

SCORE: ____ / 150 POINTS

From your textbook, remember that the Fibonacci sequence is defined by

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$$F_0 = F_1 = 1$$

$$F_k = F_{k-1} + F_{k-2} \text{ for all integers } k \text{ where } k \geq 2$$

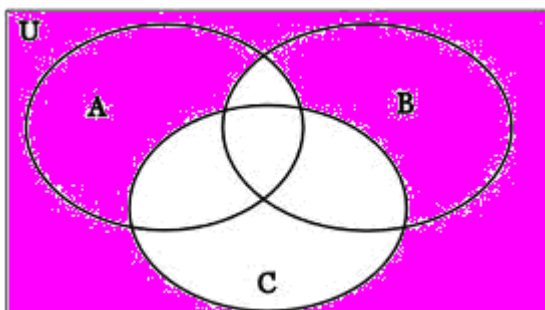
Show that the terms of the Fibonacci sequence satisfy the recurrence relation $F_k = \frac{F_{k+1} + F_{k-2}}{2}$ for all integers k where $k \geq 2$.

“Show” means you do NOT have to write a formal proof, but you must demonstrate the supporting algebra.

$$\begin{aligned} & \frac{F_{k+1} + F_{k-2}}{2} \\ &= \frac{(F_k + F_{k-1}) + F_{k-2}}{2} \\ &= \frac{F_k + (F_{k-1} + F_{k-2})}{2} \\ &= \frac{F_k + F_k}{2} \\ &= F_k \end{aligned}$$

Shade in the region(s) in the Venn diagram corresponding to $C^c - (A \cap B)$.

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Consider the sequence defined by

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$$a_0 = 4$$

$$a_1 = 8$$

$$a_k = (2+k)a_{k-1} - 2ka_{k-2} \text{ for all } k \in \mathbb{Z} \text{ such that } k \geq 2$$

[a] Find a_2 and a_3 .

$$a_2 = (2+2)a_1 - 2(2)a_0 = 4(8) - 4(4) = 16$$

$$a_3 = (2+3)a_2 - 2(3)a_1 = 5(16) - 6(8) = 32$$

[b] Based on the values of a_0 , a_1 , a_2 and a_3 , guess and prove the general formula for the sequence.

If a general formula/pattern does not seem somewhat obvious, check your arithmetic in [a].

Claim: $a_n = 2^{n+2}$

Proof by strong induction:

Basis step: $a_0 = 4 = 2^{0+2}$ and $a_1 = 8 = 2^{1+2}$

Inductive step: Assume that $a_n = 2^{n+2}$ for $n = 0, 1, \dots, k$ for some particular but arbitrary integer $k \geq 1$.

$$[\text{NEED TO SHOW: } a_{k+1} = 2^{(k+1)+2} = 2^{k+3}]$$

$$k-1 \geq 0 \text{ (since } k \geq 1) \text{ and } k-1 \leq k.$$

$$a_{k+1}$$

$$= (2 + (k+1))a_k - 2(k+1)a_{k-1}$$

$$= (3+k)2^{k+2} - (2k+2)2^{k+1}$$

$$= 3(2^{k+2}) + k2^{k+2} - 2k(2^{k+1}) - 2(2^{k+1})$$

$$= 3(2^{k+2}) + k2^{k+2} - k2^{k+2} - 2^{k+2}$$

$$= 2(2^{k+2})$$

$$= 2^{k+3}$$

So, by strong induction, $a_n = 2^{n+2}$ for $n \in \mathbb{Z}^{\text{nonneg}}$.

Simplify the expression $\binom{n+2}{n} - \binom{n}{2}$.

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$$\begin{aligned} & \frac{(n+2)!}{n!2!} - \frac{n!}{2!(n-2)!} \\ &= \frac{(n+2) \times (n+1) \times n!}{n!2!} - \frac{n \times (n-1) \times (n-2)!}{2!(n-2)!} \\ &= \frac{(n+2)(n+1)}{2} - \frac{n(n-1)}{2} \\ &= \frac{n^2 + 3n + 2}{2} - \frac{n^2 - n}{2} \\ &= \frac{4n + 2}{2} \\ &= 2n + 1 \end{aligned}$$

Use an element argument to prove the following statement.

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If $A \subseteq B^C$, then $A \cap B = \emptyset$

(Assume that A and B are subsets of some universal set U .)

Let A and B be particular but arbitrary sets such that $A \subseteq B^C$.

Suppose that $A \cap B \neq \emptyset$.

So, there exists an element $x \in A \cap B$.

So, $x \in A$ and $x \in B$, by the definition of \cap .

But, since $x \in A$ and $A \subseteq B^C$, therefore $x \in B^C$, by the definition of \subseteq .

So, $x \notin B$ by the definition of set complement.

But, $x \in B$ and $x \notin B$, a contradiction.

So, $A \cap B = \emptyset$.

Prove the statement

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$$(n^3 + 2n + 7) \bmod 3 = 1 \text{ for all integers } n \geq 2$$

in two ways:

[a] PROOF USING INDUCTION:

Basis step: $n = 2 : 2^3 + 2(2) + 7 = 19 = 3(6) + 1$

By definition of mod, $(2^3 + 2(2) + 7) \bmod 3 = 1$.

Inductive step: Assume that $(k^3 + 2k + 7) \bmod 3 = 1$ for some particular but arbitrary integer $k \geq 2$.

[NEED TO SHOW: $((k+1)^3 + 2(k+1) + 7) \bmod 3 = 1$]

By the definition of mod, $k^3 + 2k + 7 = 3m + 1$ for some $m \in \mathbb{Z}$.

$$(k+1)^3 + 2(k+1) + 7$$

$$= k^3 + 3k^2 + 5k + 10$$

$$= k^3 + 2k + 7 + 3k^2 + 3k + 3$$

$$= 3m + 1 + 3k^2 + 3k + 3$$

$$= 3(m + k^2 + k + 1) + 1$$

where $m + k^2 + k + 1 \in \mathbb{Z}$ by closure of \mathbb{Z} under multiplication and addition.

So, $(n^3 + 2n + 7) \bmod 3 = 1$ for all integers $n \geq 2$.

[b] PROOF WITHOUT USING INDUCTION:

Let n be a particular but arbitrary integer such that $n \geq 2$.

By the Quotient Remainder Theorem, $n = 3q$ or $n = 3q + 1$ or $n = 3q + 2$ for some $q \in \mathbb{Z}$.

CASE 1: $n = 3q$

$$n^3 + 2n + 7 = (3q)^3 + 2(3q) + 7 = 27q^3 + 6q + 7 = 3(9q^3 + 2q + 2) + 1$$

where $9q^3 + 2q + 2 \in \mathbb{Z}$ by closure of \mathbb{Z} under multiplication and addition.

By the definition of mod, $(n^3 + 2n + 7) \bmod 3 = 1$.

CASE 2: $n = 3q + 1$

$$n^3 + 2n + 7 = (3q + 1)^3 + 2(3q + 1) + 7 = 27q^3 + 27q^2 + 15q + 10 = 3(9q^3 + 9q^2 + 5q + 3) + 1$$

where $9q^3 + 9q^2 + 5q + 3 \in \mathbb{Z}$ by closure of \mathbb{Z} under multiplication and addition.

By the definition of mod, $(n^3 + 2n + 7) \bmod 3 = 1$.

CASE 3 $n = 3q + 2$

$$n^3 + 2n + 7 = (3q + 2)^3 + 2(3q + 2) + 7 = 27q^3 + 54q^2 + 24q + 19 = 3(9q^3 + 18q^2 + 8q + 6) + 1$$

where $9q^3 + 18q^2 + 8q + 6 \in \mathbb{Z}$ by closure of \mathbb{Z} under multiplication and addition.

By the definition of mod, $(n^3 + 2n + 7) \bmod 3 = 1$.

So, $(n^3 + 2n + 7) \bmod 3 = 1$ for all integers $n \geq 2$.

ALTERNATE SOLUTION

Let n be a particular but arbitrary integer such that $n \geq 2$.

By the Quotient Remainder Theorem, $n = 3q + r$ for some $q \in \mathbb{Z}$ and where $r = 0, 1$ or 2 .

$$\begin{aligned} n^3 + 2n + 7 &= (3q + r)^3 + 2(3q + r) + 7 = 27q^3 + 27q^2r + 9qr^2 + r^3 + 6q + 2r + 7 \\ &= 3(9q^3 + 9q^2r + 3qr^2 + 2q) + r^3 + 2r + 7 = 3m + r^3 + 2r + 7 \end{aligned}$$

where $m = 9q^3 + 9q^2r + 3qr^2 + 2q \in \mathbb{Z}$ by closure of \mathbb{Z} under multiplication and addition.

CASE 1: $r = 0$ $n^3 + 2n + 7 = 3m + 7 = 3(m + 2) + 1$

where $m + 2 \in \mathbb{Z}$ by closure of \mathbb{Z} under addition.

By the definition of mod, $(n^3 + 2n + 7) \bmod 3 = 1$.

CASE 2: $r = 1$ $n^3 + 2n + 7 = 3m + 10 = 3(m + 3) + 1$

where $m + 3 \in \mathbb{Z}$ by closure of \mathbb{Z} under addition.

By the definition of mod, $(n^3 + 2n + 7) \bmod 3 = 1$.

CASE 3: $r = 2$ $n^3 + 2n + 7 = 3m + 19 = 3(m + 6) + 1$

where $m + 6 \in \mathbb{Z}$ by closure of \mathbb{Z} under addition.

By the definition of mod, $(n^3 + 2n + 7) \bmod 3 = 1$.

So, $(n^3 + 2n + 7) \bmod 3 = 1$ for all integers $n \geq 2$.

OPTIONAL BONUS QUESTIONS ON OTHER SIDE