Find the centroid of the region bounded by $y = x^2$ and $y = 2x$. **SCORE: _____ / 25 PTS**

$$x^2 = 2x \longrightarrow x = 0, 2$$

$$\int_0^2 (2x - x^2)\,dx = \left(x^2 - \frac{1}{3}x^3\right)\Big|_0^2 = 4 - \frac{8}{3} = \frac{4}{3}$$

$$\int_0^2 x(2x - x^2)\,dx = \int_0^2 (2x^2 - x^3)\,dx = \left(\frac{2}{3}x^3 - \frac{1}{4}x^4\right)\Big|_0^2 = \frac{16}{3} - 4$$

$$= \frac{4}{3}$$

$$\frac{1}{2}\int_0^2 \left((2x)^2 - (x^2)^2\right)dx = \frac{1}{2}\int_0^2 (4x^2 - x^4)\,dx = \frac{1}{2}\left(\frac{4}{3}x^3 - \frac{1}{5}x^5\right)\Big|_0^2$$

$$= \frac{1}{2}\left(\frac{32}{3} - \frac{32}{5}\right) = \frac{1}{2}\cdot\frac{64}{15}$$

$$= \frac{32}{15}$$

$$(\bar{x}, \bar{y}) = \left(\frac{\frac{4}{3}}{\frac{4}{3}}, \frac{\frac{32}{15}}{\frac{4}{3}}\right) = \left(1, \frac{8}{5}\right)$$

[redacted]

Find the length of the parametric curve $\begin{array}{l} x = 1 + 3t + 2e^{-6t} \\ y = 5 - 4e^{-3t} \end{array}$ over $1 \le t \le 3$. **SCORE: _____ / 25 PTS**

$$\int_1^3 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_1^3 \sqrt{(3-12e^{-6t})^2 + (12e^{-3t})^2}\, dt$$

$$= \int_1^3 \sqrt{9 - 72e^{-6t} + 144e^{-12t} + 144e^{-6t}}\, dt$$

$$= \int_1^3 \sqrt{9 + 72e^{-6t} + 144e^{-12t}}\, dt$$

$$= \int_1^3 \sqrt{(3+12e^{-6t})^2}\, dt$$

$$= \int_1^3 (3+12e^{-6t})\, dt$$

$$= (3t - 2e^{-6t})\Big|_1^3$$

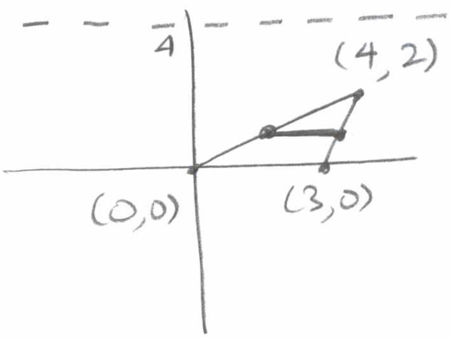
$$= 3(3-1) - 2(e^{-18} - e^{-6})$$

$$= 6 - 2e^{-18} + 2e^{-6}$$

[redacted]

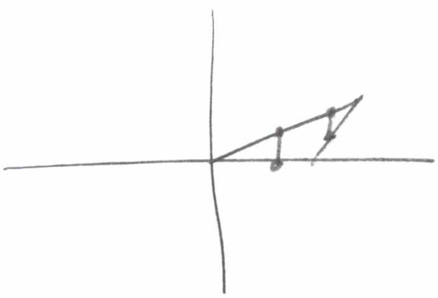
The region defined by $y \le \frac{1}{2}x$, $y \ge 2x - 6$ and $y \ge 0$ is revolved around the line $y = 4$. **SCORE: _____ / 40 PTS**

[a] Write, **BUT DO NOT EVALUATE**, an integral (or sum of integrals) for the volume of the solid using the shell method.



$$2\pi \int_0^2 (4-y)\left(\frac{y+6}{2} - 2y\right) dy$$

[b] Write, **BUT DO NOT EVALUATE**, an integral (or sum of integrals) for the volume of the solid using the disk or washer method.



$$\pi \int_0^3 \left(4^2 - \left(4 - \tfrac{1}{2}x\right)^2\right) dx$$

$$+ \pi \int_3^4 \left(\left(4 - (2x-6)\right)^2 - \left(4 - \tfrac{1}{2}x\right)^2\right) dx$$

[c] Find the volume of the solid using one of your integrals above. **DO NOT USE GEOMETRY NOR THE THEOREM OF PAPPUS.**

$$2\pi \int_0^2 (4-y)\left(3 - \tfrac{3}{2}y\right) dy$$

$$= 2\pi \int_0^2 \left(12 - 9y + \tfrac{3}{2}y^2\right) dy$$

$$= 2\pi \left(12y - \tfrac{9}{2}y^2 + \tfrac{1}{2}y^3\right)\Big|_0^2$$

$$= 2\pi (24 - 18 + 4)$$

$$= 20\pi$$

[redacted]

The snooze button on an alarm clock allows you to turn off a ringing alarm for a short time (the "snooze period"). **SCORE: _____ / 35 PTS**
At the end of the snooze period, the alarm rings again. To prevent you from getting used to a certain snooze period, Big Ben alarm clocks are designed so that the snooze period is a random time between 3 and 8 minutes.

Let X be a continuous random variable representing the length of a snooze period, and suppose that X has the probability density function

$$f(x) = \begin{cases} \dfrac{a}{\sqrt{1+x}}, & 3 \le x \le 8 \\ 0, & x < 3 \quad or \quad x > 8 \end{cases}$$ for some constant a.

[a] Find a.

$$\int_3^8 a(1+x)^{-\frac{1}{2}}\,dx = 1$$

$$u = 1+x \rightarrow du = dx$$

$$\int_4^9 a u^{-\frac{1}{2}}\,du = 1$$

$$2au^{\frac{1}{2}}\Big|_4^9 = 1$$

$$2a(3-2) = 1$$

$$2a = 1$$

$$a = \frac{1}{2}$$

[b] Find the mean length of the snooze period.

$$\int_3^8 \frac{1}{2}x(1+x)^{-\frac{1}{2}}\,dx$$

$$u = 1+x \rightarrow du = dx$$

$$x = u-1$$

$$= \int_4^9 \frac{1}{2}(u-1)u^{-\frac{1}{2}}\,du$$

$$= \frac{1}{2}\int_4^9 (u^{\frac{1}{2}} - u^{-\frac{1}{2}})\,du$$

$$= \frac{1}{2}\left(\frac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right)\Big|_4^9$$

$$= \frac{1}{3}(27-8) - (3-2)$$

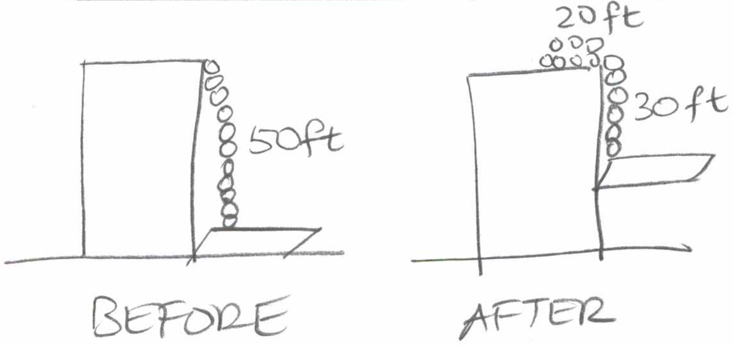
$$= \frac{19}{3} - 1$$

$$= \frac{16}{3}$$

A 50 foot long chain weighing 200 pounds hangs from the roof of a 50 foot tall building. **SCORE: _____ / 25 PTS**

The chain is used to pull a 70 pound tabletop from ground level to a window 20 feet above ground level.

Write, **<u>BUT DO NOT EVALUATE</u>**, an expression involving an integral (or sum of integrals) for the work done.



DENSITY $= \dfrac{200\,\text{lb}}{50\,\text{ft}} = 4\,\dfrac{\text{lb}}{\text{ft}}$

TOP 20 FT MOVED TO ROOF

BOTTOM 30 FT + TABLETOP MOVED 20 FT

$$\int_0^{20} 4x\,dx + (4 \cdot 30 + 70) \cdot 20$$

$$= \int_0^{20} 4x\,dx + 190 \cdot 20 \text{ ft-lb}$$