Math 22
Quiz 7
Fri Mar 2, 2012

NAMES YOU ASKED TO BE CALLED IN CLASS:

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SCORE: ___ / 30 POINTS

Use an element argument to prove that, for all sets A and B which are subsets of universal set U, if $A - B = A$, then A and B are disjoint.

SCORE: ___ / 7 PTS

NOTE: Do NOT use set algebra (identities) anywhere in your proof.

PROOF:

1 Let $A, B \subseteq U$ such that $A - B = A$.
[Prove A and B are disjoint, ie. $A \cap B = \varnothing$]
1 Suppose not; that is, suppose $A \cap B \neq \varnothing$.
1 So, there exists an element $x \in A \cap B$.
By definition of $\cap$, $x \in A$ and $x \in B$. *
Since $A - B = A$, therefore $x \in A - B$. *
By definition of $-$, $x \in A$ and $x \notin B$. *
1 But, $x \in B$ and $x \notin B$ (CONTRADICTION).
1 So, by contradiction, $A \cap B = \varnothing$.
So, by definition of disjoint, A and B are disjoint. *

ALL LINES MARKED *
ARE WORTH $\frac{1}{2}$ POINT
GRADE AGAINST
ONLY 1 VERSION

OR PROOF BY CONTRADICTION:

$1\frac{1}{2}$ Suppose not; that is, suppose there exist $A, B \subseteq U$. such that $A - B = A$ and A and B are not disjoint.
By definition of disjoint, $A \cap B \neq \varnothing$. *
1 So, there exists an element $x \in A \cap B$.
By definition of $\cap$, $x \in A$ and $x \in B$. *
Since $A - B = A$, therefore $x \in A - B$. *
By definition of $-$, $x \in A$ and $x \notin B$. *
1 But, $x \in B$ and $x \notin B$ (CONTRADICTION).
$1\frac{1}{2}$ So, by contradiction, for all $A, B \subseteq U$, if $A - B = A$, then A and B are disjoint.

OR PROOF BY CONTRAPOSITION:

$1\frac{1}{2}$ Let $A, B \subseteq U$ such that A and B are not disjoint.
1 By definition of disjoint, $A \cap B \neq \varnothing$.
1 So, there exists an element $x \in A \cap B$.
By definition of $\cap$, $x \in A$ and $x \in B$. *
$1\frac{1}{2}$ Since $x \in B$, by definition of $-$, $x \notin A - B$.
$1\frac{1}{2}$ But since $x \in A$, therefore, $A - B \neq A$.

Let B be a Boolean algebra with operations $+$ and $\cdot$, and let $a, b \in B$.

SCORE: ___ / 8 PTS

Use the definition and/or properties of a Boolean algebra **(EXCEPT the absorption laws)** to prove that $a + (a \cdot b) = a$
You do NOT need to write a formal proof. However, you must provide a justification for each step.

$a + (a \cdot b)$	
$1\frac{1}{2}$ $= (a \cdot 1) + (a \cdot b)$	by IDENTITY *
$1\frac{1}{2}$ $= a \cdot (1 + b)$	by DISTRUBUTIVE *
$= a \cdot (b + 1)$ *	by COMMUTATIVE *
$1\frac{1}{2}$ $= a \cdot 1$	by UNIVERSAL BOUNDS *
$= a$ *	by IDENTITY *

ALL ITEMS MARKED *
ARE WORTH $\frac{1}{2}$ POINT

One of the statements below is true, and the other is false. **SCORE: ___ / 9 PTS**
Prove the statement that is true, and disprove the statement that is false (ie. show that the false statement is false).

[a] For all sets A and B which are subsets of universal set U, $\wp(A \cup B) \subseteq \wp(A) \cup \wp(B)$

[b] For all sets A and B which are subsets of universal set U, $\wp(A) \cup \wp(B) \subseteq \wp(A \cup B)$

NOTES:	**The proof of the true statement may require you to use some subset relations from section 6.2. You may use those relations without proving them here.** **Do NOT use set algebra (identities) anywhere in your proof of the true statement.**

[a] is false.

COUNTEREXAMPLE:

* $\boxed{A = \{1\} \quad B = \{2\}}$

$\wp(A) = \{\varnothing, \{1\}\}$
$\wp(B) = \{\varnothing, \{2\}\}$
$\wp(A) \cup \wp(B) = \{\varnothing, \{1\}, \{2\}\}$

$A \cup B = \{1, 2\}$ $\wp(A \cup B) = \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$

* $\{1, 2\} \in \wp(A \cup B)$ BUT $\{1, 2\} \notin \wp(A) \cup \wp(B)$

ALL ITEMS MARKED * ARE WORTH ½ POINT

[b] is true.

PROOF:

* Let $A, B \subseteq U$.
* Let $X \in \wp(A) \cup \wp(B)$.
* So, by definition of $\cup$, $X \in \wp(A)$ or $X \in \wp(B)$.

1 By definition of power set, $X \subseteq A$ or $X \subseteq B$.

* Case 1: $X \subseteq A$
  1 By theorem 6.2.1.2 (inclusion in union), $A \subseteq A \cup B$.
  * By theorem 6.2.1.3 (transitivity of $\subseteq$), $X \subseteq A \cup B$.
* Case 2: $X \subseteq B$
  1 By theorem 6.2.1.2 (inclusion in union), $B \subseteq A \cup B$.
  * By theorem 6.2.1.3 (transitivity of $\subseteq$), $X \subseteq A \cup B$.

SEE ALTERNATE PROOF AT END OF PDF IF YOU DID NOT USE THEOREM 6.2.1

So, $X \subseteq A \cup B$.

1 So, by definition of power set, $X \in \wp(A \cup B)$.

* So, by definition of $\subseteq$, $\wp(A) \cup \wp(B) \subseteq \wp(A \cup B)$.

Use set algebra (identities) to justify the following statement. **SCORE: ___ / 6 PTS**
<u>You do NOT need to write a formal proof. However, you must provide a justification for each step.</u>

For all sets A and B which are subsets of universal set U, $A - (A - B) = A \cap B$

$A - (A - B)$	
$= A - (A \cap B^C)$	by SET DIFFERENCE
$= A \cap (A \cap B^C)^C$	by SET DIFFERENCE
$= A \cap (A^C \cup (B^C)^C)$	by DE MORGAN'S
$= A \cap (A^C \cup B)$	by DOUBLE COMPLEMENT
$= (A \cap A^C) \cup (A \cap B)$	by DISTRIBUTIVE
$= \varnothing \cup (A \cap B)$	by COMPLEMENT
$= (A \cap B) \cup \varnothing$	by COMMUTATIVE
$= A \cap B$	by IDENTITY

½ POINT EACH

[b] For all sets A and B which are subsets of universal set U, $\wp(A) \cup \wp(B) \subseteq \wp(A \cup B)$

ALTERNATE PROOF WITHOUT USING THEOREM 6.2.1:

★ ONLY GRADE AGAINST THIS SOLUTION IF YOU DID NOT USE "INCLUSION IN UNION" AND "TRANSITIVITY"

ALL ITEMS MARKED ✱ ARE WORTH $\frac{1}{2}$ POINT

✱ Let $A, B \subseteq U$.

✱ Let $X \in \wp(A) \cup \wp(B)$.

✱ So, by definition of $\cup$, $X \in \wp(A)$ or $X \in \wp(B)$.

1 By definition of power set, $X \subseteq A$ or $X \subseteq B$.

✱ Case 1: $X \subseteq A$

Let $y \in X$. $\frac{1}{4}$

By definition of $\subseteq$, $y \in A$. $\frac{1}{4}$

✱ So, $y \in A$ or $y \in B$.

By definition of $\cup$, $y \in A \cup B$. $\frac{1}{4}$

So, by definition of $\subseteq$, $X \subseteq A \cup B$. $\frac{1}{4}$

✱ Case 1: $X \subseteq B$

Let $y \in X$. $\frac{1}{4}$

By definition of $\subseteq$, $y \in B$. $\frac{1}{4}$

✱ So, $y \in A$ or $y \in B$.

By definition of $\cup$, $y \in A \cup B$. $\frac{1}{4}$

So, by definition of $\subseteq$, $X \subseteq A \cup B$. $\frac{1}{4}$

So, $X \subseteq A \cup B$.

1 So, by definition of power set, $X \in \wp(A \cup B)$.

✱ So, by definition of $\subseteq$, $\wp(A) \cup \wp(B) \subseteq \wp(A \cup B)$.