To find $\int \frac{N(x)}{D(x)}dx$, where $N(x)$ and $D(x)$ are both polynomials, and no cancellation is possible

Some rational integrands require partial fractions, others do not, and still others are a combination of the two.
The process below outlines when you should and should not use partial fractions.

**If degree of $N(x) \geq$ degree of $D(x)$**

Perform polynomial long division
Rewrite integrand as polynomial + remainder with degree of $N(x) <$ degree of $D(x)$
Use process below to find the integral of $\frac{remainder}{D(x)}$ (ie. set $N(x) = remainder$ and continue)

**If $N(x) = k \cdot D'(x)$ (ie. numerator is constant multiple of derivative of denominator)** **TYPE 1**

Let $u = D(x)$ & perform $u-$substitution
(or use guess & check – antiderivative is a multiple of $\ln|D(x)|$)

**If degree of $D(x) = 1$ (ie. denominator is linear)**

Let $u = D(x)$ & perform $u-$substitution
(or use guess & check – antiderivative is a multiple of $\ln|D(x)|$)

**If degree of $D(x) = 2$ and is irreducible**
**(ie. denominator is quadratic with negative discriminant, so denominator has no real roots / only complex roots)**

If $N(x) = c$ (ie. numerator is constant): **TYPE 2**

Factor leading coefficient from $D(x)$ (ie. so denominator starts with $x^2$)
Complete the square for $D(x) = (x+h)^2 + a^2$
Factor $a^2$ from denominator
Let $u = \frac{x+h}{a}$ & perform $u-$substitution
(or use $\int \frac{1}{(x+h)^2 + a^2} dx = \frac{1}{a} \tan^{-1} \frac{x+h}{a}$)

If $N(x) = ax + b$ (ie. numerator is linear): **TYPE 3**

Use technique similar to partial fractions shortcut to rewrite numerator as $A \cdot D'(x) + B$
(ie. constant multiple of derivative of denominator + constant)
Split integrand into integrand of **TYPE 1** + integrand of **TYPE 2**
Use processes above (NOTE: no absolute values required in $\ln(D(x))$)

**All other cases require partial fractions**

Factor $D(x)$ into product of linear and irreducible quadratic factors
(ie. write denominator as product of
linear factors – one for each real root (whether rational or irrational)
quadratic factors – one for each pair of complex conjugate roots)

Perform partial fractions decomposition

NOTE: for irreducible quadratic factors with denominator $d(x) = ax^2 + bx + c$

(or powers of these factors, ie. $[d(x)]^n$ or $(ax^2 + bx + c)^n$)

write numerator in $A \cdot d'(x) + B$ format for **TYPE 3** to save work later on

For all partial fractions with linear and irreducible quadratic denominators:

Use processes above

For all partial fractions with denominator $[d(x)]^n = (ax+b)^n$ (ie. power of linear factor):

Let $u = ax + b$ & perform $u-$substitution

(or use guess & check – antiderivative is a multiple of $\dfrac{1}{(ax+b)^{n-1}}$)

For all partial fractions with denominator $[d(x)]^n = (ax^2 + bx + c)^n$ (ie. power of irreducible quadratic factor):

Split integrand into integrand with numerator $A \cdot d'(x)$ + integrand with numerator $B$

For first integrand:

Let $u = ax^2 + bx + c$ & perform $u-$substitution

(or use guess & check – antiderivative is a multiple of $\dfrac{1}{(ax^2+bx+c)^{n-1}}$)

For second integrand:

Factor leading coefficient from $ax^2 + bx + c$ (ie. so irreducible quadratic starts with $x^2$)

Complete the square for $x^2 + Bx + C = (x+h)^2 + k^2$

Let $x + h = k\tan\theta$ & perform trigonometric substitution

NOTE: This is the hardest type – there will be no required problems of this type on tests

Practice against the following examples:

**TYPE 1**

$$\int \frac{7}{3x+8}\,dx = \frac{7}{3}\ln|3x+8| + C$$

**TYPE 1**

$$\int \frac{6-9x}{3x^2-4x-4}\,dx = -\frac{3}{2}\ln|3x^2-4x-4| + C$$

**TYPE 2**

$$\int \frac{5}{4x^2+24x+52}\,dx = \frac{5}{4}\int \frac{1}{x^2+6x+13}\,dx = \frac{5}{4}\int \frac{1}{(x+3)^2+2^2}\,dx = \frac{5}{8}\tan^{-1}\frac{x+3}{2} + C$$

**TYPE 3**

$$\int \frac{5x-7}{x^2+8x+25}\,dx = \int \frac{\frac{5}{2}(2x+8)-27}{x^2+8x+25}\,dx = \frac{5}{2}\int \frac{2x+8}{x^2+8x+25}\,dx - 27\int \frac{1}{(x+4)^2+3^2}\,dx$$

$$= \frac{5}{2}\ln(x^2+8x+25) - 9\tan^{-1}\frac{x+4}{3} + C$$

**MIXED**

$$\int \frac{-20x-12}{(x+1)^2(x^2+4x+7)}\,dx = \int\left(\frac{-6}{x+1} + \frac{2}{(x+1)^2} + \frac{3(2x+4)+4}{x^2+4x+7}\right)dx$$

$$= \int \frac{-6}{x+1}\,dx + \int \frac{2}{(x+1)^2}\,dx + 3\int \frac{2x+4}{x^2+4x+7}\,dx + \int \frac{4}{(x+2)^2+(\sqrt{3})^2}\,dx$$

$$= -6\ln|x+1| - \frac{2}{x+1} + 3\ln(x^2+4x+7) + \frac{4}{\sqrt{3}}\tan^{-1}\frac{x+2}{\sqrt{3}} + C$$

**MIXED**

$$\int \frac{3x^3-11x^2-49x}{x^2-5x-6}\,dx = \int\left(3x+4+\frac{-11x+24}{(x-6)(x+1)}\right)dx = \int\left(3x+4-\frac{6}{x-6}-\frac{5}{x+1}\right)dx$$

$$= \frac{3}{2}x^2 + 4x - 6\ln|x-6| - 5\ln|x+1| + C$$