Finding the domain of $f \circ g$

Because $(f \circ g)(x) = f(g(x))$,
in order for x to be in the domain of $f \circ g$,
x must first be in the domain of g, since we first need to plug x into g,
and in addition, $g(x)$ must be in the domain of f, since we then need to plug $g(x)$ into f.

Example

Let $f(x) = \dfrac{x+1}{x-2}$ and $g(x) = \dfrac{x-6}{x+3}$.
The domain of f is $\{x \neq 2\}$.
The domain of g is $\{x \neq -3\}$.

[a] If we try to find $(f \circ g)(-3)$,
we first write it as $f(g(-3))$,
and we immediately realize there's a problem
because $g(-3)$ would be $\dfrac{-3-6}{-3+3}$ which isn't defined since the denominator is 0.

The problem is that $x = -3$ isn't in the domain of g,
so $g(-3)$ doesn't give us a value,
so we have nothing to plug into f,
which means $(f \circ g)(-3) = f(g(-3))$ doesn't give us a value.

So, $x = -3$ isn't in the domain of $f \circ g$.

[b] Now, if we try to find $(f \circ g)(-12)$,
we first write it as $f(g(-12))$.
Now, $g(-12) = \dfrac{-12-6}{-12+3} = \dfrac{-18}{-9} = 2$.
So we made it past the first obstacle in part [a].

So now, $(f \circ g)(-12) = f(g(-12)) = f(2)$,
and we realize there's a second issue
because $f(2)$ would be $\dfrac{2+1}{2-2}$ which isn't defined since the denominator is 0.

The problem is that, even though -12 is in the domain of g,
$g(-12) = 2$ isn't in the domain of f.
So, even though we can plug $x = -12$ into g and get a value $g(-12) = 2$,
we can't plug that value $g(-12) = 2$ into f,
which means $(f \circ g)(-12) = f(g(-12))$ doesn't give us a value.

So, $x = -12$ isn't in the domain of $f \circ g$ either.

So, looking back on the two examples above, in order for x to be in the domain of $f \circ g$,
x must first be in the domain of g, so that we get a value when we plug x into g,
and in addition, $g(x)$ must be in the domain of f, so that we get a value when we later plug $g(x)$ into f.

To find the domain of our example above,
we first need to know the domains of both f and g.
Remember that
the domain of f is $\{x \neq 2\}$ and
the domain of g is $\{x \neq -3\}$.

For the domain of $f \circ g$, x must first be in the domain of g, which means $x \neq -3$.

In addition, $g(x)$ must be in the domain of f.
Since the domain of f is $\{x \neq 2\}$, ie. all real numbers that are not equal to 2,
that means $g(x)$ can be any real number except 2.
Or, algebraically, $g(x) \neq 2$.

Since $g(x) = \dfrac{x-6}{x+3}$,

we get $\dfrac{x-6}{x+3} \neq 2$,

or $x - 6 \neq 2(x+3)$,
or $x - 6 \neq 2x + 6$,
or $x \neq -12$.

So, putting it together, in order for x to be in the domain of $f \circ g$,
x must first be in the domain of g, ie. $x \neq -3$
and in addition, $g(x)$ must be in the domain of f, ie. $x \neq -12$.
So, the domain of $f \circ g$ is $\{x \neq -3 \text{ and } x \neq -12\}$.

Now, a common question is what happened to the condition in the domain of f that $x \neq 2$?
Why don't we have to say $x \neq 2$ in the domain of $f \circ g$?
The easiest way to understand this is to try to find $(f \circ g)(2)$ and see what happens.

Now $(f \circ g)(2) = f(g(2)) = f(-\tfrac{4}{5}) = -\tfrac{1}{14}$.
So, we can plug $x = 2$ into $f \circ g$ and get a value,
so, $x = 2$ is in the domain of $f \circ g$, even though $x = 2$ is not in the domain of f.
If you look at the calculation of $(f \circ g)(2) = f(g(2))$ above,
you see that when we plugged $x = 2$ into $f \circ g$,
we first plugged $x = 2$ into g,
got $-\tfrac{4}{5}$, and it was this $-\tfrac{4}{5}$ that we then plugged into f.
Since $-\tfrac{4}{5}$ is in the domain of f, we were able to the final value of $(f \circ g)(2)$.
The $x = 2$ we plugged into $f \circ g$ was never plugged directly into f,
so there was no problem with $x = 2$ in the domain of $f \circ g$.

NOTE: The restriction that $x \neq 2$ in the domain of f
becomes the restriction $g(x) \neq 2$ when finding the domain of $f \circ g$,
since it is $g(x)$ that gets plugged into f, not x itself.