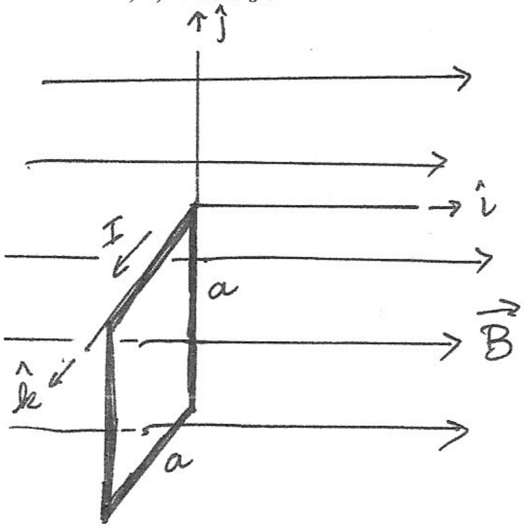
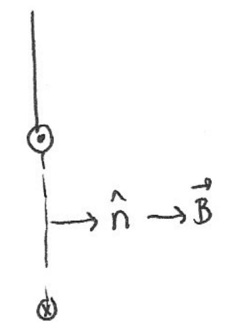
4B Winter 2016 Exam 2 Name Solutions

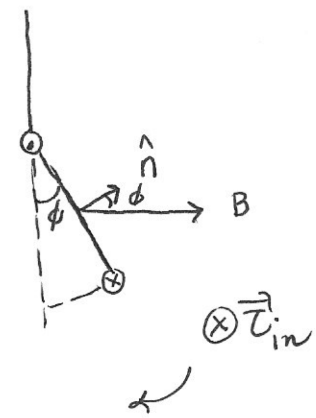
Show all your work for full credit. No scratch paper, note cards, cell phones or other electronic devices are allowed. You have one (1) hour. $\boxed{\text{PLEASE BOX YOUR FINAL ANSWER}}$

1. A single square loop of side $a$ is in free space (no gravity) hinged along the *z-axis*, with one corner at the origin as shown in the picture. The loop has uniformly distributed mass $M = 4m$ and carries current $I$ in the sense shown. The loop is immersed in a uniform magnetic field $\boldsymbol{B} = B_o\boldsymbol{i}$. The loop is displaced from the equilibrium position by a small angle $\phi$. Find the period of oscillation, $T$. (Recall $\omega = 2\pi/T$ and that the moment of inertia for a single rod about an axis located at one end is $I' = {}^1/_3 ml^2$) Your answer may be in terms of $m$, $I$, and $B_o$.




perspective view




$$\vec{\tau}_{net} = I'\vec{\alpha}$$

$$\vec{\mu} \times \vec{B} = I'\vec{\alpha}$$

$$NIA\hat{n} \times \vec{B} = I'\vec{\alpha} \qquad N = 1 \quad A = a^2$$

$$Ia^2 B \sin\phi = -I'\alpha$$

{ the torque seeks to restore equilibrium, so the acceleration is in the opposite sense }

For $\phi \ll 1$
$\sin\phi \simeq \phi$

$$Ia^2 B\phi = -I'\frac{d^2\phi}{dt^2}$$

Let $\phi = A\cos\omega t$

$$\frac{d\phi}{dt} = -A\omega\sin\omega t$$

$$\frac{d^2\phi}{dt^2} = -A\omega^2\cos\omega t$$

$$Ia^2B(\cancel{A\cos\omega t}) = +I'(+\cancel{A}\omega^2\cancel{\cos\omega t})$$

$$I' = 2\left(\tfrac{1}{3}ma^2\right) + ma^2$$

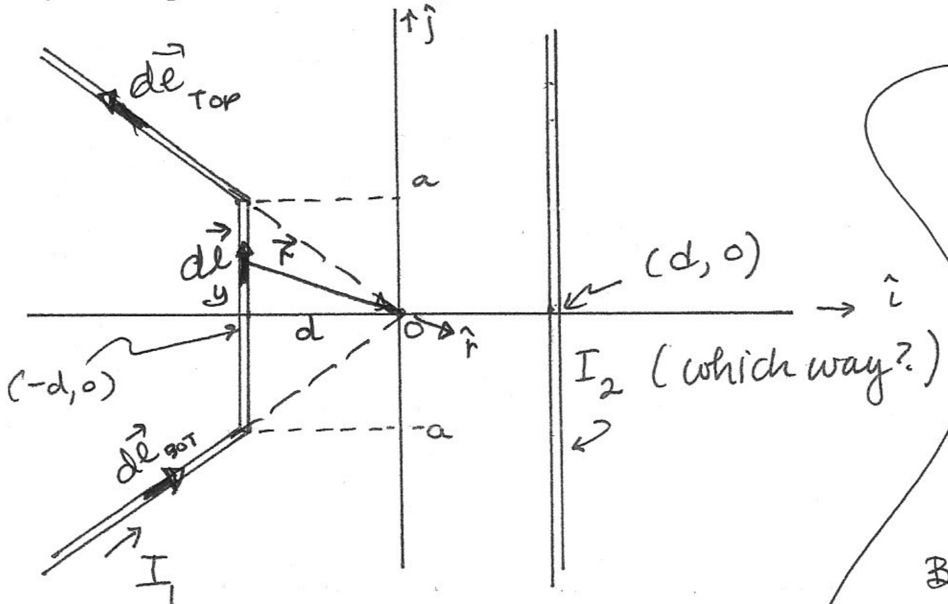
$$I' = \frac{5}{3}ma^2$$

$$\omega^2 = \frac{Ia^2B}{I'}$$

$$\omega^2 = \frac{I\cancel{a^2}B}{\frac{5}{3}m\cancel{a^2}}$$

$$\boxed{T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{5m}{3IB}}}$$

2. A very long wire with current $I_1$ is bent into three straight sections. It lies in the *x-y* plane as shown with the center section of length *2a* parallel to the *y-axis* and centered at $x = -d$. An infinitely long straight wire, also in the *x-y* plane is parallel to the y-axis and crosses the *x-axis* at $x = +d$ as shown in the diagram. If the magnetic field ***B*** at the origin is exactly zero, what is the current $I_2$? (You must use the Biot-Savart law for wire 1, but may use Ampere's law for wire 2.) Your answer should be in terms of *d, $I_1$,* and *a*




$d\vec{\ell}_{Bot} \times \hat{r} = 0$

$d\vec{\ell}_{Top} \times \hat{r} = 0$

$$d\vec{B}_1 = \frac{\mu_0 I_1}{4\pi}\frac{d\vec{\ell} \times \hat{r}}{r^2}$$

$d\vec{\ell} = dy\,\hat{j}$

$$\vec{B}_1 = \frac{\mu_0 I_1}{4\pi}\int \frac{dy\,\hat{j} \times \hat{r}}{(d^2+y^2)}$$

$$|\vec{B}_1| = \frac{\mu_0 I_1}{4\pi}\int \frac{dy\,\sin(\phi)}{d^2+y^2} \quad \{\sin\phi = \sin(\pi-\phi)\}$$

$$B_1 = \frac{\mu_0 I_1}{4\pi}\int \frac{dy\,\cos\theta}{d^2+y^2} \quad \{\sin(\pi-\phi) = \cos\theta\}$$

$y = d\tan\theta\,;\ dy = d\sec^2\theta\,d\theta$

$$B_1 = \frac{\mu_0 I_1}{4\pi}\int \frac{\cancel{d}\cos\theta\,\cancel{\sec^2\theta}\,d\theta}{d^{\cancel{2}}\cancel{(1+\tan^2\theta)}}$$

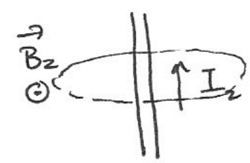
$$B_1 = \frac{\mu_0 I_1}{4\pi d}\int \cos\theta\,d\theta$$

$$B_1 = \frac{\mu_0 I_1}{4\pi d}\sin\theta\Big|_{\text{limits}}$$

but $\sin\theta = y/(d^2+y^2)^{1/2}$

$$B_1 = \frac{\mu_0 I_1}{4\pi d}\left(\frac{y}{(d^2+y^2)^{1/2}}\Bigg|_{-a}^{a}\right)$$

$$B_1 = \frac{\mu_0 I_1 2a}{4\pi d(d^2+a^2)^{1/2}}$$




$$\oint \vec{B}_2 \cdot d\vec{s} = \mu_0 I_2$$

$$B_2 \oint ds = \mu_0 I_2$$

$$B_2\, 2\pi d = \mu_0 I_2$$

$$B_2 = \frac{\mu_0 I_2}{2\pi d}$$

$\vec{B}_{net} = 0$

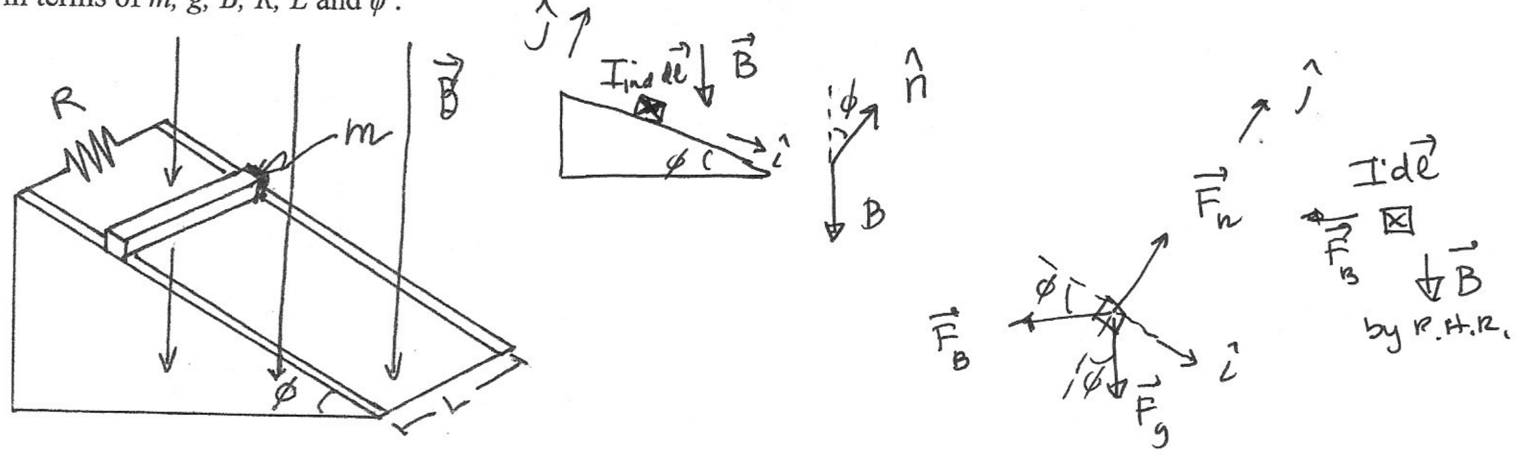
$\vec{B}_1 + \vec{B}_2 = 0$

+ out $\quad -B_1 + B_2 = 0$

$$\frac{\cancel{\mu_0} I_1 \cancel{2}a}{\cancel{4\pi d}(d^2+a^2)^{1/2}} = \frac{\cancel{\mu_0} I_2}{\cancel{2\pi d}}$$

$$\boxed{I_2 = \frac{I_1 a}{(d^2+a^2)^{1/2}}}$$

3. A metal bar of mass $m$ is released from rest on frictionless conducting rails. The conducting rails are separated by distance $L$ and are on an inclined plane which makes and $\phi$ with the horizontal. There is a uniform vertical magnetic field $\boldsymbol{B} = -B_o\boldsymbol{j}$ everywhere. Gravity is also present. The total resistance of the circuit is a constant value of $R$. What is the terminal velocity of the bar? (Terminal velocity is the maximum velocity attained. Another hint: first you must find the induced current using Faraday's Law). Your answer should be in terms of $m$, $g$, $B$, $R$, $L$ and $\phi$ .




$$\mathcal{E}_{ind} = -N\frac{d\Phi_B}{dt} \qquad \{N=1\}$$

$$\mathcal{E}_{ind} = -\frac{d}{dt}\int \vec{B}\cdot\hat{n}\,da$$

$$\mathcal{E}_{ind} = -\frac{d}{dt}\int B\,da\cos(\pi-\phi)$$

$$\mathcal{E}_{ind} = -\frac{d}{dt}B(-\cos\phi)\int_0^x L\,dx$$

$$\mathcal{E}_{ind} = BL\frac{d}{dt}x\cos\phi$$

$$\mathcal{E}_{ind} = BLv\cos\phi$$

$$I_{ind} = \frac{\mathcal{E}_{ind}}{R} = \frac{BLv\cos\phi}{R}$$

if $v \to v_{Terminal}$

$a \to 0!$

system: bar

$$\vec{F}_{net} = m\vec{a}$$

$$\vec{F}_g + \vec{F}_n + \vec{F}_B = 0$$

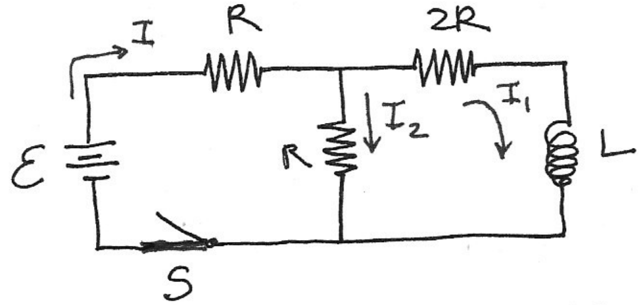
$$\hat{i}: -F_B\cos\phi + F_g\sin\phi = 0$$

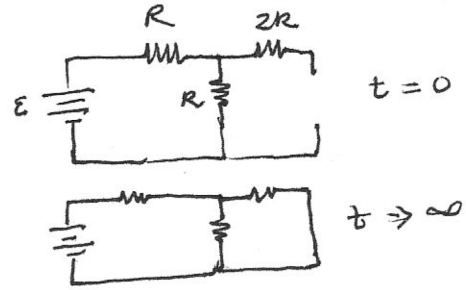
$$I_{ind}LB\cos\phi = mg\sin\phi$$

$$\frac{B^2L^2\cos^2\phi\, v_T}{R} = mg\sin\phi$$

$$\boxed{v_T = \frac{mgR\sin\phi}{B^2L^2\cos^2\phi}}$$

4. In the circuit shown, the switch $S$ is closed at $t = 0$. Find the current in the inductor as a function of time and the current in the switch as a function of time.




(1) $\mathcal{E} - IR - I_1 2R - L\dfrac{dI_1}{dt} = 0$

(2) $\mathcal{E} - IR - I_2 R = 0$

(3) $I = I_1 + I_2$

$I_2 = I - I_1$ (put in (2))

$\mathcal{E} - IR - (I - I_1)R = 0$

$\mathcal{E} - 2IR + I_1 R = 0$

* $2IR = \mathcal{E} + I_1 R$ (put in (1) × 2)

(1)×2 $2\mathcal{E} - 2IR - 2I_1 2R - 2L\dfrac{dI_1}{dt} = 0$

$2\mathcal{E} - (\mathcal{E} + I_1 R) - 4I_1 R - 2L\dfrac{dI_1}{dt} = 0$

$\mathcal{E} - 5I_1 R - 2L\dfrac{dI_1}{dt} = 0$ (Separate variables) ÷ by 5R

$-\dfrac{2L}{5R}\dfrac{dI_1}{dt} = I_1 - \dfrac{\mathcal{E}}{5R}$

$\displaystyle\int_0^{I_1} \frac{dI_1}{(I_1 - \mathcal{E}/5R)} = \int_0^t -\frac{5R}{2L}\,dt$

$\ln \dfrac{I_1 - \mathcal{E}/5R}{-\mathcal{E}/5R} = -\dfrac{5R}{2L}t$

$I_1 - \mathcal{E}/5R = -\mathcal{E}/5R\left(e^{-5R/2L\,t}\right)$

$\boxed{I_1 = \mathcal{E}/5R\left(1 - e^{-5R/2L\,t}\right)}$

INDUCTOR

use * to find I for the current thru S

* $I = \dfrac{\mathcal{E}}{2R} + \dfrac{I_1}{2}$

$I = \dfrac{\mathcal{E}}{2R} + \dfrac{\mathcal{E}}{10R} - \dfrac{\mathcal{E}}{10R}e^{-5R/2L\,t}$

$I = \dfrac{5}{10}\dfrac{\mathcal{E}}{R} + \dfrac{1}{10}\dfrac{\mathcal{E}}{R} - \dfrac{1}{10}\dfrac{\mathcal{E}}{R}e^{-5R/2L\,t}$

$I = \dfrac{3\mathcal{E}}{5R} - \dfrac{1}{10}\dfrac{\mathcal{E}}{R}e^{-5R/2L\,t}$

$\boxed{I = \dfrac{\mathcal{E}}{5R}\left(3 - \dfrac{1}{2}e^{-5R/2L\,t}\right)}$

SWITCH