Evaluate $\int \frac{x^4}{\sqrt{4+x^2}}\,dx$. **SCORE: ___ / 25 POINTS**

$$x = 2\tan\theta$$
$$dx = 2\sec^2\theta\,d\theta$$

$$\int \frac{16\tan^4\theta}{2\sec\theta} 2\sec^2\theta\,d\theta$$
$$= \int 16\tan^4\theta\sec\theta\,d\theta$$
$$= \int 16(\sec^2\theta-1)^2\sec\theta\,d\theta$$
$$= 16\int(\sec^5\theta - 2\sec^3\theta + \sec\theta)\,d\theta$$
$$= 16\int\sec^5\theta\,d\theta - 32\int\sec^3\theta\,d\theta + 16\int\sec\theta\,d\theta$$
$$= 16\left[\frac{1}{4}\sec^3\theta\tan\theta + \frac{3}{4}\int\sec^3\theta\,d\theta\right] - 32\int\sec^3\theta\,d\theta + 16\int\sec\theta\,d\theta$$
$$= 4\sec^3\theta\tan\theta - 20\int\sec^3\theta\,d\theta + 16\int\sec\theta\,d\theta$$
$$= 4\sec^3\theta\tan\theta - 20\left[\frac{1}{2}\ln|\sec\theta+\tan\theta| + \frac{1}{2}\sec\theta\tan\theta\right] + 16\ln|\sec\theta+\tan\theta| + C$$
$$= 4\sec^3\theta\tan\theta + 6\ln|\sec\theta+\tan\theta| - 10\sec\theta\tan\theta + C$$
$$= 4\left(\frac{\sqrt{4+x^2}}{2}\right)^3\frac{x}{2} + 6\ln\left|\frac{\sqrt{4+x^2}}{2}+\frac{x}{2}\right| - 10\frac{\sqrt{4+x^2}}{2}\frac{x}{2} + C$$
$$= \frac{x(4+x^2)\sqrt{4+x^2}}{4} - \frac{5x\sqrt{4+x^2}}{2} + 6\ln(\sqrt{4+x^2}+x) + C$$
$$= \frac{(x^3-6x)\sqrt{4+x^2}}{4} + 6\ln(\sqrt{4+x^2}+x) + C$$

Evaluate $\int 2x\tanh^{-1}x\,dx$. **SCORE: ___ / 20 POINTS**

| $u$ | $dv$ |
|---|---|
| $\tanh^{-1}x$ | $2x$ |
| $\frac{1}{1-x^2}$ | $x^2$ |
| ------------------------------ | |
| $1$ | $\frac{x^2}{1-x^2} = -1 + \frac{1}{1-x^2}$ |
| $0$ | $-x + \tanh^{-1}x$ |

$$\int 2x\tanh^{-1}x\,dx$$
$$= x^2\tanh^{-1}x - (-x + \tanh^{-1}x) + C$$
$$= (x^2-1)\tanh^{-1}x + x + C$$

Evaluate the following integrals. **SCORE: ___ / 50 POINTS**

[a] $\displaystyle\int_{-1}^{1} \frac{1+x^2}{x}\,dx$

$$= \int_{-1}^{0} \frac{1+x^2}{x}\,dx + \int_{0}^{1} \frac{1+x^2}{x}\,dx$$

➡ $\displaystyle\int_{0}^{1} \frac{1+x^2}{x}\,dx$

$$= \lim_{N\to 0^+} \int_{N}^{1} \left(\frac{1}{x} + x\right) dx$$

$$= \lim_{N\to 0^+} \left( \ln|x| + \frac{1}{2}x^2 \right)\Bigg|_{N}^{1}$$

$$= \lim_{N\to 0^+} \left( \frac{1}{2} - \ln|N| - \frac{1}{2}N^2 \right)$$

CAN ALSO COMPARE TO $\displaystyle\int_{0}^{1} \frac{1}{x}\,dx$

DOES NOT EXIST SINCE

⬇

$$\frac{1}{2} - \frac{1}{2}N^2 \to \frac{1}{2} \text{ BUT } -\ln|N| \to \infty$$

SO, $\displaystyle\int_{0}^{1} \frac{1+x^2}{x}\,dx$ DIVERGES AND $\displaystyle\int_{-1}^{1} \frac{1+x^2}{x}\,dx$ DIVERGES

[b] $\displaystyle\int_{e}^{\infty} \frac{\ln x}{x^2}\,dx$

| $u$ | $dv$ |
|---|---|
| $\ln x$ | $\frac{1}{x^2}$ |
| $\frac{1}{x}$ | $-\frac{1}{x}$ |
| $1$ | $-\frac{1}{x^2}$ |
| $0$ | $\frac{1}{x}$ |

$$\int_{e}^{\infty} \frac{\ln x}{x^2}\,dx$$

$$= \lim_{N\to\infty} \left( -\frac{\ln x}{x} - \frac{1}{x} \right)\Bigg|_{e}^{N}$$

$$= \lim_{N\to\infty} \left( -\frac{\ln N}{N} - \frac{1}{N} + \frac{2}{e} \right)$$

$$\lim_{N\to\infty} \frac{\ln N}{N} = \lim_{N\to\infty} \frac{\frac{1}{N}}{1} = 0$$

$$= 0 - 0 + \frac{2}{e}$$

$$= \frac{2}{e}$$

[c] $\displaystyle\int_{-e}^{e} \frac{\sin x}{1+x^4}\,dx$

$$\frac{\sin(-x)}{1+(-x)^4} = \frac{-\sin x}{1+x^4} = -\frac{\sin x}{1+x^4}$$

SINCE THE INTEGRAND IS ODD AND CONTINUOUS, AND THE INTERVAL IS SYMMETRIC, $\displaystyle\int_{-e}^{e} \frac{\sin x}{1+x^4}\,dx = 0$

Determine if $\int_0^1 \frac{\sec^2 x}{x\sqrt{x}}\,dx$ converges or diverges. **SCORE: ___ / 15 POINTS**

$$\sec^2 x \geq 1$$

$$\frac{\sec^2 x}{x\sqrt{x}} \geq \frac{1}{x\sqrt{x}}$$

$$\int_0^1 \frac{1}{x\sqrt{x}}\,dx = \int_0^1 \frac{1}{x^{\frac{3}{2}}}\,dx \text{ DIVERGES SINCE } p = \frac{3}{2} \geq 1$$

SO, $\int_0^1 \frac{\sec^2 x}{x\sqrt{x}}\,dx$ DIVERGES

Evaluate $\int \frac{29x - x^3}{x^2 - 5x - 6}\,dx$. **SCORE: ___ / 25 POINTS**

$$= \int \left( -x - 5 + \frac{-2x - 30}{x^2 - 5x - 6} \right) dx$$

$$= \int \left( -x - 5 + \frac{-6}{x - 6} + \frac{4}{x + 1} \right) dx$$

$$= -\frac{1}{2}x^2 - 5x - 6\ln|x - 6| + 4\ln|x + 1| + C$$

$$\frac{-2x - 30}{x^2 - 5x - 6} = \frac{A}{x - 6} + \frac{B}{x + 1}$$

$$-2x - 30 = A(x + 1) + B(x - 6)$$

$$x = -1: \quad -28 = -7B \quad \Rightarrow B = 4$$

$$x = 6: \quad -42 = 7A \quad \Rightarrow A = -6$$

*SANITY CHECK* $\quad x = 2:$

$$\frac{-2x - 30}{x^2 - 5x - 6} = \frac{-34}{-12} = \frac{17}{6}$$

$$\frac{-6}{x - 6} + \frac{4}{x + 1} = \frac{-6}{-4} + \frac{4}{3} = \frac{3}{2} + \frac{4}{3} = \frac{9 + 8}{6} = \frac{17}{6}$$

Using the table of integrals, evaluate $\int \frac{\sqrt{4-x^6}}{x}\,dx$. **SCORE: ___ / 15 POINTS**

State the number of the rule(s) used. **<u>YOU MUST USE THE TABLE OF INTEGRALS TO RECEIVE CREDIT.</u>**
**<u>NO CREDIT IF YOU SOLVE THE PROBLEM USING A TRIGONOMETRIC SUBSTITUTION (SECTION 7.3).</u>**

$u = x^3$

$du = 3x^2\,dx$

$\frac{1}{3x^2}\,du = dx$

$\frac{\sqrt{4-x^6}}{x}\frac{1}{3x^2}\,du = \frac{\sqrt{4-x^6}}{3x^3}\,du = \frac{\sqrt{4-u^2}}{3u}\,du$

RULE #32: $a = 2$

$$\int \frac{\sqrt{4-u^2}}{3u}\,du = \frac{1}{3}\left(\sqrt{4-u^2} - 2\ln\left|\frac{2+\sqrt{4-u^2}}{u}\right|\right) + C$$

$$\int \frac{\sqrt{4-x^6}}{x}\,dx = \frac{1}{3}\left(\sqrt{4-x^6} - 2\ln\left|\frac{2+\sqrt{4-x^6}}{x^3}\right|\right) + C$$

**OR**

$u = x^6$

$du = 6x^5\,dx$

$\frac{1}{6x^5}\,du = dx$

$\frac{\sqrt{4-x^6}}{x}\frac{1}{6x^5}\,du = \frac{\sqrt{4-x^6}}{6x^6}\,du = \frac{\sqrt{4-u}}{6u}\,du$

RULE #58: $a = 4, b = -1$

$$\int \frac{\sqrt{4-u}}{6u}\,du = \frac{1}{6}\left(2\sqrt{4-u} + 4\int \frac{du}{u\sqrt{4-u}}\right) + C$$

RULE #57: $a = 4, b = -1$

$$\int \frac{\sqrt{4-u}}{6u}\,du = \frac{1}{6}\left(2\sqrt{4-u} + 4\left(\frac{1}{2}\ln\left|\frac{\sqrt{4-u}-2}{\sqrt{4-u}+2}\right|\right)\right) + C$$

$$\int \frac{\sqrt{4-x^6}}{x}\,dx = \frac{1}{6}\left(2\sqrt{4-x^6} + 4\left(\frac{1}{2}\ln\left|\frac{\sqrt{4-x^6}-2}{\sqrt{4-x^6}+2}\right|\right)\right) + C$$

$$\int \frac{\sqrt{4-x^6}}{x}\,dx = \frac{1}{3}\left(\sqrt{4-x^6} + \ln\left|\frac{\sqrt{4-x^6}-2}{\sqrt{4-x^6}+2}\right|\right) + C$$