Linear Algebra, Spring 2010

Augmented matrices

We often use “augmented matrices” to simplify calculations involved in solving the matrix equation $AX = B$, where A is an m by n matrix, X is an unknown vector in R^n, and B is a given vector in R^m. The augmented matrix for this calculation is written
$(C_1 | C_2 | \ldots | C_n | B)$.

Here the C_i are the columns of matrix A. Note that $AX = B$ corresponds to m linear equations in n variables, and elementary row operations on the augmented matrix correspond to either (1) switching two equations, (2) multiplying an equation by a non-zero constant, or (3) adding a non-zero constant times one equation added to another. These operations are encoded by doing the same operations to the augmented matrix.

We then put the augmented matrix [A|B] in row reduced echelon form. The equation $AX = B$ will have a solution if and only if the equations corresponding to the augmented matrix have the same solution. This is because placing the matrix A in row reduced echelon form is accomplished by multiplying A on the left by a series of invertible elementary matrices $E_1 E_2 .. E_n = R$, and doing the same to B. That is, the solutions of $AX = B$ are the same as the solutions to $RAX = RB$. (If $AX = B$ for some vector X, then also $RAX = RB$; and if $RAX = RB$, then $R^{-1} RAX = R^{-1} RB$, or $AX = B$.)

We can easily see whether there is a solution: if the number of non-zero rows in RA is the same as the number of non-zero rows in $R[A|B]$, then there is a solution (the system of equations is said to be “*consistent*.”) If there are more non-zero rows in $R[A|B]$ than in RA, then there is no solution (the system of equations is said to be “*inconsistent*.”)

Here are two related examples $AX = B$, both in row-reduced echelon form:

Example 1: $\begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \vec{X} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ or $\left(\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 3 & 2 \\ 0 & 0 & 0 & 0 & 3 \end{array}\right)$

and

Example 2: $\begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \vec{X} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ or $\left(\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

The first example clearly has no solution, since the product of X and the third row of the matrix cannot be 3. The second example has the obvious solution $X = (1,2,0,0)^t$. If S is the set of all solutions to the homogeneous system $AX = 0$, then the set of all solutions to the second example above is the sum of a particular solution plus any solution to the homogeneous case, as we saw in R^3 and R^2. That is, we simply add $X = (1,2,0,0)^t$ to V, for any vector V in S. As we saw before, we can represent the set of solutions to $AX = 0$ as follows, using the parameters r and s for the 3rd and 4th columns of the reduced matrix variables:

$$X = r\begin{pmatrix}-1\\-2\\1\\0\end{pmatrix} + s\begin{pmatrix}-2\\-3\\0\\1\end{pmatrix}$$

Therefore, the set of solutions to $AX = B$, for the second example, is:

$$X = \begin{pmatrix}1\\2\\0\\0\end{pmatrix} + r\begin{pmatrix}-1\\-2\\1\\0\end{pmatrix} + s\begin{pmatrix}-2\\-3\\0\\1\end{pmatrix}.$$

We should also pay attention to relevant dimensions. The dimension of the domain space is 4, the domension of the range space is 3. However, the dimension of the column space is only 2, and the rank of the matrix is therefore 2 also. (Note the dimension of the row space must be the same, 2.) The nullity, or dimension of the null space is the number of non-leading one columns, or 2 also. The set of vectors in the column space, which represents the set of vectors mapped onto by the matrix, is 2 also, even though it exists in a 3 dimensional space. The rank of the augmented matrix in the first example is 3, which is greater than that of the non-augmented matrix, so it has no solution. The dimension of the augmented matrix in example 2 is 2, the same as the rank of A, so there are an infinite number of solutions.