Math 22 4
Quiz ~~3~~ Version B
Fri Feb 10, 2012

NAMES YOU ASKED TO BE CALLED IN CLASS:

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SCORE: ___ / 30 POINTS

One of the statements below is true, and the other is false.
Prove the statement that is true, and disprove the statement that is false (ie. show that the false statement is false).

SCORE: ___ / 6 POINTS

[a] For all positive integers a, b, and c, if $ab \mid c$, then $a \mid c$ and $b \mid c$.

[b] For all positive integers a, b, and c, if $a \mid bc$, then $a \mid b$ or $a \mid c$.

[a] is true.

Proof: Let a, b, and c be particular but arbitrarily chosen positive integers such that $ab \mid c$. 1
By definition of $\mid$, $c = abk$ for some $k \in Z$. 1
So, $c = a(bk) = b(ak)$ where $bk, ak \in Z$ by closure of Z under $\times$. 1 ½ ½
Therefore, by the definition of $\mid$, $a \mid c$ and $b \mid c$. 1

[b] is false.

Counterexample: $a = 4$, $b = 2$ and $c = 2$
$4 \mid (2 \times 2)$ since $4 = (2 \times 2) \times 1$
but $4 \mid 2$.

← GRADED BY ME (DON'T FORGET TO ADD THESE POINTS TO YOUR TOTAL)

Prove the following statement.

SCORE: ___ / 8 POINTS

For all positive integers c, if $3 \nmid c$, then $(c^2 - 3c) \bmod 3 = 1$.

NOTE: Do NOT use any properties of "mod" unless they were proven in this class or you prove them here (NOT recommended).

Proof: Let c be a particular but arbitrarily chosen positive integer such that $3 \nmid c$. 1
By definition of $\mid$, $c \neq 3q$ for any $q \in Z$. ½
By the quotient remainder theorem, $c = 3q + 1$ or $c = 3q + 2$ for some $q \in Z$. 1
CASE 1: $c = 3q + 1$

$$c^2 - 3c = (3q+1)^2 - 3(3q+1) = 9q^2 + 6q + 1 - 9q - 3 = 9q^2 - 3q - 2 = 3(3q^2 - q - 1) + 1$$ 1

where $3q^2 - q - 1 \in Z$ by closure of Z under $\times$ and $+$. 1
By definition of mod, $(c^2 - 3c) \bmod 3 = 1$. ½

CASE 2: $c = 3q + 2$

$$c^2 - 3c = (3q+2)^2 - 3(3q+2) = 9q^2 + 12q + 4 - 9q - 6 = 9q^2 + 3q - 2 = 3(3q^2 + q - 1) + 1$$ 1

where $3q^2 + q - 1 \in Z$ by closure of Z under $\times$ and $+$. 1
By definition of mod, $(c^2 - 3c) \bmod 3 = 1$. ½

½ Therefore, $c^2 \bmod 3 = 1$.

One of the statements below is true, and the other is false.

SCORE: ___ / 8 POINTS

Prove the statement that is true, and disprove the statement that is false (ie. show that the false statement is false).

NOTE: Do NOT use any properties of "mod" unless they were proven in this class or you prove them here (NOT recommended).

[a] $\forall n \in Z^+,\quad n \bmod 3 = 2 \rightarrow n^2 \bmod 3^2 = 2^2$

[b] For all integers a and b, if $a \bmod 8 = 7$ and $b \bmod 12 = 5$, then $ab \bmod 4 = 3$.

HINT: The shortest, simplest proof for the true statement is a direct proof from definitions.

[a] is false.

Counterexample: $n = 5$

$5 \bmod 3 = 2$ since $5 = 3(1) + 2$

but $5^2 \bmod 3^2 = 25 \bmod 9 = 7$ since $25 = 9(2) + 7$

← GRADED BY ME (DON'T FORGET TO ADD THESE POINTS TO YOUR TOTAL)

[b] is true.

Proof: Let a and b be particular but arbitrarily chosen integers such that $a \bmod 8 = 7$ and $b \bmod 12 = 5$. 1

By definition of mod, $a = 8m + 7$ for some $m \in Z$. 1

By definition of mod, $b = 12n + 5$ for some $n \in Z$. 1

$ab = (8m+7)(12n+5) = 96mn + 40m + 84n + 35 = 4(24mn + 10m + 21n + 8) + 3$ 2

where $24mn + 10m + 21n + 8 \in Z$ by closure of Z under $\times$ and $+$. 1

Therefore, by definition of mod, $ab \bmod 4 = 3$. 1

Prove the following statement.

SCORE: ___ / 8 POINTS

The quotient of a non-zero rational number divided by an irrational number is irrational.

Proof by contradiction: Suppose not.

That is, suppose there exist a non-zero rational number x and an irrational number y

2 such that $\frac{x}{y}$ is rational.

1 By the definition of rational, $x = \frac{a}{b}$ and $\frac{x}{y} = \frac{c}{d}$ for some $a, b, c, d \in Z$ where $b, d \neq 0$. $\frac{1}{2}$

1 So, $xd = cy$.

Since $x, d \neq 0$, by zero product property, $xd \neq 0$, and by zero product property, $c \neq 0$.

$y = \frac{x}{\frac{x}{y}} = \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{ad}{bc}$ where $ad, bc \in Z$ by closure of Z under $\times$, 1

and $bc \neq 0$ by zero product property. $\frac{1}{2}$

So, by definition of rational, y is rational. $\frac{1}{2}$

But, y is rational and y is irrational. (CONTRADICTION) 1

So, by contradiction, the quotient of a non-zero rational number divided by an irrational number is irrational.

$\frac{1}{2}$