To graph a rational function $f(x) = \frac{n(x)}{d(x)}$ (where $n(x)$ are $d(x)$ are both polynomials)

***n*(*x*)** and ***d*(*x*)** are used to refer to the numerator & denominator polynomial respectively.

[0] Factor $n(x)$ and $d(x)$ into linear and irreducible quadratic factors.

[1] Find the domain of f, by solving for the denominator $\neq 0$.

[2] Find the $x-$intercepts of f, by solving for the numerator $= 0$,
where x must be in the domain (ie. denominator $\neq 0$).

[3] Find the $y-$intercepts of f, by finding $f(0)$.

[4] Find the long run behavior, which is based on the degrees / leading terms of the numerator and the denominator.

If degree of numerator < degree of denominator,
then the graph has a horizontal asymptote at $y = 0$. ★
If degree of numerator = degree of denominator,
then the graph has a horizontal asymptote at $y = \frac{\textit{leading coefficient of } n(x)}{\textit{leading coefficient of } d(x)}$. ★
If degree of numerator = 1 + degree of denominator,
then the graph has a slant/oblique asymptote,
which is the quotient after polynomial long division.
If degree of numerator > 1 + degree of denominator,
then the long run behavior resembles $y = \frac{\textit{leading term of } n(x)}{\textit{leading term of } d(x)}$. ★

★ These three cases all come down to the same principle,
that the long run behavior resembles $y = \frac{\textit{leading term of } n(x)}{\textit{leading term of } d(x)}$.

[5] Find the vertical asymptotes of f, by simplifying f (by cancelling),
and solving for the simplified denominator $= 0$.

f has holes at $x-$values which are not in the domain, but also do not correspond to vertical asymptotes.
Find the $y-$coordinates of those holes by substituting the $x-$values into the simplified version of f.

[6] Find the behavior of f at each $x-$intercept and vertical asymptote by replacing x in each factor of f with the $x-$value at that intercept / vertical asymptote, except for the factor that becomes 0.
You can use the simplified version of f.

For $x-$intercepts, this tells what polynomial the graph looks like as it meets the $x-$axis.
For vertical asymptotes, this tells whether the graph goes up or down on each side of the vertical asymptote.

[7] Plot all intercepts, vertical / horizontal / slant asymptotes and holes.
Sketch the behavior of the graph at each $x-$intercept, and on each side of each vertical asymptote.
Connect the points and sketches of curves together,
so that the long run behavior matches any horizontal / slant asymptotes
without introducing any "new" $x-$intercepts that are not there.
NOTE: These graphs CAN cross their horizontal / slant asymptotes, but NOT their vertical asymptotes.

Example 1

$$f(x) = \frac{x^2 - 2x - 3}{4 - x^2}$$

[0] FACTOR: $f(x) = -\frac{(x-3)(x+1)}{(x-2)(x+2)}$

[1] DOMAIN: denominator $(x-2)(x+2) \neq 0 \quad \Rightarrow \quad x \neq 2 \text{ and } x \neq -2$

[2] $x-$INTERCEPTS: numerator $(x-3)(x+1) = 0 \quad \Rightarrow \quad x = 3 \text{ or } x = -1$

[3] $y-$INTERCEPTS: $f(0) = -\frac{3}{4}$

[4] LONG RUN BEHAVIOR: degree of numerator = degree of denominator

horizontal asymptote at $y = \frac{\text{leading coefficient of } x^2 - 2x - 3}{\text{leading coefficient of } 4 - x^2} = \frac{1}{-1} = -1$

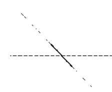
[5] VERTICAL ASYMPTOTES: f is already simplified

simplified denominator $(x-2)(x+2) = 0 \quad \Rightarrow \quad x = 2 \text{ and } x = -2$

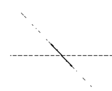
HOLES: no holes (all $x-$values not in the domain correspond to vertical asymptotes)

[6] BEHAVIOR AT $x-$INTERCEPTS:

at $x = 3$, $f(x) \approx -\frac{(x-3)(3+1)}{(3-2)(3+2)} = -\frac{4}{5}(x-3)$

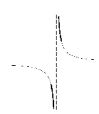


at $x = -1$, $f(x) \approx -\frac{(-1-3)(x+1)}{(-1-2)(-1+2)} = -\frac{4}{3}(x+1)$

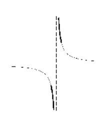


BEHAVIOR ON EACH SIDE OF VERTICAL ASYMPTOTES:

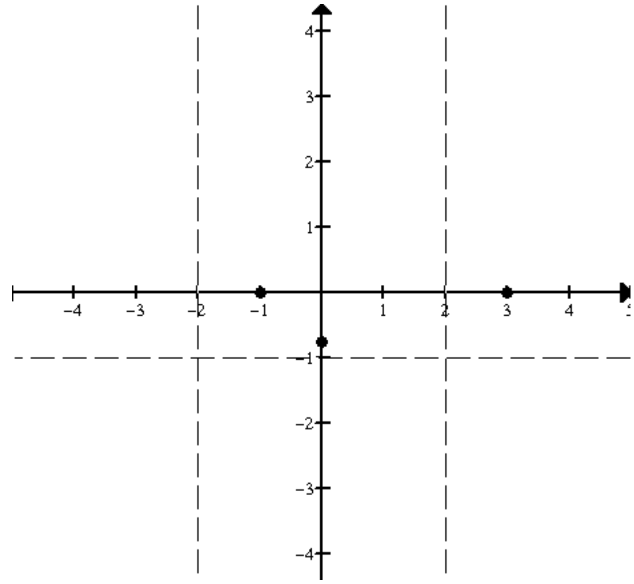
at $x = 2$, $f(x) \approx -\frac{(2-3)(2+1)}{(x-2)(2+2)} = \frac{3}{4(x-2)}$



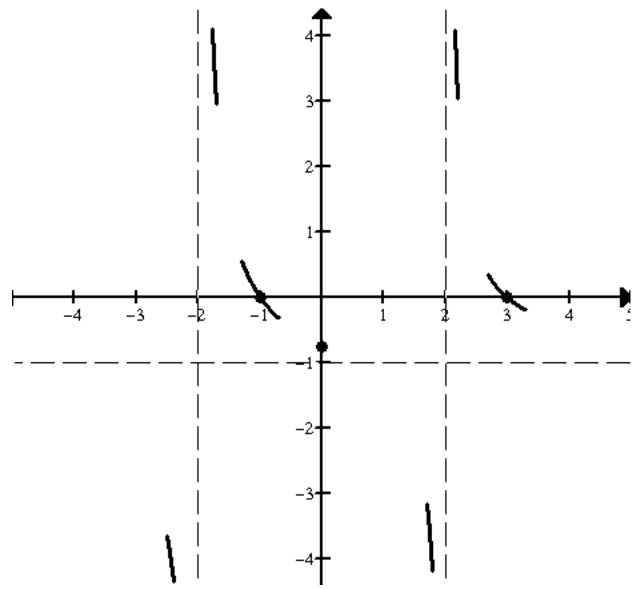
at $x = -2$, $f(x) \approx -\frac{(-2-3)(-2+1)}{(-2-2)(x+2)} = \frac{5}{4(x+2)}$



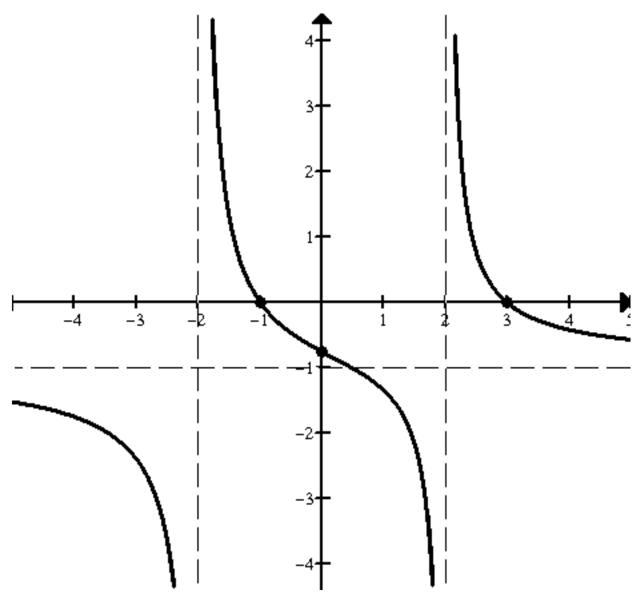
[7] INTERCEPTS, ASYMPTOTES & HOLES:



BEHAVIOR AT INTERCEPTS & ASYMPTOTES:



CONNECT & APPROACH HORIZONTAL / SLANT ASYMPTOTES WITHOUT "NEW" INTERCEPTS:



Example 2

$f(x) = \dfrac{2x^3 - 2x^2}{x^2 - 4x + 4}$

[0] FACTOR: $f(x) = \dfrac{2x^2(x-1)}{(x-2)^2}$

[1] DOMAIN: denominator $(x-2)^2 \neq 0 \Rightarrow x \neq 2$

[2] $x-$INTERCEPTS: numerator $2x^2(x-1) = 0 \Rightarrow x = 0$ *or* $x = 1$

[3] $y-$INTERCEPTS: $f(0) = 0$

[4] LONG RUN BEHAVIOR: degree of numerator = 1 + degree of denominator

$f(x) = \dfrac{2x^3 - 2x^2}{x^2 - 4x + 4} = 2x + 6 + \dfrac{16x - 24}{x^2 - 4x + 4}$ by polynomial long division

slant/oblique asymptote at $y = 2x + 6$

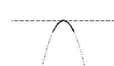
[5] VERTICAL ASYMPTOTES: f is already simplified

simplified denominator $(x-2)^2 = 0 \Rightarrow x = 2$

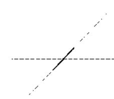
HOLES: no holes (all $x-$values not in the domain correspond to vertical asymptotes)

[6] BEHAVIOR AT $x-$INTERCEPTS:

at $x = 0$, $f(x) \approx \dfrac{2x^2(0-1)}{(0-2)^2} = -\dfrac{1}{2}x^2$

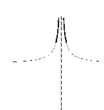


at $x = 1$, $f(x) \approx \dfrac{2(1)^2(x-1)}{(1-2)^2} = 2(x-1)$



BEHAVIOR ON EACH SIDE OF VERTICAL ASYMPTOTES:

at $x = 2$, $f(x) \approx \dfrac{2(2)^2(2-1)}{(x-2)^2} = \dfrac{8}{(x-2)^2}$



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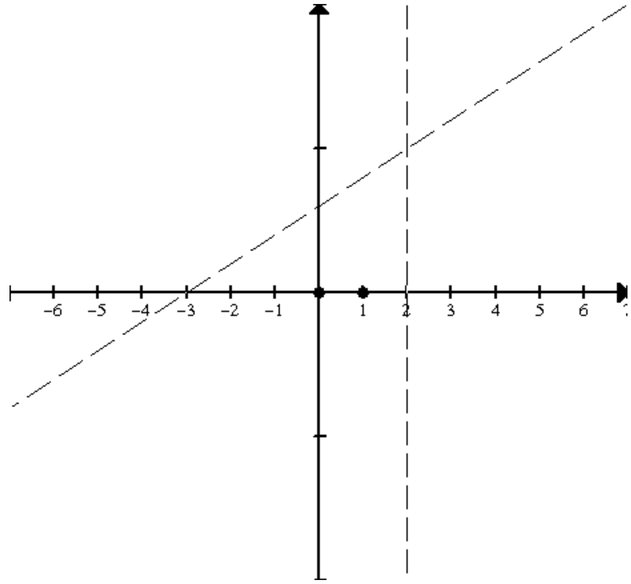
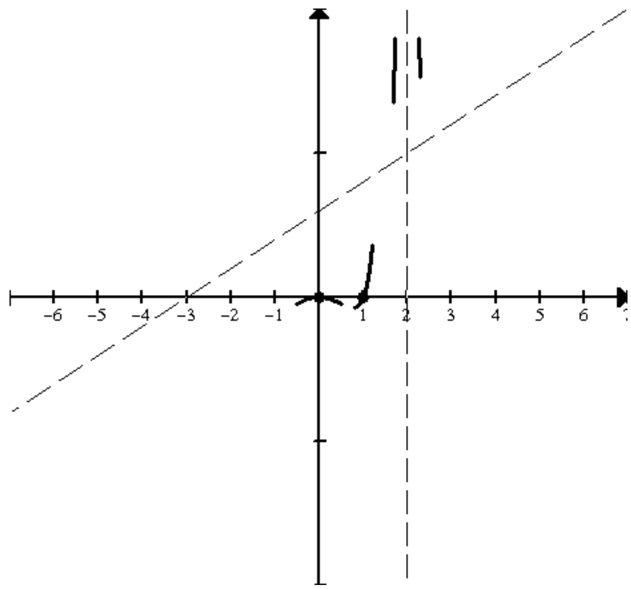
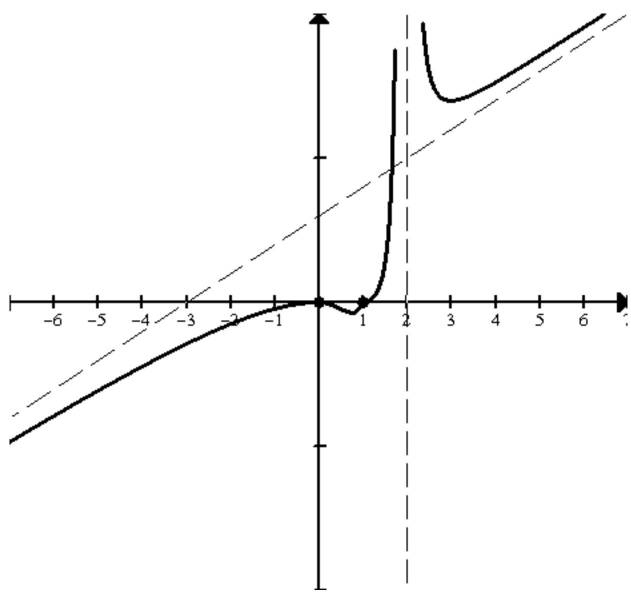


BEHAVIOR AT INTERCEPTS & ASYMPTOTES:



CONNECT & APPROACH HORIZONTAL / SLANT ASYMPTOTES WITHOUT "NEW" INTERCEPTS:

This graph is NOT to scale
Various features were exaggerated
since they cannot be seen all together
when drawn to scale



Example 3

$$f(x)=\frac{x^2-2x-3}{x^3-x}$$

[0] FACTOR: $f(x)=\dfrac{(x-3)(x+1)}{x(x+1)(x-1)}$

[1] DOMAIN: denominator $x(x+1)(x-1)\neq 0 \;\Rightarrow\; x\neq 0 \text{ and } x\neq -1 \text{ and } x\neq 1$

[2] $x-$INTERCEPTS: numerator $(x-3)(x+1)=0 \;\Rightarrow\; x=3 \text{ or } x=-1$
but $x=-1$ is not in the domain, so only $x=3$

[3] $y-$INTERCEPTS: $f(0)$ does not exist, since $x=0$ is not in the domain, so no $y-$intercept

[4] LONG RUN BEHAVIOR: degree of numerator < degree of denominator
horizontal asymptote at $y=0$

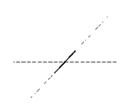
[5] VERTICAL ASYMPTOTES: $f(x)=\dfrac{(x-3)(x+1)}{x(x+1)(x-1)}=\dfrac{x-3}{x(x-1)}$
simplified denominator $x(x-1)=0 \;\Rightarrow\; x=0 \text{ and } x=1$

HOLES: hole at $x=-1$ (not in the domain and does not correspond to vertical asymptote)
$y-$coordinate at hole $=\dfrac{-1-3}{-1(-1-1)}=-2$

[6] USING SIMPLIFIED VERSION OF f

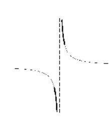
BEHAVIOR AT $x-$INTERCEPTS:

at $x=3$, $f(x)\approx\dfrac{x-3}{3(3-1)}=\dfrac{1}{6}(x-3)$

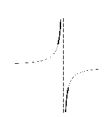


BEHAVIOR ON EACH SIDE OF VERTICAL ASYMPTOTES:

at $x=0$, $f(x)\approx\dfrac{0-3}{x(0-1)}=\dfrac{3}{x}$



at $x=1$, $f(x)\approx\dfrac{1-3}{1(x-1)}=-\dfrac{2}{x-1}$



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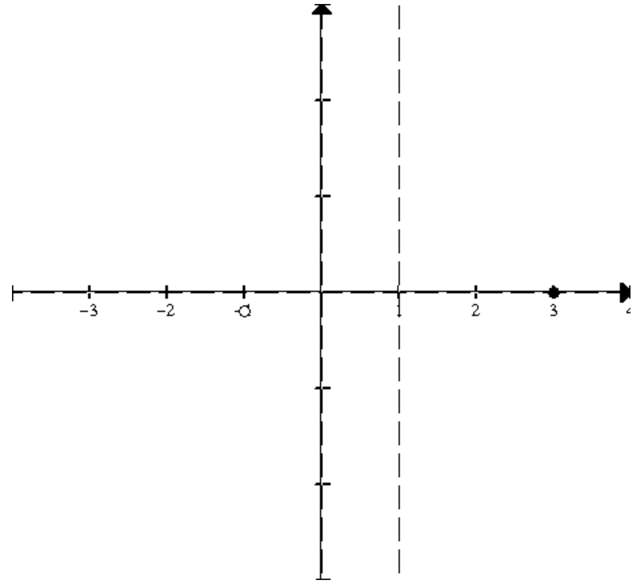
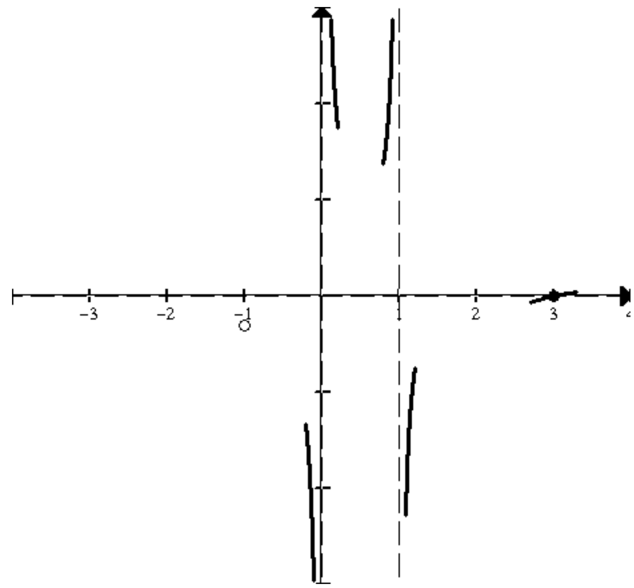


BEHAVIOR AT INTERCEPTS & ASYMPTOTES:



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This graph is NOT to scale
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The graph heads back down to the *x*-axis
on the right end