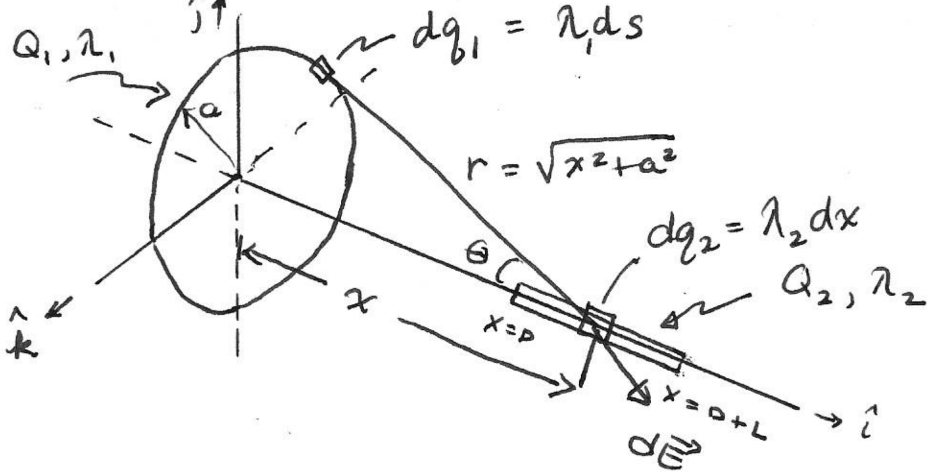
4B Exam 1 Winter 2020 Name SOLUTION

You have one (1) hour for 4 questions. No note cards, scratch papers, cell phones, calculators, or other electronic devices are allowed. For full credit, show all your work. Please box your final answer. If part of your work is on the back of the page, you must indicate you want the back graded also.

1. A ring of radius *a* and positive, uniformly distributed charge Q_1 is in the *y-z plane* with its center at *x = 0*. A line of length *L* and positive, uniformly distributed charge Q_2 lies on the *x-axis* with one end at *x = D*, the other end at *x = D + L*. Find the force the ring exerts on the line of charge.



STEP 1: FIND $\vec{E}$ at some position x due to the ring

$$\vec{E} = \int d\vec{E}$$

$$\vec{E} = \int \left(dE_x \hat{i} + \cancel{dE_\perp \hat{i}}\right)$$

by symm $\int \to$ ZERO

$$\vec{E} = \int dE \cos\theta \, \hat{i}$$

$$\vec{E} = \int \frac{k\, dq_1}{r^2} \frac{x}{r} \hat{i}$$

$$\vec{E} = \int_0^{2\pi a} \frac{k \lambda_1 ds \, x}{r^3} \hat{i}$$

$$\lambda_1 = \frac{Q_1}{2\pi a}$$

$$\vec{E} = \frac{k\, 2\pi a \lambda_1 \, x}{(x^2+a^2)^{3/2}} \hat{i}$$

$$\vec{E} = \frac{kQ_1 x \hat{i}}{(x^2+a^2)^{3/2}}$$

STEP 2: FIND THE FORCE OF RING ON LINE system (line)

$$\vec{F}_{net} = \int d\vec{F}$$

$$\vec{F}_{net} = \int \vec{E}\, dq_2$$

$$\vec{F}_{net} = \int \frac{x k Q_1 \lambda_2 dx}{(x^2+a^2)^{3/2}} \hat{i}$$

Let $u = x^2 + a^2$

$du = 2x\,dx$

$$\vec{F}_{net} = \frac{kQ_1\lambda_2}{2} \int \frac{du}{u^{3/2}} \hat{i}$$

$$\vec{F}_{net} = \frac{kQ_1\lambda_2}{\cancel{2}} \left.\frac{(x^2+a^2)^{-1/2}}{-1/\cancel{2}}\right|_D^{D+L} \hat{i}$$

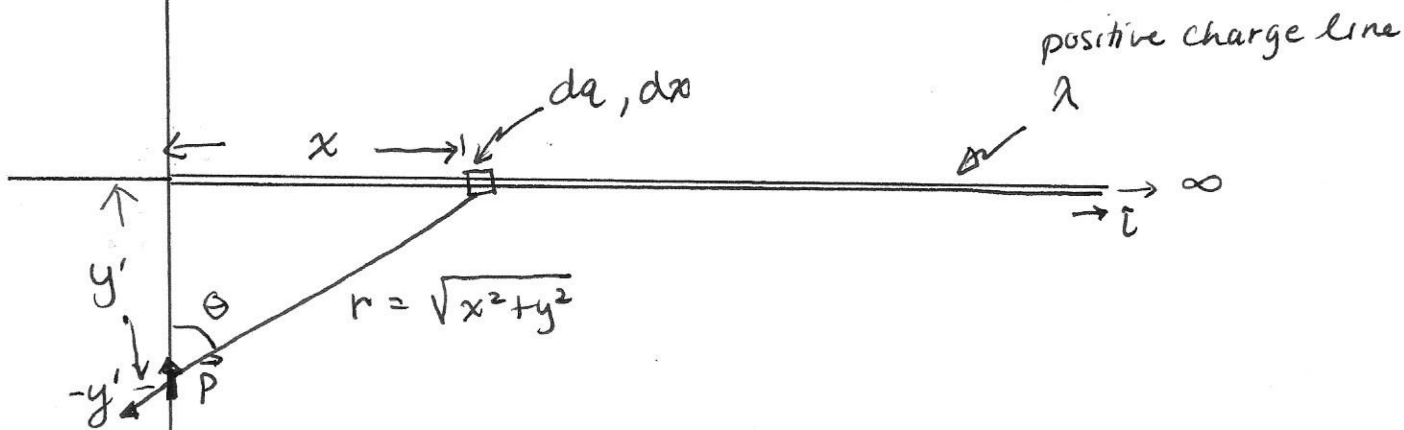
$$\lambda_2 = \frac{Q_2}{L}$$

$$\vec{F}_{net} = -kQ_1\lambda_2 \left(\frac{1}{\sqrt{(D+L)^2+a^2}} - \frac{1}{\sqrt{D^2+a^2}} \right) \hat{i}$$

$$\boxed{\vec{F}_{net} = \frac{kQ_1Q_2}{L} \left(\frac{1}{\sqrt{D^2+a^2}} - \frac{1}{\sqrt{(D+L)^2+a^2}} \right) \hat{i}}$$

2. A semi-infinite line of positive charge lies along the x-axis. The charge density, λ, is uniform. An electric dipole, $\vec{p} = 2aq\hat{j}$, is located at the position on the negative *y-axis* at *y* = -*y'* as shown. Find the torque on the dipole due to the line of charge.

$\vec{\tau} = \vec{p} \times \vec{E}$ $\vec{p}$ is given. Find $\vec{E}$ @ $\vec{p}$



$d\vec{E} = dE_x\hat{i} + dE_y\hat{j}$

$$\vec{E} = \int dE\sin\theta(-\hat{i}) + dE\cos\theta(-\hat{j})$$

since $\vec{\tau} = p\hat{j} \times \vec{E}$ the $\hat{j}$ component does not contribute and we only need E_x

$$E_x = -\int dE\sin\theta$$

$$E_x = -\int \frac{k\,dq}{r^2}\frac{x}{r}$$

$$E_x = -\int \frac{k\lambda x\,dx}{(x^2+y'^2)^{3/2}}$$

u-substitution (again!)

$$E_x = -\frac{k\lambda}{2}\int \frac{du}{u^{3/2}}$$

$$E_x = -\frac{k\lambda}{\cancel{2}}\frac{u^{-1/2}}{-1/\cancel{2}}\Big|$$

$$E_x = +k\lambda\left(\frac{1}{\sqrt{x^2+y'^2}}\right)\Bigg|_0^\infty$$

$$E_x = +k\lambda\left(\cancel{\frac{1}{\sqrt{\ }}}^{\,0} - \frac{1}{\sqrt{y'^2}}\right)$$

$$\vec{E} = -\frac{k\lambda}{y'}\hat{i} + (E_y)\hat{j}$$

$$\vec{\tau} = \vec{p} \times \vec{E}$$

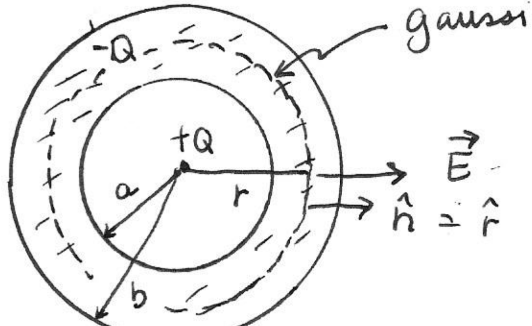
$$\vec{\tau} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & p & 0 \\ -\frac{k\lambda}{y'} & E_y & 0 \end{vmatrix}$$

$$\boxed{\vec{\tau} = \frac{pk\lambda}{y'}\hat{k}}$$

or

$$\boxed{\vec{\tau} = \frac{2aqk\lambda}{y'}\hat{k}}$$

3. A dielectric sphere has a hollow at the center. At the center of the hollow is charge $+Q$. The hollow has radius a, and the outside of the sphere has a radius b. The dielectric is uniformly charged with a total charge of $-Q$. The zero position of the electric potential to be at $r = b$. ($V(b) = 0$.) Find the electric potential, $V(r)$, in the region of the dielectric. That is, find $V(r)$ where $a < r < b$.



STEP 1: FIND $\vec{E}$

$$\oint \vec{E}\cdot\hat{n}\,dA = \frac{Q_{in}}{\epsilon_0}$$

$$\oint E_r\hat{r}\cdot\hat{r}\,dA = \frac{+Q - \frac{QV_{in}}{V}}{\epsilon_0}$$

$$E_r\oint dA = \frac{Q(V - V_{in})}{V\epsilon_0}$$

$$E_r\,4\pi r^2 = \frac{Q}{\epsilon_0}\left(\frac{V - V_{in}}{V}\right)$$

$$E_r = \frac{Q}{4\pi\epsilon_0}\left(\frac{V - V_{in}}{Vr^2}\right)$$

$$E_r = \frac{Q}{4\pi\epsilon_0 V_d}\left(\frac{V}{r^2} - \frac{V_{in}}{r^2}\right)$$

$$E_r = \frac{Q}{4\pi\epsilon_0 V_d}\left(\frac{V}{r^2} - \frac{\frac{4}{3}\pi r^3}{r^2} + \frac{\frac{4}{3}\pi a^3}{r^2}\right)$$

$$E_r = \frac{Q}{4\pi\epsilon_0 V_d}\left[\frac{\frac{4}{3}\pi b^3 - \cancel{\frac{4}{3}\pi a^3} - \frac{4}{3}\pi r^3 + \cancel{\frac{4}{3}\pi a^3}}{r^2}\right]$$

The portion of negative charge enclosed By the Gaussian

$$\rho = \frac{Q_{in}}{V_{in}} = \frac{-Q}{V}$$

$$Q_{in} = \frac{-QV_{in}}{V_d}$$

$$Q_{in} = \frac{-Q\left(\frac{4}{3}\pi r^3 - \frac{4}{3}\pi a^3\right)}{\frac{4}{3}\pi b^3 - \frac{4}{3}\pi a^3}$$

$$\Delta V = -\int \vec{E}\cdot d\vec{s}$$

$$\cancel{V(b)}^{\,0} - V(r) = -\int_r^b \vec{E}\cdot d\vec{r}$$

$$-V(r) = -\int_r^b \frac{Q}{4\pi\epsilon_0 V_d}\left(\frac{\frac{4}{3}\pi b^3}{r^2} - \frac{4}{3}\pi r\right)\hat{r}\cdot dr\,\hat{r}$$

$$V(r) = \int_r^b \frac{Q}{3\epsilon_0 V_d}\left(\frac{b^3}{r^2} - r\right)dr$$

$$V(r) = \frac{Q}{3\epsilon_0 V}\left(\left.\frac{-b^3}{r}\right|_r^b - \left.\frac{r^2}{2}\right|_r^b\right)$$

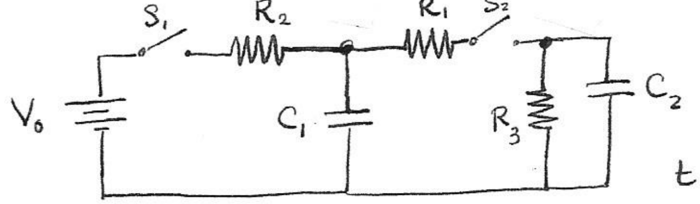
$$V(r) = \frac{-Q}{3\epsilon_0 V_d}\left(b^2 - \frac{b^3}{r} + \frac{b^2}{2} - \frac{r^2}{2}\right)$$

$$V(r) = \frac{-Q\left(\frac{3}{2}b^2 - \frac{b^3}{3r} - \frac{r^2}{2}\right)}{\cancel{3}\epsilon_0\left(\frac{4}{\cancel{3}}\pi b^3 - \frac{4}{\cancel{3}}\pi a^3\right)}$$

$$\boxed{V(r) = \frac{kQ(2b^3 - 3b^2 r + r^3)}{2(b^3 - a^3)r}}$$

4. In the circuit shown, the capacitors C_1 and C_2 are initially uncharged. The battery voltage is V_o and the resistors are: $R_1 = R$, $R_2 = 2R$ and $R_3 = 3R$. Switch S_2 is closed and then switch S_1 is closed.

a) (5 points) What is the battery current immediately after both switches are closed AND after both switches have been closed for a very long time? Your answer should be in terms of V_o and R.



$t = 0$ – NO CURRENT TO R_1

$$V_0 - I_i R_2 = 0 \rightarrow \boxed{I_i = \frac{V_0}{2R}}$$

$t \rightarrow \infty$ – NO CHARGE TO C_1 or C_2

$$V_0 - I_f R_2 - I_f R_1 - I_f R_3 = 0$$

$$I_f = \frac{V_0}{R_1 + R_2 + R_3} \qquad \boxed{I_f = \frac{V_0}{6R}}$$

b) (10 points) With both switches have been closed for a long time, what is the voltage across C_1 and the voltage across C_2? Your answer must be in terms of V_o

$$V_0 - I_f R_2 - V_{C_1} = 0$$

$$V_{C_1} = V_0 - \frac{V_0}{6R} \cdot 2R$$

$$V_{C_1} = V_0 - \frac{V_0}{3}$$

$$\boxed{V_{C_1} = \frac{2}{3} V_0}$$

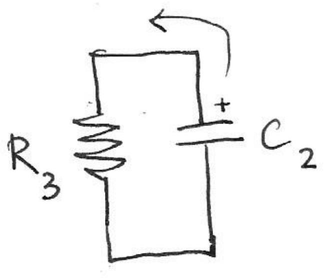
$$V_0 - I_f R_2 - I_f R_1 - V_{C_2} = 0$$

$$V_{C_2} = V_0 - \frac{V_0}{6R}(2R + R)$$

$$V_{C2} = V_0 - \frac{V_0}{2}$$

$$\boxed{V_{C_2} = \frac{1}{2} V_0}$$

c) (10 points) Last, switch S_2 is opened again after being closed for a long time. What is the current through R_3 as a function of time?



$$V_{C_2} - IR_3 = 0$$

Let the charge on the Cap at any instant be

$$Q = Q_{2i} - q_2'$$

the original charge

the charge that left

$$\frac{Q_{2i} - q_2'}{R_3 C_2} = \frac{dq_2'}{dt} \cdot \frac{R_3}{R_3}$$

$$Q_{2i} = C_2 V_{C2}$$

$$Q_{2i} = C_2 \frac{V_0}{2}$$

$$\int_0^{q_2} \frac{dq_2'}{q_2' - Q_{2i}} = \int_0^t -\frac{dt}{R_3 C_2}$$

$$\ln\left|\frac{q_2 - Q_{2i}}{-Q_{2i}}\right| = -\frac{t}{R_3 C_2}$$

$$q_2 = Q_{2i}\left(1 - e^{-t/R_3C_2}\right)$$

this is the charge which left

$$I = \frac{dq_2}{dt} = \frac{Q_{2i}}{R_3 C_2} e^{-t/R_3C_2} = \frac{\cancel{C_2} V_0}{2 \cdot 3R\cancel{C_2}} e^{-t/R_3C_2}$$

$$\boxed{I = \frac{V_0}{6R} e^{-t/3RC_2}}$$