Math 22 (9:30am – 10:20am) **NAME YOU ASKED TO BE CALLED IN CLASS: ____________**
Midterm 2 Version V
Thu Nov 10, 2011

SCORE: ___ / 150 POINTS

Use an element argument to prove the following statement. **SCORE: ___ / 20 POINTS**

If $A \subseteq B^C$, then $A \cap B = \varnothing$

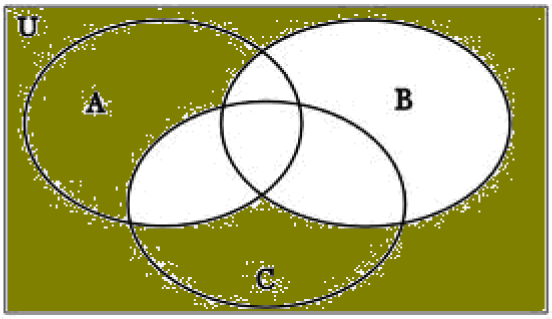
(Assume that A and B are subsets of some universal set U.)

Let A and B be particular but arbitrary sets such that $A \subseteq B^C$.
Suppose that $A \cap B \neq \varnothing$.
So, there exists an element $x \in A \cap B$.
So, $x \in A$ and $x \in B$, by the definition of $\cap$.
But, since $x \in A$ and $A \subseteq B^C$, therefore $x \in B^C$, be the definition of $\subseteq$.
So, $x \notin B$ by the definition of set complement.
But, $x \in B$ and $x \notin B$, a contradiction.
So, $A \cap B = \varnothing$.

Simplify the expression $\binom{n+2}{n} - \binom{n}{2}$. **SCORE: ___ / 15 POINTS**

$$\frac{(n+2)!}{n!2!} - \frac{n!}{2!(n-2)!}$$
$$= \frac{(n+2)\times(n+1)\times n!}{n!2!} - \frac{n\times(n-1)\times(n-2)!}{2!(n-2)!}$$
$$= \frac{(n+2)(n+1)}{2} - \frac{n(n-1)}{2}$$
$$= \frac{n^2+3n+2}{2} - \frac{n^2-n}{2}$$
$$= \frac{4n+2}{2}$$
$$= 2n+1$$

Shade in the region(s) in the Venn diagram corresponding to $B^C - (C \cap A)$. **SCORE: ___ / 10 POINTS**



From your textbook, remember that the Fibonacci sequence is defined by **SCORE: ___ / 15 POINTS**

$$F_0 = F_1 = 1$$
$$F_k = F_{k-1} + F_{k-2} \text{ for all integers } k \text{ where } k \geq 2$$

Show that the terms of the Fibonacci sequence satisfy the recurrence relation $F_k = \dfrac{F_{k+1} + F_{k-2}}{2}$ for all integers k where $k \geq 2$.

"Show" means you do NOT have to write a formal proof, but you must demonstrate the supporting algebra.

$$\frac{F_{k+1} + F_{k-2}}{2}$$
$$= \frac{(F_k + F_{k-1}) + F_{k-2}}{2}$$
$$= \frac{F_k + (F_{k-1} + F_{k-2})}{2}$$
$$= \frac{F_k + F_k}{2}$$
$$= F_k$$

Consider the sequence defined by **SCORE: ___ / 35 POINTS**

$$a_0 = 4$$
$$a_1 = 8$$
$$a_k = (2+k)a_{k-1} - 2k\,a_{k-2} \text{ for all } k \in Z \text{ such that } k \geq 2$$

[a] Find a_2 and a_3.

$$a_2 = (2+2)a_1 - 2(2)\,a_0 = 4(8) - 4(4) = 16$$
$$a_3 = (2+3)a_2 - 2(3)\,a_1 = 5(16) - 6(8) = 32$$

[b] Based on the values of a_0, a_1, a_2 and a_3, guess and prove the general formula for the sequence.

If a general formula/pattern does not seem somewhat obvious, check your arithmetic in [a].

Claim: $a_n = 2^{n+2}$

Proof by strong induction:

Basis step: $a_0 = 4 = 2^{0+2}$ and $a_1 = 8 = 2^{1+2}$

Inductive step: Assume that $a_n = 2^{n+2}$ for $n = 0, 1, \ldots, k$ for some particular but arbitrary integer $k \geq 1$.

[NEED TO SHOW: $a_{k+1} = 2^{(k+1)+2} = 2^{k+3}$]

$k - 1 \geq 0$ (since $k \geq 1$) and $k - 1 \leq k$.

$$\begin{aligned}
&a_{k+1} \\
&= (2+(k+1))a_k - 2(k+1)\,a_{k-1} \\
&= (3+k)2^{k+2} - (2k+2)\,2^{k+1} \\
&= 3(2^{k+2}) + k2^{k+2} - 2k(2^{k+1}) - 2(2^{k+1}) \\
&= 3(2^{k+2}) + k2^{k+2} - k2^{k+2} - 2^{k+2} \\
&= 2(2^{k+2}) \\
&= 2^{k+3}
\end{aligned}$$

So, by strong induction, $a_n = 2^{n+2}$ for $n \in Z^{nonneg}$.

Prove the statement **SCORE: ___ / 55 POINTS**

$(n^3 + 2n + 7) \bmod 3 = 1$ for all integers $n \geq 2$

in two ways:

[a] PROOF USING INDUCTION:

Basis step: $n = 2$: $2^3 + 2(2) + 7 = 19 = 3(6) + 1$

By definition of mod, $(2^3 + 2(2) + 7) \bmod 3 = 1$.

Inductive step: Assume that $(k^3 + 2k + 7) \bmod 3 = 1$ for some particular but arbitrary integer $k \geq 2$.

[NEED TO SHOW: $((k+1)^3 + 2(k+1) + 7) \bmod 3 = 1$]

By the definition of mod, $k^3 + 2k + 7 = 3m + 1$ for some $m \in Z$.

$(k+1)^3 + 2(k+1) + 7$

$= k^3 + 3k^2 + 5k + 10$

$= k^3 + 2k + 7 + 3k^2 + 3k + 3$

$= 3m + 1 + 3k^2 + 3k + 3$

$= 3(m + k^2 + k + 1) + 1$

where $m + k^2 + k + 1 \in Z$ by closure of Z under multiplication and addition.

So, $(n^3 + 2n + 7) \bmod 3 = 1$ for all integers $n \geq 2$.

[b] PROOF WITHOUT USING INDUCTION:

Let n be a particular but arbitrary integer such that $n \geq 2$.
By the Quotient Remainder Theorem, $n = 3q$ or $n = 3q+1$ or $n = 3q+2$ for some $q \in Z$.

CASE 1: $n = 3q$

$n^3 + 2n + 7 = (3q)^3 + 2(3q) + 7 = 27q^3 + 6q + 7 = 3(9q^3 + 2q + 2) + 1$

where $9q^3 + 2q + 2 \in Z$ by closure of Z under multiplication and addition.

By the definition of mod, $(n^3 + 2n + 7) \bmod 3 = 1$.

CASE 2: $n = 3q+1$

$n^3 + 2n + 7 = (3q+1)^3 + 2(3q+1) + 7 = 27q^3 + 27q^2 + 15q + 10 = 3(9q^3 + 9q^2 + 5q + 3) + 1$

where $9q^3 + 9q^2 + 5q + 3 \in Z$ by closure of Z under multiplication and addition.

By the definition of mod, $(n^3 + 2n + 7) \bmod 3 = 1$.

CASE 3 $n = 3q+2$

$n^3 + 2n + 7 = (3q+2)^3 + 2(3q+2) + 7 = 27q^3 + 54q^2 + 24q + 19 = 3(9q^3 + 18q^2 + 8q + 6) + 1$

where $9q^3 + 18q^2 + 8q + 6 \in Z$ by closure of Z under multiplication and addition.

By the definition of mod, $(n^3 + 2n + 7) \bmod 3 = 1$.

So, $(n^3 + 2n + 7) \bmod 3 = 1$ for all integers $n \geq 2$.

ALTERNATE SOLUTION

Let n be a particular but arbitrary integer such that $n \geq 2$.
By the Quotient Remainder Theorem, $n = 3q + r$ for some $q \in Z$ and where $r = 0, 1$ or 2.

$n^3 + 2n + 7 = (3q+r)^3 + 2(3q+r) + 7 = 27q^3 + 27q^2r + 9qr^2 + r^3 + 6q + 2r + 7$

$= 3(9q^3 + 9q^2r + 3qr^2 + 2q) + r^3 + 2r + 7 = 3m + r^3 + 2r + 7$

where $m = 9q^3 + 9q^2r + 3qr^2 + 2q \in Z$ by closure of Z under multiplication and addition.

CASE 1: $r = 0$ $n^3 + 2n + 7 = 3m + 7 = 3(m+2) + 1$

where $m + 2 \in Z$ by closure of Z under addition.

By the definition of mod, $(n^3 + 2n + 7) \bmod 3 = 1$.

CASE 2: $r = 1$ $n^3 + 2n + 7 = 3m + 10 = 3(m+3) + 1$

where $m + 3 \in Z$ by closure of Z under addition.

By the definition of mod, $(n^3 + 2n + 7) \bmod 3 = 1$.

CASE 3: $r = 2$ $n^3 + 2n + 7 = 3m + 19 = 3(m+6) + 1$

where $m + 6 \in Z$ by closure of Z under addition.

By the definition of mod, $(n^3 + 2n + 7) \bmod 3 = 1$.

So, $(n^3 + 2n + 7) \bmod 3 = 1$ for all integers $n \geq 2$.

OPTIONAL BONUS QUESTIONS ON OTHER SIDE