STRONG INDUCTION

The difference between (regular) induction and strong induction is
how much you assume in the inductive step.

For instance,
if you wanted to prove a predicate $P(n)$ is true for all integers $n \geq 1$,
during the inductive step of a (regular) induction proof,
you would assume that $P(k)$ is true
for some particular but arbitrary integer $k \geq 1$;
whereas, during the inductive step of a strong induction proof,
you would assume that $P(1), P(2), P(3), \ldots, P(k)$ are all true
for some particular but arbitrary integer $k \geq 1$.
In both types of proofs, you would then prove $P(k+1)$ is true.
In the (regular) induction proof,
you would need to find a connection between $P(k+1)$ and $P(k)$.
In a strong induction proof,
you could find a connection between $P(k+1)$ and one or any of $P(1), P(2), P(3), \ldots, P(k)$.
So, it would seem there are more ways to construct a strong induction proof
than a (regular) induction proof.

Also, because the list of assumptions in a strong induction proof
includes the assumption of a (regular) induction proof,
any statement that can be proved by a (regular) induction proof
can automatically be proved by a strong induction proof
You would simply state, but never use, the fact that $P(1), P(2), P(3), \ldots, P(k-1)$ are true,
and simply use the fact that $P(k)$ is true to do the inductive step.

So, it would seem that strong induction is the more powerful proof method. (*)
The tricky part of strong induction proofs, though, is that
it is easier to make an easy-to-overlook mistake during the inductive step.

The examples on the following page show incorrect “proofs”
that certain false statements are true
by using strong induction proofs that contain subtle errors.
Counterexamples are provided for you.
Do not try to rewrite the proof,
or explain why a particular counterexample proves the statement is false.
Find the statement in the proof that is not, and cannot be, justified properly.

(*) In reality, though, it can also be shown that
anything that can be proven by a strong induction proof
can automatically be proven by a (regular) induction proof,
which means the two proof methods are equally powerful.

<u>TWO EXAMPLES OF INCORRECT STRONG INDUCTION PROOFS</u>

Claim:

Any integral amount of postage greater than or equal to 5 cents can be made by using only 3 cent and 5 cent stamps.

Note:

The claim is false since 7 cents postage cannot be made using only 3 cent and 5 cent stamps.

“Proof” by strong induction:

Basis step: 5 cents postage can be made using one 5 cent stamp.

Inductive step: Assume that 5, 6, … k cents postage can be made using only 3 cent and 5 cent stamps.
To make $k+1$ cents postage,
note that $k+1=(k-2)+3$.
Since $k-2 \le k$, by the hypothesis of the inductive step,
$k-2$ cents postage can be made using only 3 cent and 5 cent stamps.
By adding one more 3 cent stamp, we have a total of $k+1$ cents postage.

So, by strong induction, any integral amount of postage greater than or equal to 5 cents can be made by using only 3 cent and 5 cent stamps.

Claim:

Every non-negative integer is even.

Note:

The claim is obviously false.

“Proof” by strong induction:

Basis step: 0 is even.

Inductive step: Assume that all non-negative integers from 0 to k are even.
To prove that $k+1$ is even,
note that $k+1=(k-1)+2$.
Since $k-1 \le k$, by the hypothesis of the inductive step, $k-1$ is even.
By the definition of even, $k-1=2m$ for some $m \in Z$.
So, $k+1=(k-1)+2=2m+2=2(m+1)$ where $m+1 \in Z$ by closure.
So, by the definition of even, $k+1$ is even.

So, by strong induction, every non-negative integer is even.

If you found the errors in both proofs,
you should notice that the errors are of the same basic type. (See also 5.4 exercise 19.)
In fact, this is a very common type of error in writing strong induction proofs,
and something you need to be very careful about when writing your own strong induction proofs.