Math 22
Quiz 5
Thu Feb 16, 2012

NAMES YOU ASKED TO BE CALLED IN CLASS: ______________ ______________

SCORE: ___ / 30 POINTS

One of the statements below is true, and the other is false.
Prove the statement that is true, and disprove the statement that is false (ie. show that the false statement is false).

SCORE: ___ / 14 PTS

[a] $\sqrt[3]{2}$ is irrational.

[b] The quotient of an irrational number divided by a rational number is an irrational number.

[a] is true.

EACH UNDERLINED SENTENCE IS WORTH $\frac{1}{2}$ POINT, EXCEPT THOSE LABELLED AS WORTH 1 POINT.

Proof by contradiction:

Suppose not.

That is, suppose that $\sqrt[3]{2}$ is rational. 1

By definition of rational, $\sqrt[3]{2} = \dfrac{a}{b}$ for some $a, b \in Z$.

Without loss of generality, we may assume that the fraction $\dfrac{a}{b}$ is in simplest reduced form, ie. the greatest common factor of a and b is 1. 1

Since $\sqrt[3]{2} = \dfrac{a}{b}$, therefore $2 = \dfrac{a^3}{b^3}$ and $2b^3 = a^3$.

$b^3 \in Z$ by the closure of Z under $\times$.

So, by definition of $|$, $2 \mid 2b^3$ and therefore, $2 \mid a^3$.

By the lemma below, $2 \mid a$, so $a = 2c$ for some $c \in Z$.

So, $2b^3 = (2c)^3 = 8c^3$ and $b^3 = 4c^3 = 2(2c^3)$ where $2c^3 \in Z$ by the closure of Z under $\times$.

So, by definition of $|$, $2 \mid 4c^3$ and therefore, $2 \mid b^3$.

By the lemma below, $2 \mid b$.

Since $2 \mid a$ and $2 \mid b$, the greatest common factor of a and b is at least 2.

But, the greatest common factor of a and b is 1, and the greatest common factor of a and b is at least 2.
(CONTRADICTION)

So, by contradiction, $\sqrt[3]{2}$ is irrational.

Lemma: For all integers x, if $2 \mid x^3$, then $2 \mid x$. 1

Proof by contraposition:

Let $x \in Z$ such that $2 \nmid x$.

By definition of $|$, $x \neq 2q$ for any $q \in Z$.

By the quotient remainder theorem, $x = 2q + 1$ for some $q \in Z$.

So, $x^3 = (2q+1)^3 = 8q^3 + 12q^2 + 6q + 1 = 2(4q^3 + 6q^2 + 3q) + 1$ 1

where $4q^3 + 6q^2 + 3q \in Z$ by the closure of Z under $\times$ and $+$.

So, by the definition of mod, $x^3 \bmod 2 = 1$, and by exercise 26 in section 4.4, $2 \nmid x^3$.

[b] is true.

Counterexample: The irrational number $\sqrt{2}$ divided by the rational number 0 is not a number, and therefore not an irrational number. 1

Prove the following statement by mathematical induction. **SCORE: ___ / 8 PTS**

For all integers a greater than or equal to 2, $a^3 + 2a$ is divisible by 3.

NOTE: You will NOT earn any points if you do not use mathematical induction.

Proof by induction:

Basis step: $a = 2$: $2^3 + 2(2) = 12 = 3(4)$, so $2^3 + 2(2)$ is divisible by 3. 1

Inductive step: 1 Assume that $k^3 + 2k$ is divisible by 3 for some $k \in Z$ where $k \geq 2$. 1

1 By definition of divisible, $k^3 + 2k = 3m$ for some $m \in Z$.

$(k+1)^3 + 2(k+1) = k^3 + 3k^2 + 3k + 1 + 2k + 2 = k^3 + 2k + 3k^2 + 3k + 3$

$= 3m + 3k^2 + 3k + 3 = 3(m + k^2 + k + 1)$ $1\frac{1}{2}$

where $m + k^2 + k + 1 \in Z$ by the closure of Z under $\times$ and $+$. 1

By the definition of divisible, $(k+1)^3 + 2(k+1)$ is divisible by 3. 1

So, by mathematical induction, $a^3 + 2a$ is divisible by 3 for all integers a greater than or equal to 2. $\frac{1}{2}$

Prove the following statement by mathematical induction. **SCORE: ___ / 8 PTS**

For all integers k greater than or equal to 3, $2^k + 10 < 3^k$.

NOTE: You will NOT earn any points if you do not use mathematical induction.

Proof by induction:

Basis step: $k = 3$: $2^3 + 10 = 18 < 27 = 3^3$. 1

Inductive step: 1 Assume that $2^m + 10 < 3^m$ for some $m \in Z$ where $m \geq 3$. 1

1 $3^{m+1} = 3(3^m)$

1 $> 3(2^m + 10) = 3(2^m) + 30$

1 $> 2(2^m) + 30$

1 $> 2(2^m) + 10 = 2^{m+1} + 10$

So, $2^{m+1} + 10 < 3^{m+1}$. $\frac{1}{2}$

So, by mathematical induction, $2^k + 10 < 3^k$ for all integers k greater than or equal to 3. $\frac{1}{2}$