Math 22 (9:30am – 10:20am)
Quiz 7 (Take Home)
Due Mon Nov 28, 2011 @ start of class

NAME YOU ASKED TO BE CALLED IN CLASS: ____________

NO LATE QUIZZES ACCEPTED
You are strongly encouraged to send a scan of your solution to lobert@deanza.edu no later than Sun Nov 27

SCORE: ___ / 30 POINTS

Define $f : Z^{nonneg} \to Z$ by the rule $f(x) = x^2 + 4x + 2$. **SCORE: ___ / 6 POINTS**
Write a formal proof that f is one-to-one, using algebra. (Hint: Consult the lecture notes.)
DO NOT USE CALCULUS, GRAPHS OR THE HORIZONTAL LINE TEST.

PROOF: Let $x, y \in Z^{nonneg}$ such that $f(x) = f(y)$ ie. $x^2 + 4x + 2 = y^2 + 4y + 2$
So, $x^2 - y^2 + 4x - 4y = 0$.
So, $(x + y)(x - y) + 4(x - y) = 0$.
So, $(x + y + 4)(x - y) = 0$.
Since $x, y > 0$, therefore, $x + y + 4 > 0$ ie. $x + y + 4 \neq 0$.
So, $x - y = 0$ by zero product property.
So, $x = y$.
So, f is one-to-one.

Let R be a relation on Z defined by **SCORE: ___ / 9 POINTS**

xRy if and only if $5 \mid (2x + 3y)$

[a] Is R reflexive ? If yes, write a formal proof. If no, give a counterexample and show how it indicates R is not reflexive.

PROOF: Let $x \in Z$.
$2x + 3x = 5x$.
Since $x \in Z$, $5 \mid (2x + 3x)$.
So, xRx.
So, R is reflexive.

[b] Is R symmetric ? If yes, write a formal proof. If no, give a counterexample and show how it indicates R is not symmetric.

PROOF: Let $x, y \in Z$ such that xRy ie. $5 \mid (2x + 3y)$.
So, $2x + 3y = 5k$ for some $k \in Z$.
$2y + 3x = 5x + 5y - (2x + 3y) = 5x + 5y - 5k = 5(x + y - k)$.
Since $x + y - k \in Z$ by closure, therefore, $5 \mid (2y + 3x)$.
So, yRx.
So, R is symmetric.

[c] Is R transitive ? If yes, write a formal proof. If no, give a counterexample and show how it indicates R is not transitive.

PROOF: Let $x, y, z \in Z$ such that xRy and yRz ie. $5 \mid (2x + 3y)$ and $5 \mid (2y + 3z)$.
So, $2x + 3y = 5k$ for some $k \in Z$ and $2y + 3z = 5m$ for some $m \in Z$.
$2x + 3z = (2x + 3y) + (2y + 3z) - 5y = 5k + 5m - 5y = 5(k + m - y)$.
Since $k + m - y \in Z$ by closure, therefore, $5 \mid (2x + 3z)$.
So, xRz.
So, R is transitive.

Let $A = \{1, 2, 3\}$ and let $B = \wp(A) - \{\varnothing\}$. Define $f : B \to A$ by the rule **SCORE: ___ / 6 POINTS**

$f(X) =$ the smallest element of X

[a] Is f one-to-one ? Justify your answer with a sufficient number of examples.

No. $f(\{1\}) = f(\{1, 2\}) = 1$.

[b] Is f onto ? Justify your answer with a sufficient number of examples.

Yes. $f(\{1\}) = 1,\quad f(\{2\}) = 2,\quad f(\{3\}) = 3$.

[c] Find $f^{-1}(\{1, 3\})$. **Write your answer in proper notation.**

$\{\{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}, \{3\}\}$

[d] If $D = \wp(Z^+) - \{\varnothing\}$ and $f : D \to Z^+$ is defined by the rule $f(X) =$ the smallest element of X, what is the theorem that guarantees that $f(X)$ exists for all $X \in D$?

Well Ordering Principle (every non-empty set of integers greater than some fixed integer (in this case, 0) has a least element)

Let $A = \{x \in R \mid 1 < x < 3\}$ and $B = \{x \in R \mid 5 < x < 9\}$. **SCORE: ___ / 9 POINTS**

Prove that A and B have the same cardinality by finding a one-to-one correspondence $f : A \to B$.

You must prove that your one-to-one correspondence *f* is [a] a function, [b] one-to-one, and [c] onto.

PROOF: Let $f : A \to B$ be defined by $(x, y) \in f$ iff $y = 2x + 3$.

Let $x \in A$ ie. $x \in R$ and $1 < x < 3$.
Let $y = 2x + 3$.
So, $2 < 2x < 6$ and $5 < 2x + 3 < 9$ ie. $5 < y < 9$.
So, $y \in B$ and $(x, y) \in f$.

Let $x \in A$ and $y, z \in B$ such that $(x, y) \in f$ and $(x, z) \in f$.
So, $y = 2x + 3$ and $z = 2x + 3$.
So, $y = z$. So, f is a function from A to B.

Let $x, y \in A$ such that $f(x) = f(y)$ ie. $2x + 3 = 2y + 3$.
So, $2x = 2y$ and $x = y$. So, f is one-to-one.

Let $y \in B$ ie. $y \in R$ and $5 < y < 9$.

Let $x = \dfrac{y-3}{2}$.

So, $2 < y - 3 < 6$ and $1 < \dfrac{y-3}{2} < 3$ ie. $1 < x < 3$ or $x \in A$.

And $f(x) = f\left(\dfrac{y-3}{2}\right) = 2\left(\dfrac{y-3}{2}\right) + 3 = (y-3) + 3 = y$ So, f is onto.

Since f is a one-to-one onto function from A to B, therefore, f is a one-to-one correspondence, and A and B have the same cardinality.