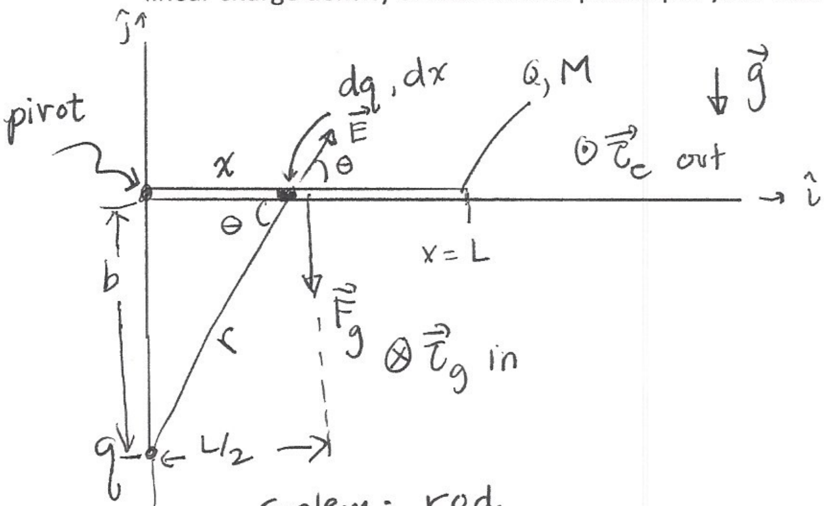
4B Exam 1 – Winter 2017 Name________________________________

Show all your work for full credit. No scratch papers, note cards, calculators, cell phones or other electronic devices are allowed. You have one (1) hour for the exam.

1. A rod of positive charge Q, mass M and length L extending along the positive *x-axis* is pivoted about the origin as shown. Gravity is present. A point charge is located a distance b along the negative *y-axis* as shown. **What value must the point charge be** if the rod does not rotate? (You may use λ for the linear charge density of the rod. but please put your final answer in terms of Q.)



System: rod

$$\vec{\tau} = I\cancel{\alpha}^{\,0}$$

$$\vec{\tau}_g + \vec{\tau}_e = 0$$

$$\vec{r}\times\vec{F}_g + \vec{r}\times\vec{F}_e = 0$$

in +: $\frac{L}{2} Mg \sin 90 + \int -x\,dq\,E\sin\theta = 0$

$$\frac{L}{2}Mg - \int \frac{x\,\lambda\,dx\;kq\,\sin\theta}{b^2+x^2} = 0 \qquad \sin\theta = \frac{b}{r}$$

$$\frac{L}{2}Mg = k\lambda q b \int_0^L \frac{x\,dx}{(b^2+x^2)^{3/2}}$$

but $u = b^2+x^2$, $du = 2x\,dx$

$$\frac{L}{\cancel{2}}Mg = \frac{k\lambda q b}{\cancel{2}} \int \frac{du}{u^{3/2}}$$

$$LMg = k\lambda q b \left(\frac{u^{-1/2}}{-1/2}\right)$$

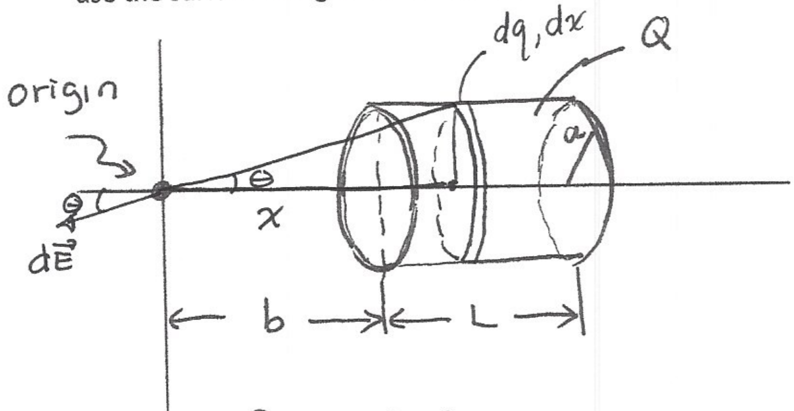
$$LMg = -2k\lambda q b \left.\frac{1}{\sqrt{x^2+b^2}}\right|_0^L$$

$$LMg = -2k\lambda q b\left[\frac{1}{\sqrt{L^2+b^2}} - \frac{1}{b}\right]$$

$$q = \frac{LMg}{2k\lambda b\left[\frac{1}{b} - \frac{1}{\sqrt{L^2+b^2}}\right]}$$

$$\boxed{q = \frac{Mg\sqrt{L^2+b^2}}{2kQ\left[\sqrt{L^2+b^2} - b\right]}}$$

2. A uniformly charged, thin-walled right circular cylindrical shell has total charge Q, radius a and is centered on the *x-axis* from $x = b$ to $x = b + L$ as shown. **Find the electric field at the origin.** (You may use the surface charge density σ, but please put your final answer in terms of Q.)



$$\sigma = \frac{Q}{A} = \frac{dq}{2\pi a\,dx} = \frac{dq}{da} = \frac{Q}{2\pi a L}$$

$$\vec{E} = \int d\vec{E}$$

$$\vec{E} = \int dE_x(+\hat{i}) + dE_y(\hat{j}) \quad \{\text{by symmetry } \int dE_y = 0$$

$$\vec{E} = \int -dE\cos\theta\,\hat{i}$$

$$\vec{E} = \int -\frac{k\,dq\,x}{r^3}\,\hat{i}$$

$$\vec{E} = \int_b^{L+b} \frac{-k\sigma\,2\pi a\,dx\cdot x}{(x^2+a^2)^{3/2}}\,\hat{i} \qquad \text{let } u = x^2+a^2,\quad du = 2x\,dx$$

$$\vec{E} = -k\sigma a\pi \int \frac{du}{u^{3/2}}\,\hat{i}$$

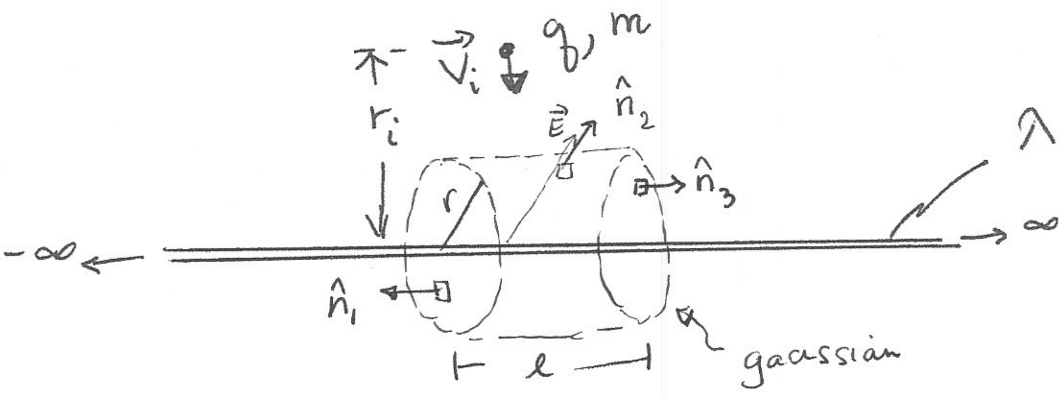
$$\vec{E} = -k\sigma a\pi \left(\frac{u^{-1/2}}{-1/2}\right)\Big|\,\hat{i}$$

$$\vec{E} = 2k\sigma a\pi \frac{1}{\sqrt{x^2+a^2}}\Big|_b^{L+b}\,\hat{i}$$

$$\vec{E} = 2k\sigma a\pi \left[\frac{1}{\sqrt{(L+b)^2+a^2}} - \frac{1}{\sqrt{b^2+a^2}}\right]\hat{i}$$

$$\boxed{\vec{E} = \frac{kQ}{L}\left[\frac{1}{\sqrt{(L+b)^2+a^2}} - \frac{1}{\sqrt{b^2+a^2}}\right]\hat{i}}$$

3. An infinite, uniform line of charge lies along the *x-axis*. The positive linear charge density is λ. A point charge of mass *m* and charge *q* is given an initial velocity pointed toward the *x-axis*. When the distance to the line charge is half of the original distance or, $r_f = \frac{1}{2} r_i$, the velocity is instantaneously zero. **Find the initial velocity.** (You may begin with Gauss's Law to find the electric field)



$$\oint \vec{E}\cdot\hat{n}\,dA = \frac{q_{in}}{\epsilon_0}$$

$$\int_1 \vec{E}\cdot\cancel{\hat{n}_1}\,dA_1 + \int_2 \vec{E}\cdot\hat{n}_2\,dA_2 + \int_3 \vec{E}\cdot\cancel{\hat{n}_3\,dA_3}^{\,0} = \frac{\lambda l}{\epsilon_0}$$

$\vec{E}\perp\hat{n}_1$ $\qquad \int_2 E_r\hat{r}\cdot\hat{r}\,dA_2$ $\qquad \vec{E}\perp\hat{n}_3$ $\qquad = \frac{\lambda l}{\epsilon_0}$

$$E_r\,2\pi r\cancel{l} = \frac{\lambda\cancel{l}}{\epsilon_0}$$

$$E_r = \frac{\lambda}{2\pi\epsilon_0 r}$$

$$V_f - V_i = -\int_i^f \vec{E}\cdot d\vec{l}$$

$$V_f - V_i = \int_{r_f}^{r_i} \vec{E}\cdot d\vec{r}$$

$$V_f - V_i = \int_{r_f}^{r_i} E_r\,\hat{r}\cdot\hat{r}\,dr$$

$$V_f - V_i = \frac{\lambda}{2\pi\epsilon_0}\int_{r_f}^{r_i}\frac{dr}{r}$$

$$V_f - V_i = \frac{\lambda}{2\pi\epsilon_0}\ln\frac{r_i}{r_f}$$

System: charge q and line charge λ

$$W_{net} = \Delta E$$

$$0 = \Delta K + \Delta U_e$$

$$0 = \cancel{K_f}^{\,0} - K_i + q\Delta V$$

$$0 = -\frac{1}{2}mv_i^2 + q(V_f - V_i)$$

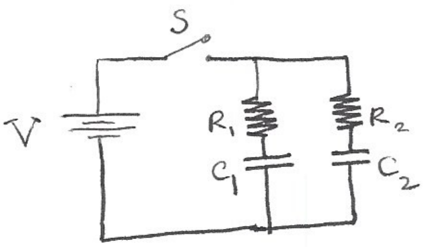
$$\frac{1}{\cancel{2}}mv_i^2 = q\frac{\lambda}{\cancel{2}\pi\epsilon_0}\ln\frac{r_i}{\frac{1}{2}r_i}$$

$$v_i^2 = \frac{q\lambda}{\pi m\epsilon_0}\ln 2$$

$$\boxed{v_i = \sqrt{\frac{q\lambda\ln 2}{\pi\epsilon_0 m}}}$$

4. In the circuit shown, capacitors C_1 and C_2 are both initially uncharged. The resistors are R_1, R_2, and the power supply has potential V.

a. (20 points) At time $t = 0$, switch S is closed. **Find the battery current I** as a function of time.



$$I = I_1 + I_2$$

$$V - I_1 R_1 - \frac{q}{C_1} = 0$$

$$V - I_2 R_2 - \frac{q}{C_2} = 0$$

$$\frac{VC_1}{R_1C_1} - I_1 - \frac{q}{R_1C_1} = 0$$

$$\frac{q - VC_1}{R_1C_1} = -I_1$$

$$\frac{q - VC_1}{R_1C_1} = -\frac{dq}{dt}$$

$$\int_0^Q \frac{dq}{q - VC_1} = \int_0^t -\frac{dt}{R_1C_1}$$

$$\ln \frac{Q - VC_1}{-VC_1} = \frac{-t}{R_1C_1}$$

$$\frac{Q - VC_1}{-VC_1} = e^{-t/R_1C_1}$$

$$Q_1 = VC_1(1 - e^{-t/R_1C_1})$$

$$I_1 = \frac{dq_1}{dt} = \frac{V\cancel{C_1}}{R_1\cancel{C_1}} e^{-t/R_1C_1}$$

$$I_1 = \frac{V}{R_1} e^{-t/R_1C_1} \qquad I_2 = \frac{V}{R_2} e^{-t/R_2C_2}$$

$$\boxed{I = \frac{V}{R_1} e^{-t/R_1C_1} + \frac{V}{R_2} e^{-t/R_2C_2}}$$

b. (5 pts.) The switch has been closed for a long (long) time. Now switch S is opened. What is the final charge on capacitor C_2? Briefly explain your reasoning.

$$Q_{max_2} = VC_2$$

when the switch is opened, both caps are already charged to potential V, no charge moves

$$\boxed{Q_{f_2} = Q_{max_2} = VC_2}$$