Evaluate $\int (x^2 - 4)^{\frac{3}{2}}\, dx$. **SCORE: ___ / 25 POINTS**

$$x = 2\sec\theta$$
$$dx = 2\sec\theta\tan\theta\, d\theta$$

$$\int (4\tan^2\theta)^{\frac{3}{2}}\, 2\sec\theta\tan\theta\, d\theta$$
$$= \int 16\tan^4\theta\sec\theta\, d\theta$$
$$= \int 16(\sec^2\theta - 1)^2\sec\theta\, d\theta$$
$$= 16\int (\sec^5\theta - 2\sec^3\theta + \sec\theta)\, d\theta$$
$$= 16\int \sec^5\theta\, d\theta - 32\int \sec^3\theta\, d\theta + 16\int \sec\theta\, d\theta$$
$$= 16\left[\frac{1}{4}\sec^3\theta\tan\theta + \frac{3}{4}\int \sec^3\theta\, d\theta\right] - 32\int \sec^3\theta\, d\theta + 16\int \sec\theta\, d\theta$$
$$= 4\sec^3\theta\tan\theta - 20\int \sec^3\theta\, d\theta + 16\int \sec\theta\, d\theta$$
$$= 4\sec^3\theta\tan\theta - 20\left[\frac{1}{2}\ln\left|\sec\theta + \tan\theta\right| + \frac{1}{2}\sec\theta\tan\theta\right] + 16\ln\left|\sec\theta + \tan\theta\right| + C$$
$$= 4\sec^3\theta\tan\theta + 6\ln\left|\sec\theta + \tan\theta\right| - 10\sec\theta\tan\theta + C$$
$$= 4\left(\frac{x}{2}\right)^3\frac{\sqrt{x^2-4}}{2} + 6\ln\left|\frac{x}{2} + \frac{\sqrt{x^2-4}}{2}\right| - 10\frac{x}{2}\frac{\sqrt{x^2-4}}{2} + C$$
$$= \frac{x^3\sqrt{x^2-4}}{4} - \frac{5x\sqrt{x^2-4}}{2} + 6\ln\left|x + \sqrt{x^2-4}\right| + C$$
$$= \frac{(x^3 - 10x)\sqrt{x^2-4}}{4} + 6\ln\left|x + \sqrt{x^2-4}\right| + C$$

Evaluate $\int 2x\tanh^{-1} x\, dx$. **SCORE: ___ / 20 POINTS**

u	dv
$\tanh^{-1} x$	$2x$
$\frac{1}{1-x^2}$	x^2
1	$\frac{x^2}{1-x^2} = -1 + \frac{1}{1-x^2}$
0	$-x + \tanh^{-1} x$

$$\int 2x\tanh^{-1} x\, dx$$
$$= x^2\tanh^{-1} x - (-x + \tanh^{-1} x) + C$$
$$= (x^2 - 1)\tanh^{-1} x + x + C$$

Evaluate the following integrals. **SCORE: ___ / 50 POINTS**

[a] $\displaystyle\int_{-e}^{e} \frac{1+x^4}{x^3}\,dx$

$$= \int_{-e}^{0} \frac{1+x^4}{x^3}\,dx + \int_{0}^{e} \frac{1+x^4}{x^3}\,dx \quad \Rightarrow \quad \int_{0}^{e} \frac{1+x^4}{x^3}\,dx$$

$$= \lim_{N\to 0^+} \int_{N}^{e} \left(\frac{1}{x^3}+x\right) dx$$

$$= \lim_{N\to 0^+} \left(-\frac{1}{2x^2}+\frac{1}{2}x^2\right)\Bigg|_{N}^{e}$$

$$= \lim_{N\to 0^+} \left(-\frac{1}{2e^2}+\frac{1}{2}e^2+\frac{1}{2N^2}-\frac{1}{2}N^2\right)$$

CAN ALSO COMPARE TO $\displaystyle\int_{0}^{e} \frac{1}{x^3}\,dx$

DOES NOT EXIST SINCE

$$-\frac{1}{2e^2}+\frac{1}{2}e^2-\frac{1}{2}N^2 \to -\frac{1}{2e^2}+\frac{1}{2}e^2 \text{ BUT } \frac{1}{2N^2} \to \infty$$

SO, $\displaystyle\int_{0}^{e} \frac{1+x^4}{x^3}\,dx$ DIVERGES AND $\displaystyle\int_{-e}^{e} \frac{1+x^4}{x^3}\,dx$ DIVERGES

[b] $\displaystyle\int_{e}^{\infty} \frac{\ln x}{x^2}\,dx$

u	dv
$\ln x$	$\frac{1}{x^2}$
$\frac{1}{x}$	$-\frac{1}{x}$
1	$-\frac{1}{x^2}$
0	$\frac{1}{x}$

$$\int_{e}^{\infty} \frac{\ln x}{x^2}\,dx$$

$$= \lim_{N\to\infty} \left(-\frac{\ln x}{x}-\frac{1}{x}\right)\Bigg|_{e}^{N}$$

$$= \lim_{N\to\infty} \left(-\frac{\ln N}{N}-\frac{1}{N}+\frac{2}{e}\right) \qquad \lim_{N\to\infty} \frac{\ln N}{N} = \lim_{N\to\infty} \frac{\frac{1}{N}}{1} = 0$$

$$= 0 - 0 + \frac{2}{e}$$

$$= \frac{2}{e}$$

[c] $\displaystyle\int_{-1}^{1} \frac{x}{\ln(4+x^2)}\,dx$

$$\frac{-x}{\ln(4+(-x)^2)} = -\frac{x}{\ln(4+x^2)}$$

SINCE THE INTEGRAND IS ODD AND CONTINUOUS, AND THE INTERVAL IS SYMMETRIC,

$$\int_{-1}^{1} \frac{x}{\ln(4+x^2)}\,dx = 0$$

Determine if $\int_0^1 \frac{1+e^{-x}}{x^2}\,dx$ converges or diverges. **SCORE: ___ / 15 POINTS**

$$e^{-x} > 0$$
$$1+e^{-x} > 1$$
$$\frac{1+e^{-x}}{x^2} > \frac{1}{x^2}$$
$$\int_0^1 \frac{1}{x^2}\,dx \text{ DIVERGES SINCE } p = 2 \geq 1$$
$$\text{SO, } \int_0^1 \frac{1+e^{-x}}{x^2}\,dx \text{ DIVERGES}$$

Evaluate $\int \frac{2x^3+13x^2-1}{x^2+5x-6}\,dx$. **SCORE: ___ / 25 POINTS**

$$= \int \left(2x+3+\frac{-3x+17}{x^2+5x-6} \right) dx$$
$$= \int \left(2x+3+\frac{A}{x+6}+\frac{B}{x-1} \right) dx$$
$$= x^2+3x-5\ln|x+6|+2\ln|x-1|+C$$

$$\frac{-3x+17}{x^2+5x-6} = \frac{A}{x+6}+\frac{B}{x-1}$$
$$-3x+17 = A(x-1)+B(x+6)$$
$$x=1: \quad 14 = 7B \quad \Rightarrow B = 2$$
$$x=-6: \quad 35 = -7A \quad \Rightarrow A = -5$$
$$\textit{SANITY CHECK} \quad x = 2:$$
$$\frac{-3x+17}{x^2+5x-6} = \frac{11}{8}$$
$$\frac{-5}{x+6}+\frac{2}{x-1} = \frac{-5}{8}+\frac{2}{1} = \frac{-5+16}{8} = \frac{11}{8}$$

Using the table of integrals, evaluate $\int \frac{\sqrt{x^8-9}}{x^5}\,dx$. **SCORE: ___ / 15 POINTS**

State the number of the rule(s) used. **<u>YOU MUST USE THE TABLE OF INTEGRALS TO RECEIVE CREDIT.</u>**
<u>NO CREDIT IF YOU SOLVE THE PROBLEM USING A TRIGONOMETRIC SUBSTITUTION (SECTION 7.3).</u>

$$u = x^4$$
$$du = 4x^3\,dx$$
$$\frac{1}{4x^3}\,du = dx$$
$$\frac{\sqrt{x^8-9}}{x^5}\frac{1}{4x^3}\,du = \frac{\sqrt{x^8-9}}{4x^8}\,du = \frac{\sqrt{u^2-9}}{4u^2}\,du$$

RULE #42: $a = 3$

$$\int \frac{\sqrt{u^2-9}}{4u^2}\,du = \frac{1}{4}\left(-\frac{\sqrt{u^2-9}}{u} + \ln\left|u+\sqrt{u^2-9}\right| \right) + C$$
$$\int \frac{\sqrt{x^8-9}}{x^5}\,dx = \frac{1}{4}\left(-\frac{\sqrt{x^8-9}}{x^4} + \ln\left|x^4+\sqrt{x^8-9}\right| \right) + C$$
$$\int \frac{\sqrt{x^8-9}}{x^5}\,dx = \frac{1}{4}\left(-\frac{\sqrt{x^8-9}}{x^4} + \ln(x^4+\sqrt{x^8-9}) \right) + C$$