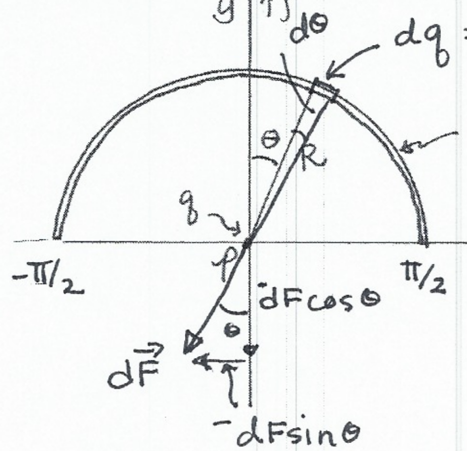
4B Exam 1 – Winter 2018 Name Solutions

Show all your work for full credit. No scratch papers, note cards, calculators, cell phones or other electronic devices are allowed. You have one (1) hour for the exam.

1. A line of positive charge is formed into a semicircle of radius *R*. The charge per unit length along the semicircle is described by $\lambda = \lambda_o \cos\theta$. What is the force **F** (magnitude and direction) on a charge, *q*, placed at the center of curvature? (Recall $\sin 2x = 2\sin x\cos x$ and $\int \cos^2 ax\,dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$)

$$dq = \lambda ds = \lambda R d\theta = \lambda_o R\cos\theta\, d\theta$$



positive charge density $\lambda = \lambda_o \cos\theta$

x $\rightarrow \hat{i}$

$$\vec{F} = \int d\vec{F}$$

$$\vec{F} = \int dF_x \hat{i} + \int dF_y \hat{j}$$

$$F_x = -\int \frac{kq\,dq \sin\theta}{R^2}$$

$$F_x = -\int_{-\pi/2}^{\pi/2} \frac{kq\lambda_o \cancel{R}\cos\theta\sin\theta\, d\theta}{R^{\cancel{2}}}$$

$$F_x = -\frac{kq\lambda_o}{2R}\int_{-\pi/2}^{\pi/2} \sin 2\theta\, d\theta$$

$$F_x = +\frac{kq\lambda_o}{4R}\left(\cos 2\theta \Big|_{-\pi/2}^{\pi/2}\right)$$

$$F_x = \frac{kq\lambda_o}{4R}\left(\cos 2(\pi/2) - \cos 2(-\pi/2)\right)$$

$$F_x = \frac{kq\lambda_o}{4R}\left(-1 - -1\right)$$

$F_x = 0$ as expected due to the symmetry of the charge distribution

$$F_y = -\int \frac{kq\,dq\cos\theta}{R^2}$$

$$F_y = -\int_{-\pi/2}^{\pi/2} \frac{kq\lambda_o \cancel{R}\cos^2\theta\, d\theta}{R^{\cancel{2}}}$$

{ here you may change the limits $\int_{-\pi/2}^{\pi/2} \rightarrow 2\int_0^{\pi/2}$ but not in the x-direction

$$F_y = -\frac{kq\lambda_o}{R}\int_{-\pi/2}^{\pi/2} \cos^2\theta\, d\theta$$

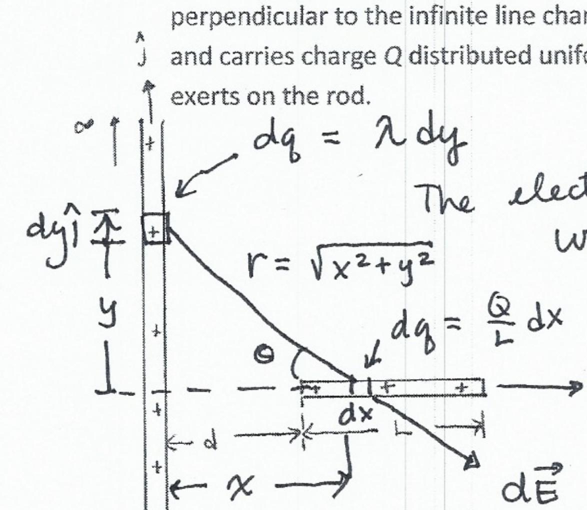
$$F_y = -\frac{kq\lambda_o}{R}\left[\frac{\theta}{2}\Big|_{-\pi/2}^{\pi/2} + \frac{\sin 2\theta}{4}\Big|_{-\pi/2}^{\pi/2}\right]$$

$$F_y = -\frac{kq\lambda_o}{R}\left[\frac{\pi}{4} - -\frac{\pi}{4} + \cancel{\frac{\sin 2(\pi/2)}{4}}^{\,0} - \cancel{\frac{\sin 2(-\pi/2)}{4}}^{\,0}\right]$$

$$F_y = -\frac{kq\lambda_o}{R}\left(\frac{\pi}{2}\right)$$

$$\boxed{\vec{F} = -\frac{kq\pi\lambda_o}{2R}\hat{j}}$$

2. An infinite line of uniform positive density charge density λ lies along the y-axis. A rod of length L lies perpendicular to the infinite line charge. The near end of the rod is a distance d above the line charge and carries charge Q distributed uniformly along its length. Find the force that the infinitely long charge exerts on the rod.



The electric field due to the infinitely long wire is only in the x-direction at (or radial) the charged rod.

$$d\vec{E} = dE_x\,\hat{i} + dE_y\,\hat{j}$$

$$\therefore \int dE_y = 0$$

$$\vec{F} = \int d\vec{F}$$

$$\vec{F} = \int \vec{E}\, dq$$

$$\vec{F} = \int_d^{d+L} \frac{2k\lambda}{x}\frac{Q}{L}\, dx\ \hat{i}$$

$$\vec{F} = \frac{2k\lambda Q}{L}\int_d^{d+L} \frac{dx}{x}\ \hat{i}$$

$$\boxed{\vec{F} = \frac{2k\lambda Q}{L}\ln\left(\frac{d+L}{d}\right)\hat{i}}$$

$$\boxed{\vec{F} = \frac{\lambda Q}{2\pi\epsilon_0 L}\ln\left(\frac{d+L}{d}\right)\hat{i}}$$

$$E_x = \int dE_x$$

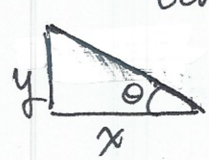
$$E_x = \int dE\cos\theta$$

$$E_x = \int \frac{k\, dq\cos\theta}{r^2}$$

$$E_x = \int_{-a}^{+a} \frac{k\lambda\, dy\cos\theta}{r^2}$$

$$E_x = 2\int_0^a \frac{k\lambda\, dy\cos\theta}{x^2+y^2}$$

$$\tan\theta = \frac{y}{x}$$



$$dy = x\sec^2\theta\, d\theta$$

$$x^2\tan^2\theta = y^2$$

$$E_x = 2k\lambda\int \frac{\cancel{x}\,\cancel{\sec^2\theta}\, d\theta\cos\theta}{\cancel{x^2}\,\cancel{(1+\tan^2\theta)}}$$

$$E_x = \frac{2k\lambda}{x}\sin\theta\,\Big|_{\text{limits}}$$

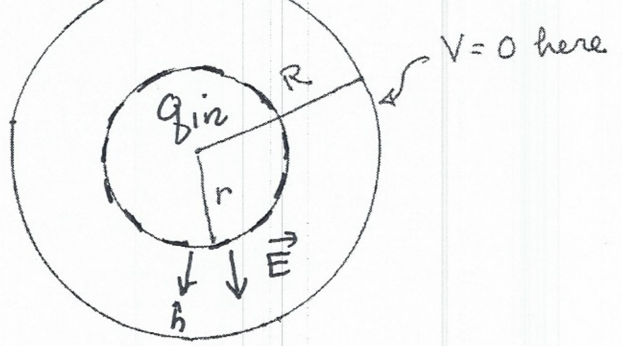
$$E_x = \frac{2k\lambda}{x}\frac{y}{\sqrt{y^2+x^2}}\Big|_0^a = \frac{2k\lambda a}{x\sqrt{x^2+a^2}}$$

as $a \to \infty$

$$E_x = \frac{2k\lambda}{x}$$

$$\vec{E} = \frac{2k\lambda}{x}\hat{i} = \frac{\lambda}{2\pi\epsilon_0 x}\hat{i}$$

3. A sphere of radius R contains a total charge Q of uniform volume charge density ρ. The electric potential is equal to zero at the surface of the sphere, that is, $V(R) = 0$. Find the electric potential, V, as a function of r for $r < R$. (Please find the electric field inside the sphere using Gauss's Law.)



$$q_{in} = \rho V_{in}$$

$$q_{in} = \rho \tfrac{4}{3}\pi r^3$$

$$\Phi_e = \oint \vec{E}\cdot\hat{n}\,dA = \frac{q_{in}}{\epsilon_0}$$

$$\oint E_r\,\hat{n}\cdot\hat{r}\,dA = \frac{\rho V_{in}}{\epsilon_0}$$

$$E_r \oint dA = \frac{\rho \frac{4}{3}\pi r^3}{\epsilon_0}$$

$$E_r\left(\cancel{4\pi} r^{\cancel{2}}\right) = \frac{\rho \frac{\cancel{4}}{3}\cancel{\pi} r^{\cancel{3}}}{\epsilon_0}$$

$$E_r = \frac{\rho r}{3\epsilon_0}$$

$$\vec{E} = \frac{\rho r\,\hat{r}}{3\epsilon_0}$$

$$\Delta V = -\int_r^R \vec{E}\cdot d\vec{s}$$

$$\overset{0}{\cancel{V(R)}} - V(r) = -\int_r^R E_r\,\hat{r}\cdot dr\,\hat{r}$$

$$+V(r) = +\int_r^R \frac{\rho r}{3\epsilon_0}\,dr$$

$$V(r) = \frac{\rho r^2}{6\epsilon_0}\Bigg|_r^R$$

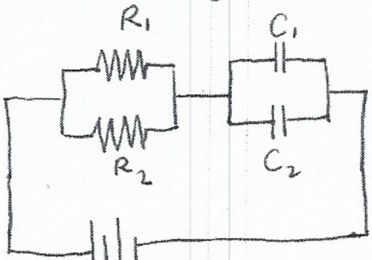
$$\boxed{V(r) = \frac{\rho}{6\epsilon_0}\left(R^2 - r^2\right)}$$ acceptable

$$\rho = \frac{Q_{TOTAL}}{V_{TOTAL}} = \frac{Q}{\frac{4}{3}\pi R^3}$$

$$V(r) = \frac{Q}{2\cancel{6}\cdot\frac{4}{\cancel{3}}\pi R^3\epsilon_0}\left(R^2 - r^2\right)$$

$$\boxed{V(r) = \frac{Q}{8\pi\epsilon_0}\left(\frac{1}{R} - \frac{r^2}{R^3}\right)}$$ alternatively

4. The circuit shown contains two resistors $R_1 = R$ and $R_2 = 2R$ and two capacitors $C_1 = 2\,C$ and $C_2 = 3C$ connected to a battery with terminal voltage V. The capacitors are uncharged at time $t = 0$. What are the charges Q_1 and Q_2 on the two capacitors, C_1 and C_2, as functions of time. (You must derive the expression for the charge as a function of time from Kirchoff's Loop equation.)



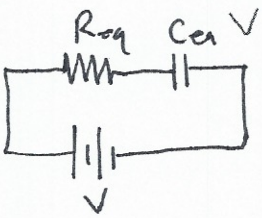
$$C_{eq} = C_1 + C_2$$

$$C_{eq} = 2C + 3C = 5C$$

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}$$

$$\frac{1}{R_{eq}} = \frac{1}{R} + \frac{1}{2R} = \frac{2}{2R} + \frac{1}{2R} = \frac{3}{2R}$$

$$R_{eq} = \frac{2R}{3} \qquad \tau = R_{eq}C_{eq} = \frac{2R}{3} \cdot 5C = \frac{10}{3}RC$$



$$V - IR_{eq} - \frac{Q}{C_{eq}} = 0$$

$$\frac{V}{R_{eq}} - I - \frac{Q}{R_{eq}C_{eq}} = 0$$

$$-I = \frac{Q - VC_{eq}}{R_{eq}C_{eq}}$$

$$-\frac{dQ}{dt} = \frac{Q - VC_{eq}}{R_{eq}C_{eq}}$$

$$\int_0^Q \frac{+dQ}{Q - VC_{eq}} = \int_0^t \frac{-dt}{R_{eq}C_{eq}}$$

$$\ln\left(\frac{Q - VC_{eq}}{-VC_{eq}}\right) = \frac{-t}{R_{eq}C_{eq}}$$

$$\frac{Q - VC_{eq}}{-VC_{eq}} = e^{-t/R_{eq}C_{eq}}$$

$$Q(t) = VC_{eq}\left(1 - e^{-t/R_{eq}C_{eq}}\right)$$

$$Q(t) = Q_{max}\left(1 - e^{-3t/10RC}\right)$$

$$Q = Q_1 + Q_2$$

$$V_1 = V_2$$

$$\frac{Q_1}{C_1} = \frac{Q_2}{C_2}$$

alternatively loop 2

$$\frac{Q_1}{C_1} - \frac{Q_2}{C_2} = 0$$

$$\frac{Q_1}{2C} = \frac{Q_2}{3C} \rightarrow Q_1 = \frac{2}{3}Q_2$$

$$Q = \frac{2}{3}Q_2 + Q_2 = \frac{5}{3}Q_2$$

$$Q_2 = \frac{3}{5}Q \;;\; Q_1 = \frac{2}{5}Q$$

$$\boxed{\begin{aligned} Q_1 &= \frac{2}{5}Q_{max}\left(1 - e^{-3t/10RC}\right) \\ Q_2 &= \frac{3}{5}Q_{max}\left(1 - e^{-3t/10RC}\right) \end{aligned}}$$