2 - y^2 = b$$

$$y + x\frac{dy}{dx} = 0 \qquad\qquad 2x - 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{y}{x} \qquad\qquad \frac{dy}{dx} = \frac{x}{y}$$

$$\frac{-y}{x} \cdot \frac{x}{y} = -1 \quad \text{T.L. ARE } \perp$$

If $q(t) = (\tan^{-1} t)^{\sec t}$, find $q'(t)$. **SCORE: * / 14 POINTS**

$$y = (\tan^{-1} t)^{\sec t}$$

$$\ln y = (\sec t)\ln \tan^{-1} t$$

$$\frac{1}{y}\frac{dy}{dx} = \sec t \tan t \ln \tan^{-1} t + \sec t \frac{1}{\tan^{-1} t}\frac{1}{1+t^2}$$

$$\frac{dy}{dx} = y \sec t \left(\tan t \ln \tan^{-1} t + \frac{1}{(1+t^2)\tan^{-1} t}\right)$$

$$= (\tan^{-1} t)^{\sec t} \sec t \left(\tan t \ln \tan^{-1} t + \frac{1}{(1+t^2)\tan^{-1} t}\right)$$

Find the equation of the tangent line to the curve $2^{xy^2} = x - y$ at $(1, -1)$. **SCORE: * / 14 POINTS**

$$(\ln 2) 2^{xy^2}\left(y^2 + 2xy\frac{dy}{dx}\right) = 1 - \frac{dy}{dx}$$

$$(\ln 2)\, 2\left(1 - 2\frac{dy}{dx}\bigg|_{(1,-1)}\right) = 1 - \frac{dy}{dx}\bigg|_{(1,-1)}$$

$$2\ln 2 - 1 = (4\ln 2 - 1)\frac{dy}{dx}\bigg|_{(1,-1)}$$

$$\frac{dy}{dx}\bigg|_{(1,-1)} = \frac{2\ln 2 - 1}{4\ln 2 - 1} \qquad \text{T.L. } y + 1 = \frac{2\ln 2 - 1}{4\ln 2 - 1}(x - 1)$$

If $g(x)=\dfrac{x^3+2x-4}{\sqrt[3]{x}}$, find $g''(x)$. **SIMPLIFY YOUR ANSWER.**

SCORE: ____ / 8 POINTS

$$g(x)=\frac{x^3+2x-4}{x^{\frac{1}{3}}}=x^{\frac{8}{3}}+2x^{\frac{2}{3}}-4x^{-\frac{1}{3}}$$

$$g'(x)=\frac{8}{3}x^{\frac{5}{3}}+\frac{4}{3}x^{-\frac{1}{3}}+\frac{4}{3}x^{-\frac{4}{3}}$$

$$g''(x)=\frac{40}{9}x^{\frac{2}{3}}-\frac{4}{9}x^{-\frac{4}{3}}-\frac{16}{9}x^{-\frac{7}{3}}=\frac{4}{9}x^{-\frac{7}{3}}(10x^3-x-4)$$

The position of an object at time t is given by $s(t)=\ln\sqrt{1+t^2}$.
Find the **acceleration** of the object at time $t=2$.

SCORE: ____ / 14 POINTS

$$s(t)=\frac{1}{2}\ln(1+t^2)$$

$$v(t)=s'(t)=\frac{1}{2}\frac{1}{1+t^2}2t=\frac{t}{1+t^2}$$

$$a(t)=v'(t)=\frac{1+t^2-t(2t)}{(1+t^2)^2}=\frac{1-t^2}{(1+t^2)^2}$$

$$a(2)=\frac{1-4}{(1+4)^2}=\frac{-3}{25}$$

A television camera is positioned 3000 ft from the base of a rocket launching pad. A rocket rises vertically from the launching pad at 900 ft/s. If the camera is always aimed at the rocket, how quickly is the camera tilting upward when the rocket has risen 4000 ft?

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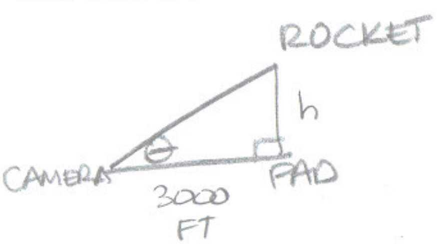
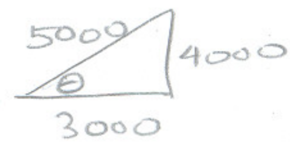


$$\frac{dh}{dt}=900\text{ ft/s}$$

WANT $\frac{d\theta}{dt}$ WHEN $h=4000$ ft



$$\tan\theta=\frac{h}{3000}$$

$$\sec^2\theta\frac{d\theta}{dt}=\frac{1}{3000}\frac{dh}{dt}$$

$$\left(\frac{5}{3}\right)^2\frac{d\theta}{dt}=\frac{1}{\cancel{3000}_{10}}\ {}^{3}\cancel{900}$$

$$\frac{d\theta}{dt}=\frac{3}{10}\cdot\frac{9}{25}=\frac{27}{250}\ \frac{\text{RADIANS}}{\text{SEC}}$$