Write the following formally. If you wish, you may use correct mathematical symbols/notation where appropriate. **SCORE: ___ / 15 PTS**
<u>Your answers must be in complete sentences, and all variables in your definition must be properly declared.</u>

[a] the formal definition of the term "prime"

AN INTEGER n IS PRIME IF AND ONLY IF $n > 1$
AND FOR ALL POSITIVE INTEGERS r, s, IF $n = rs$, THEN $r = 1$ OR $s = 1$

[b] the quotient-remainder theorem

FOR ALL INTEGERS n AND POSITIVE INTEGERS d,
THERE EXIST UNIQUE INTEGERS q, r SUCH THAT $n = qd + r$
AND $0 \le r < d$

Let $A = \{x \in Z : x \bmod 3 \neq 2\}$ and $B = \{x \in Z : 3 \mid (x - x^2)\}$. Prove that $A = B$. **SCORE: ___ / 35 PTS**

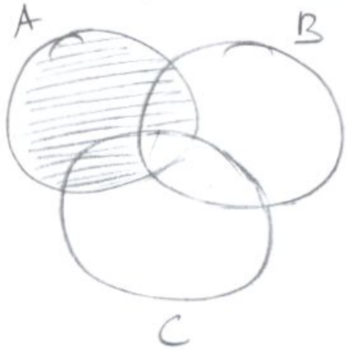
PROOF:
LET $x \in A$
SO, $x \in \mathbb{Z}$ AND $x \bmod 3 \neq 2$
BY QRT AND DEF'N OF MOD, $x = 3q$ OR $x = 3q+1$ FOR SOME $q \in \mathbb{Z}$
CASE 1: $x = 3q$
$x - x^2 = 3q - 9q^2 = 3(q - 3q^2)$ WHERE $q - 3q^2 \in \mathbb{Z}$ BY CLOSURE OF $\mathbb{Z}$ UNDER $*$ AND $-$.
SO $3 \mid (x - x^2)$ BY DEF'N OF $\mid$
CASE 2: $x = 3q+1$
$x - x^2 = (3q+1) - (9q^2 + 6q + 1) = -9q^2 - 3q = 3(-3q^2 - q)$
WHERE $-3q^2 - q \in \mathbb{Z}$ BY CLOSURE OF $\mathbb{Z}$ UNDER $*$ AND $-$.
SO $3 \mid (x - x^2)$ BY DEF'N OF $\mid$
SO, $3 \mid (x - x^2)$ AND $x \in \mathbb{Z}$
SO, $x \in B$
SO, $A \subseteq B$ BY DEF'N OF $\subseteq$

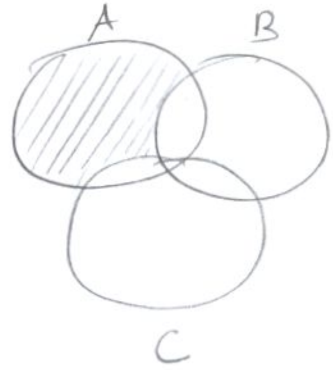
NOW, LET $x \in B$
SO $x \in \mathbb{Z}$ AND $3 \mid (x - x^2)$
SUPPOSE $x \notin A$
SO $x \bmod 3 = 2$
BY DEF'N OF MOD, $x = 3q+2$ FOR SOME $q \in \mathbb{Z}$
$x - x^2 = (3q+2) - (9q^2 + 12q + 4) = -9q^2 - 9q - 2 = 3(-3q^2 - 3q - 1) + 1$
WHERE $-3q^2 - 3q - 1 \in \mathbb{Z}$ BY CLOSURE OF $\mathbb{Z}$ UNDER $*$ AND $-$.
SO, $(x - x^2) \bmod 3 = 1$ BY DEF'N OF MOD
SO, $3 \nmid (x - x^2)$ BY EX 26 IN SECTION 4.4

SO, $x \notin B$ BUT $x \in B$ (CONTRADICTION)
SO $x \in A$ SO $B \subseteq A$ SO $A = B$

Prove that the following statement is **false**. **SCORE: ___ / 10 PTS**

For all sets A, B and C which are subsets of universal set U, $A-(B\cap C)=(A-B)\cap(A-C)$





$A=\{1\}$ $B\cap C=\emptyset$

$B=\{1\}$ $A-(B\cap C)=\{1\}$

$C=\emptyset$

$A-B=\emptyset$

$A-C=\{1\}$

$(A-B)\cap(A-C)=\emptyset$

You have a collection of 4 ¢ and 5 ¢ postage stamps. **SCORE: ___ / 30 PTS**

[a] Use **strong induction** to prove that you can obtain n ¢ of postage using a combination of the 4 ¢ and 5 ¢ postage stamps for all integers $n\geq 14$.

NOTE: You MUST use strong induction, not (regular) mathematical induction.
ie. It MUST NOT be possible to get a (regular) mathematical induction proof by simply changing a minimal number of lines of your proof.

BASIS STEP: $14¢ = 2(5¢) + 4¢$
$15¢ = 3(5¢)$
$16¢ = 4(4¢)$
$17¢ = 3(4¢) + 5¢$

INDUCTIVE STEP: ASSUME $14¢, 15¢, \ldots, k¢$ CAN BE OBTAINED USING $4¢ + 5¢$ STAMPS FOR SOME INTEGER $k \geq 17$

$(k+1)¢ = (k-3)¢ + 4¢$ AND $k-3 \geq 14$

SO $(k+1)¢$ CAN BE OBTAINED BY ADDING A $4¢$ STAMP TO THE COMBINATION OF STAMPS USED TO OBTAIN $(k-3)¢$

BY STRONG INDUCTION, $n¢$ CAN BE OBTAINED USING $4¢ + 5¢$ STAMPS FOR ALL INTEGERS $n \geq 14$

[b] Show that [a] is false if the condition is changed to "$n\geq 8$".

$11¢$ CANNOT BE MADE USING $4¢$ AND $5¢$

Prove the following statement using an element argument. **SCORE: ___ / 25 PTS**

For all sets A, B and C which are subsets of universal set U,
if $A \subseteq B$, and B and C are disjoint, then A and C are disjoint.

PROOF: LET $A, B, C \subseteq U$ SUCH THAT $A \subseteq B$ AND B + C ARE DISJOINT
SUPPOSE A + C ARE NOT DISJOINT
SO, $A \cap C \neq \emptyset$ BY DEF'N OF DISJOINT
SO, THERE IS AN ELEMENT $x \in A \cap C$
SO, $x \in A$ AND $x \in C$ BY DEF'N OF $\cap$
SO, $x \in A$
SO, $x \in B$ BY DEF'N OF $\subseteq$
SO, $x \in B$ AND $x \in C$
SO, $x \in B \cap C$ BY DEF'N OF $\cap$
SO, $B \cap C \neq \emptyset$
SO, B + C ARE NOT DISJOINT BY DEF'N OF DISJOINT
BUT B + C ARE DISJOINT (CONTRADICTION)
SO, A + C ARE DISJOINT

Let B be a Boolean algebra with operations $+$ and $\cdot$, and let $a, b \in B$. **SCORE: ___ / 20 PTS**

Use the definition and/or properties of a Boolean algebra **<u>(EXCEPT De Morgan's laws)</u>** to prove that $(a+b)+(\bar{a}\cdot\bar{b})=1$

<u>You do NOT need to write a formal proof. However, you must provide a justification for each step.</u>

$(a+b)+(\bar{a}\cdot\bar{b})$

$= ((a+b)+\bar{a})\cdot((a+b)+\bar{b})$ DISTRIBUTIVE

$= ((b+a)+\bar{a})\cdot((a+b)+\bar{b})$ COMMUTATIVE

$= (b+(a+\bar{a}))\cdot(a+(b+\bar{b}))$ ASSOCIATIVE

$= (b+1)\cdot(a+1)$ COMPLEMENT

$= 1\cdot 1$ UNIVERSAL BOUNDS

$= 1$ IDENTITY

[redacted]

Fill in the blanks. **<u>You do NOT need to show work.</u>** SCORE: ___ / 15 PTS

[a] $(-29)\ div\ 9 =$ -4

$-29 = -4(9) + 7$

[b] $(-29) \bmod 9 =$ 7

[c] If $A = \{a\}$, $\wp(\wp(A)) =$ $\{\emptyset, \{\emptyset\}, \{\{a\}\}, \{\emptyset, \{a\}\}\}$ $\wp(A) = \{\emptyset, \{a\}\}$

[d] The **<u>symbolic</u>** translation of the statement "$A \not\subseteq B - C$" is $\exists x \in A : x \notin B$ OR $x \in C$

⬆ **NOTE: Do not use $\cup$, $\cap$, $-$ or ^{C} in your answer** ⬆

$\sim(\forall x \in A, x \in B$ AND $x \notin C)$