Math 1B Midterm 1 Review Answers

[1] Using the "Pythagorean", reciprocal and quotient identities, $\cosh x = \frac{5}{4}$ and $\operatorname{csch} x = -\frac{4}{3}$.

[2] $\lim\limits_{x\to\infty} \frac{e^{2x}+1}{e^{2x}-1} = \lim\limits_{x\to\infty} \frac{2e^{2x}}{2e^{2x}} = 1$

[3] $y = \cosh^{-1} x$

$\Rightarrow \cosh y = x$ and $y \ge 0$

$\Rightarrow (\sinh y)\frac{dy}{dx} = 1$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sinh y}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{\cosh^2 y - 1}}$ (NOTE: $y \ge 0 \Rightarrow \sinh y \ge 0$)

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{x^2-1}}$

[4] [a] By Net Change Theorem,

$$s(13) - s(1) = \int_1^{13} s'(t)\,dt = \int_1^{13} v(t)\,dt \quad \Rightarrow \quad s(13) = s(1) + \int_1^{13} v(t)\,dt \quad = \quad 21 + \int_1^{13} v(t)\,dt \text{ feet}$$

[b] [i] $21 + (12+18+17)*4 = 209$ feet

[ii] $21 + (14+24+8)*4 = 205$ feet

[iii] $21 + (18+17+5)*4 = 181$ feet

[5] $\int_{\pi}^{\frac{3\pi}{2}} 2\sin x\,dx = -2$ or $\int_{2\pi}^{3\pi} \sin\frac{1}{2}x\,dx = -2$

[6] $\lim\limits_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\left[\left(-1+\frac{3i}{n}\right)^2 + 3\left(-1+\frac{3i}{n}\right) + 2\right]$

$= \lim\limits_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left(\frac{3i}{n} + \frac{9i^2}{n^2}\right)$

$= \lim\limits_{n\to\infty} \frac{3}{n} \left(\frac{3}{n}\sum_{i=1}^{n} i + \frac{9}{n^2}\sum_{i=1}^{n} i^2\right)$

$= \lim\limits_{n\to\infty} \frac{3}{n} \left(\frac{3}{n}\frac{n(n+1)}{2} + \frac{9}{n^2}\frac{n(n+1)(2n+1)}{6}\right)$

$= \lim\limits_{n\to\infty} 3\left(\frac{3(n+1)}{2n} + \frac{3(n+1)(2n+1)}{2n^2}\right)$

$= 3\left(\frac{3}{2} + \frac{6}{2}\right)$

$= \frac{27}{2}$

[7] $\int_{-2}^{8} x\,dx - \int_{-2}^{8} \sqrt{25-(x-3)^2}\,dx$ = area of 8×8 triangle – area of 2×2 triangle

– area of semicircle of radius 5 (with center at $(3, 0)$)

$= \frac{1}{2}(8\times 8) - \frac{1}{2}(2\times 2) - \frac{1}{2}\pi(5)^2$

$= 30 - \frac{25}{2}\pi$

[8] $\frac{1}{2} \le \sin x \le 1$ on $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$

$\Rightarrow \frac{1}{2}x \le x\sin x \le x$ on $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$

$$\Rightarrow \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{2}x\,dx \le \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x\sin x\,dx \le \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x\,dx$$

$$\Rightarrow \frac{1}{4}x^2\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}} \le \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x\sin x\,dx \le \frac{1}{2}x^2\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$\Rightarrow \frac{1}{4}\left[\left(\frac{\pi}{2}\right)^2 - \left(\frac{\pi}{6}\right)^2\right] \le \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x\sin x\,dx \le \frac{1}{2}\left[\left(\frac{\pi}{2}\right)^2 - \left(\frac{\pi}{6}\right)^2\right]$$

$$\Rightarrow \frac{1}{4}\left(\frac{\pi}{6}\right)^2(3^2-1^2) \le \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x\sin x\,dx \le \frac{1}{2}\left(\frac{\pi}{6}\right)^2(3^2-1^2)$$

$$\Rightarrow \frac{1}{4}\frac{\pi^2}{36}(8) \le \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x\sin x\,dx \le \frac{1}{2}\frac{\pi^2}{36}(8)$$

$$\Rightarrow \frac{\pi^2}{18} \le \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x\sin x\,dx \le \frac{\pi^2}{9}$$

[9] $\int_{7}^{-5} f(x)\,dx + \int_{-1}^{10} f(x)\,dx + \int_{10}^{7} f(x)\,dx = \int_{7}^{-5} f(x)\,dx + \int_{-1}^{7} f(x)\,dx = \int_{-1}^{-5} f(x)\,dx$

[10] [a] $g(3) = \int_{-2}^{3} f(t)\,dt = \frac{1}{2}(3\times 3) - \frac{1}{2}(2\times 1) = \frac{7}{2}$ and $g'(3) = f(3) = -1$

[b] Critical numbers of g occur where $g'(x) = 0$ or is undefined in the domain of g.
By FTC Part 1, since f is continuous on $[-5, 5]$, g is differentiable on $(-5, 5)$, and continuous on $[-5, 5]$.
So the domain of g includes $[-5, 5]$.
So, the critical numbers of g occur where $g'(x) = f(x) = 0$ or is undefined in $[-5, 5]$ ie. at $x = -4$, -2, 1 and 4.

[c] g is decreasing where $g'(x) < 0$, so g is decreasing on $[-\infty, -4]$ and $[1, 4]$.

[d] Local minima of g occur at critical numbers of g where $g'(x)$ changes from negative to positive, so the local minima of g are $x=-4$ and 4.

[e] g is concave up where $g'(x)$ is increasing, so g is concave up on $[-\infty,-3]$, $[-2,0]$ and $[3,\infty]$.

[f] Inflection points of g occur where g is continuous and $g'(x)$ changes from increasing to decreasing, or vice versa, so the inflection points of g are $x=-3$, -2, 0 and 3.

[11] $$g'(x)=3x^2\ln(1+x^6)-2x\ln(1+x^4)$$

$$g''(x)=6x\ln(1+x^6)+3x^2\frac{6x^5}{1+x^6}-\left(2\ln(1+x^4)+2x\frac{4x^3}{1+x^4}\right)$$

$$g''(1)=6\ln 2+3\left(\frac{6}{2}\right)-2\ln 2-2\left(\frac{4}{2}\right)=4\ln 2+5$$

[12] $$\frac{d}{dx}\left(4+\int_a^x \frac{1}{f(t)}\,dt\right)=\frac{d}{dx}2\sqrt{x}$$

$$\Rightarrow \frac{1}{f(x)}=\frac{1}{\sqrt{x}}$$

$$\Rightarrow f(x)=\sqrt{x}$$

$$4+\int_a^x \frac{1}{\sqrt{t}}\,dt=2\sqrt{x}$$

$$\Rightarrow 4+2\sqrt{t}\Big|_a^x=2\sqrt{x}$$

$$\Rightarrow 4+2\sqrt{x}-2\sqrt{a}=2\sqrt{x}$$

$$\Rightarrow 4-2\sqrt{a}=0$$

$$\Rightarrow a=4$$

[13] the number of pounds Morgan gained from when he was 8 years old to when he was 15 years old

[14] [a] $$v(t)-v(0)=\int_0^t v'(x)\,dx=\int_0^t a(x)\,dx \quad\Rightarrow\quad v(t)=v(0)+\int_0^t a(x)\,dx \;=\; 4+(3x-x^2)\Big|_0^t \;=\; 4+3t-t^2$$

$$\int_1^6 v(t)\,dt \;=\; \left(4t+\frac{3}{2}t^2-\frac{1}{3}t^3\right)\Bigg|_1^6 \;=\; 4(6-1)+\frac{3}{2}(6^2-1^2)-\frac{1}{3}(6^3-1^3) \;=\; 20+\frac{105}{2}-\frac{215}{3}$$

$$=\; 20+52\frac{1}{2}-71\frac{2}{3} \;=\; \frac{5}{6}\text{ meters}$$

[b] $v(t)=4+3t-t^2=(4-t)(1+t)\ge 0$ only on $[-1,4]$

$$\int_1^6 |\,v(t)\,|\,dt \;=\; \int_1^4 (4+3t-t^2)\,dt+\int_4^6 -(4+3t-t^2)\,dt \;=\; \left(4t+\frac{3}{2}t^2-\frac{1}{3}t^3\right)\Bigg|_1^4 + -\left(4t+\frac{3}{2}t^2-\frac{1}{3}t^3\right)\Bigg|_4^6$$

$$=\; 4(4-1)+\frac{3}{2}(4^2-1^2)-\frac{1}{3}(4^3-1^3)-\left(4(6-4)+\frac{3}{2}(6^2-4^2)-\frac{1}{3}(6^3-4^3)\right)$$

$$=\; 12+\frac{45}{2}-21-\left(8+30-\frac{152}{3}\right) \;=\; 12+22\frac{1}{2}-21-8-30+50\frac{2}{3} \;=\; 26\frac{1}{6}\text{ meters}$$

[15] Let $u=2t+1$

$$\frac{du}{dt}=2 \quad\Rightarrow\quad dt=\frac{1}{2}du$$

$$t=-1 \quad\Rightarrow\quad u=-1$$

$$t=2 \quad\Rightarrow\quad u=5$$

$$\int_{-1}^{2} f(2t+1)\,dt \;=\; \int_{-1}^{5}\frac{1}{2}f(u)\,du \;=\; \frac{1}{2}\int_{-1}^{5} f(u)\,du \;=\; \frac{7}{2}$$