MORE MARTIN GARDNER MATHEMATICS

30 Years of Bulgarian Solitaire

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"Oh, you're a mathematician! Let me show you something interesting."

But I'm trying to work on my talk, I thought. As the train sped along, the man sitting across from me looked eager. *OK, let's get this over with.*

"Here are fifteen playing cards. Arrange them into piles; as many piles as you like, each with as many cards as you like."

I made five piles with heights 3, 1, 4, 1, 6. The π reference went unnoticed.

"Now take one card from each pile to make a new pile."

The operation left me with piles of $3 - 1 = 2$ cards, $4 - 1 = 3$ cards, $6 - 1 = 5$ cards, two empty piles from $1 - 1 = 0$, and a new pile of 5 cards. I realized that the order of the piles does not matter, so for consistency I put them in non-increasing order, 5, 5, 3, 2.

"Now do it again and again. I know what will happen!" He looked away.

Did he already think through the iterations? How long will this go? I was curious now. Here is the sequence of pile sizes:

$$\begin{aligned}(6,4,3,1,1) \quad \to \quad (5,5,3,2) \quad &\to \quad (4,4,4,2,1) \quad \to \quad (5,3,3,3,1) \\ \to \quad (5,4,2,2,2) \quad &\to \quad (5,4,3,1,1,1) \quad \to \quad (6,4,3,2)\end{aligned}$$

Oh no, that's almost where I started. When will this end? But then, suddenly, it did end:

$$(6,4,3,2) \quad \to \quad (5,4,3,2,1) \quad \to \quad (5,4,3,2,1) \text{ again.}$$

"Hmm," I said.

"You ended with one pile of 5 cards, one of 4, one of 3, one of 2, and one of 1, didn't you?" He looked at the cards. "Yes! That's always what happens. Try again!"

I started with a single pile of 15 cards. It took more moves, but I did indeed end up with the 5, 4, 3, 2, 1 pattern. Then I started with three piles of 5 cards. It took even more moves, but ended at the same fixed point.

Three examples is not a proof, but the claim was now reasonable. *Why would it always go there? How long could it take to reach this fixed point? What happens with other numbers of cards?*

"This *is* interesting," I admitted.

http://dx.doi.org/10.4169/college.math.j.43.2.135
MSC: 01A60, 11P81, 91A46

This puzzle was popularized by Martin Gardner in 1983 [**9**], with the unusual name Bulgarian solitaire. In this article, we examine the earlier history of the puzzle including its name, summarize subsequent research, and consider a new two-player variation.

Before Gardner

Around 1980, Konstantin Oskolkov of the Steklov Mathematical Institute in Moscow traveled by train to give a talk in Leningrad (now Saint Petersburg). A man on the train told him of the problem, although the details of the dialog above are fictional. Oskolkov shared this with his colleagues at the institute; reportedly when one number theorist heard about it "his face took a Satanic expression, he ran to his office, closed the door and did not come out until he solved the problem."

The puzzle reached Victor Gutenmacher, editor of the journal *Kvant*. The problem appeared there in 1980, in the context of stacks of books. Andrei Toom, a research scientist at Moscow State University, published a solution in 1981 [**19**]. The material was also included in a 1981 book on mathematics olympiads [**20**] whose authors include Gutenmacher and Toom.

Later in 1980, Anatolii Alexeevich Karatsuba traveled from Steklov to the Institute of Mathematics of the Bulgarian Academy of Sciences in Sofia. After a lecture on approximation theory, he shared Oskolkov's story and the puzzle. Again, it captured the attention of many mathematicians, and Milko Petkov published it in the "competition problems" section of a high school mathematics journal in 1980, in the context of heaps of balls. No student solved the puzzle, so the solution of Petkov's institute colleague Borislav Bojanov was published in 1981 [**4**].

Gert Almkvist of Lund University was visiting Sofia while Karatsuba was there. When Almkvist returned to Sweden, he shared the puzzle with colleagues, including Henrik Eriksson of the KTH Royal Institute of Technology, who published a solution also in 1981 [**7**] with the name "Bulgarisk patiens," in the context of piles of cards.

Jørgen Brandt, finishing his masters at Aarhus University, somehow learned of the puzzle as well (without the name). "The problem appeared so pure, that I did not think much about where it came from," he explained recently. His thorough solution, in the setting of integer partitions, was submitted to *Proceedings of the American Mathematical Society* in 1981 and published in 1982 [**5**].

Meanwhile, Eriksson traveled from Stockholm to California, where he called the puzzle Bulgarian solitaire. He recently explained, "The silly name is my invention, silly because it is neither Bulgarian nor a solitaire." Donald Knuth started a fall 1982 programming and problem-solving seminar with Bulgarian solitaire [**12**]. Ron Graham passed it on to Gardner, who used Bulgarian solitaire as the culminating example of "tasks you cannot help finishing no matter how hard you try to block finishing them" [**9**]. The silly name has stuck.

Gardner and beyond

A partition of n is a collection of positive integers $\lambda = (\lambda_1, \dots, \lambda_t)$ whose sum is n; as the order does not matter, we index the parts in non-increasing order, i.e., $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_t$. Bulgarian solitaire can be considered as an operation on partitions, $B(\lambda) = (t, \lambda_1 - 1, \dots, \lambda_t - 1)$ where there may be zeros to remove and the parts may need to be reordered. This can also be visualized through a graphic representation of partitions. The Ferrers diagram of a partition has a column of λ_1 dots followed by a column of λ_2

dots, etc. Figure 1 shows the Bulgarian solitaire operation on (6, 4, 3, 1, 1), choosing to remove the bottom dot of each column, that is, the bottom row.

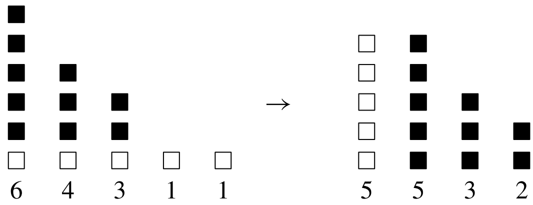

Figure 1. Example of the Bulgarian solitaire operation.

The number of partitions of n grows very quickly as n increases, but it is helpful to look at the effect of B on all partitions of n for small n. Figure 2 shows all partitions of 6 under the operation.

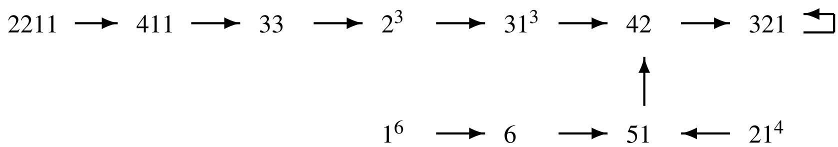

Figure 2. Bulgarian solitaire on all partitions of 6; exponents denote repetition.

Like the 15 card puzzle of the introduction, Bulgarian solitaire on partitions of 6 leads to a fixed point, the partition $\tau_3 = (3, 2, 1)$. Both 6 and 15 are triangular numbers, i.e., can be written as $T_k = 1 + \cdots + k = k(k+1)/2$. The effect of Bulgarian solitaire can be very different on other values of n.

Exercise. Work out Bulgarian solitaire on the 22 partitions of 8. (For the solution, see [**13**, Figure 4].)

The three 1981 solutions in European journals [**4, 7, 19**] use very similar ideas to show that, for $n = T_k$, there is indeed just the single fixed point $\tau_k = (k, k-1, \ldots, 2, 1)$. Each author argues that a ball / book / card cannot move to a longer diagonal in the configuration, and will eventually move into the shortest diagonal possible, filling any gaps and leaving a triangular shape. Brandt [**5**] and Toom [**19**] showed that for other values of n, the "almost triangular" partitions form cycles. For example, the partitions (4, 2, 2) and (3, 3, 1, 1) form a 2-cycle in one connected component of the partitions of 8 from the exercise; see Figure 3. From this characterization, Brandt used Pólya enumeration theory to determine the number of connected components. For instance, the partitions are in a single component exactly when n is within one of a triangular number. See also [**11**], which revisits and generalizes Toom's solution, and [**1**], which elaborates on Brandt's work.

The partitions of Figure 3 illustrate another relationship. Reading the row lengths of (4, 2, 2) rather than the column heights gives (3, 3, 1, 1). Considered another way, reflecting the Ferrers diagram for (3, 3, 1, 1) across the diagonal line $y = x$ gives (4, 2, 2). These partitions are said to be *conjugates*.

Gardner's article [**9**] focuses on the fixed point result for triangular numbers, with a discussion of using 45 cards, and the analog of Figure 2 for partitions of 10. He

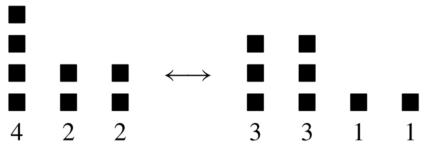

Figure 3. A two-cycle among the partitions of 8.

mentions two other conjectures, which were both proven within a few years. First, for $n = T_k$, data suggest that the greatest possible distance to the fixed point is linear in n, specifically $k(k-1)$. Igusa [**15**] shows that the partition $\gamma_k = (k-1, k-1, k-2, k-3, \ldots, 3, 2, 1, 1)$ is at distance $k(k-1)$ from τ_k and that this distance is maximal; see also [**3**] and [**8**]. Second, look carefully at the sequence of partitions from $\gamma_3 = (2, 2, 1, 1)$ to $\tau_3 = (3, 2, 1)$ in Figure 2: until the fixed point, the sequence consists of nested conjugate pairs, shown in Figure 4. Bentz [**3**] proves that this occurs along the "main trunk" of the diagram for all $n = T_k$, which distinguishes γ_k among the partitions at maximal length from the fixed point (for $k \geq 4$, there are several partitions at that distance). See also [**2**].

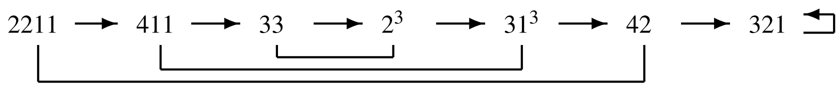

Figure 4. Conjugate pairs on the "main trunk" of Figure 2

Gardner also mentioned "Eden" partitions that have no predecessor under the operation, such as $\gamma_2 = (2, 2, 1, 1)$, $(2, 1, 1, 1, 1)$, and $(1, 1, 1, 1, 1, 1)$ in Figure 2. Recently, Hopkins and Sellers [**14**] determined a formula for the number of these partitions for general n that resembles a recurrence result of Euler.

There remain many open questions about Bulgarian solitaire, several of which are surveyed in Hopkins and Jones [**13**]. Griggs and Ho [**10**] conjectured maximal lengths to a cycle partition for $n \neq T_k$. How many partitions are at given length from a cycle partition or fixed point? What is the relation between B and conjugation outside the "main trunk"? When there are multiple components, is there an easy way to tell if two partitions are in the same component? What can be said about component sizes? There are many questions about asymptotic and random behavior; some are addressed by Popov [**18**].

Bulgarian solitaire continues to arise in the teaching literature, such as Nicholson [**16**] and Dorée [**6**]. The operation has been rediscovered at least once, see [**17**]. Several variants have been considered in the literature, but there is not space here to discuss them.

A new game

Figure 1 shows the effect of changing the bottom row of the Ferrers diagram of $(6, 4, 3, 1, 1)$ to a column. Suppose you could change *any* row to a column. Figure 5 shows the effect of changing the length 2 row to a column. Under this operation, $(6, 4, 3, 1, 1)$ could be followed by one of three partitions: $(5, 5, 3, 2)$ by selecting the bottom row, $(5, 3, 3, 2, 1, 1)$ selecting the length 2 or a length 3 row, or $(5, 4, 3, 1, 1, 1)$ selecting a length 1 row.

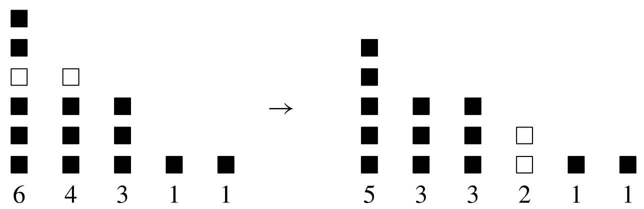

Figure 5. Changing the length 2 row of (6, 4, 3, 1, 1) to a column gives (5, 3, 3, 2, 1, 1).

For a 2-player game, we start with the single-part partition (n). In turn, each player chooses a single row to change into a column. The loser is the first player who creates a partition that has occurred before in play.

The first player can only chose a row of length 1, giving the partition $(n-1, 1)$. The second player then can choose a row of length 1 or 2, making the partition $(n-2, 1, 1)$ or $(n-2, 2)$, respectively (for sufficiently large n). All possible moves for partitions of 6 are shown in Figure 6.

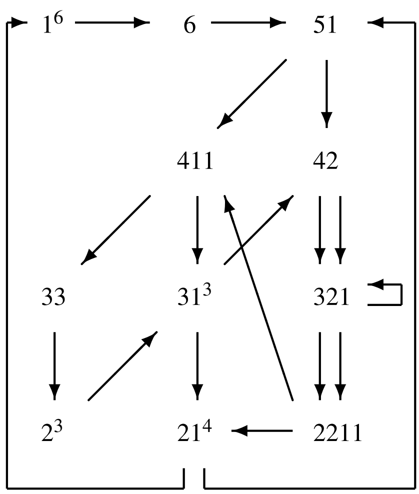

Figure 6. All possible moves in the two-player game on partitions of 6. Double arrows indicate where selecting either of two different row lengths produces the same partition.

Here are some open questions about this game.

- As a finite impartial two-player game of perfect information with no draws, either the first or second player has a winning strategy. Which player? What is the strategy? Do the answers depend on n?
- Notice that there is a Hamiltonian cycle in the directed graph of Figure 6. That is, there is a sequence of play where every partition arises before there is repetition. I conjecture that this occurs for all n. If so, is there a nice way to describe the corresponding sequence of moves?

Conclusion

A mathematical recreation that reportedly arose from a conversation in a train has gone on to inspire several lines of research and student activities. This is due largely to

Martin Gardner's attention to it in his mathematics column from *Scientific American*. Bulgarian solitaire is just one example of the impact his work has had on all of us who enjoy mathematics.

Acknowledgment. The history of Bulgarian solitaire comes from personal communication with Borislav Bojanov, Jørgen Brandt, Vesselin Drensky, Victor Gutenmacher, Pencho Petrushev, Andrei Toom, and especially Henrik Eriksson. Recollections of events thirty years ago can be imperfect; hopefully the account provided here approaches what occurred. Aleksandar Nikolov kindly provided translations from Bulgarian. Suggestions from two anonymous referees improved the article. I would like to thank Eriksson, Suzanne Dorée, and Mizan Khan for their interest and help.

Summary. Bulgarian solitaire is a puzzle popularized by Martin Gardner that can be considered an operation on integer partitions. We explore its history in Russia, Bulgaria, Sweden, and the United States before revisiting Gardner's treatment and considering subsequent research. The article concludes by introducing a related combinatorial game.

References

1. E. Akin and M. Davis, Bulgarian solitaire, *Amer. Math. Monthly* **92** (1985) 237–250; available at http://dx.doi.org/10.2307/2323643.
2. T. Bending, Bulgarian Solitaire, *Eureka* **50** (1990) 12–20.
3. H.-J. Bentz, Proof of the Bulgarian solitaire conjectures, *Ars Combin.* **23** (1987) 151–170.
4. B. Bojanov, Problem solution 4, *Obuchenieto po matematika* (Mathematics Education) **24**(5) (1981) 59–60.
5. J. Brandt, Cycles of partitions, *Proc. Amer. Math. Soc.* **85** (1982) 483–486; available at http://dx.doi.org/10.1090/S0002-9939-1982-0656129-5.
6. S. Dorée, Bulgarian Solitaire, *Resources for Teaching Discrete Mathematics*, B. Hopkins, ed., Mathematical Association of America, Washington DC, 2009, 83–92.
7. H. Eriksson, Bulgarisk patiens, *Elementa* **64**(4) (1981) 186–188.
8. G. Etienne, Tableaux de Young et solitaire bulgare. *J. Combin. Theory Ser. A* **58** (1991) 181–197; available at http://dx.doi.org/10.1016/0097-3165(91)90059-P.
9. M. Gardner, Mathematical Games: Tasks you cannot help finishing no matter how hard you try to block finishing them, *Sci. Amer.* **249** (1983) 12–21; available at http://dx.doi.org/10.1038/scientificamerican0883-12. Also available as Bulgarian solitaire & other seemingly endless tasks, *The Last Recreations*, Springer-Verlag NY, 2007, 27–43.
10. J. Griggs and C.-C. Ho, The cycling of partitions and compositions under repeated shifts, *Adv. in Appl. Math.* **21** (1998) 205–227; available at http://dx.doi.org/10.1006/aama.1998.0597.
11. T. Hart, G. Khan, and M. Khan, Revisiting Toom's proof of Bulgarian solitaire, *Ann. Sci. Math. Qu.*, to appear.
12. J. Hobby and D. Knuth, *A Programming and Problem-Solving Seminar*, Stanford University Department of Computer Science report, 1983.
13. B. Hopkins and M. Jones, Shift-Induced Dynamical Systems on Partitions and Compositions, *Electr. J. Comb.* **13** (2006) R80 19 pp.
14. B. Hopkins and J. Sellers, Exact Enumeration of Garden of Eden Partitions, *Integers* **7**(2) (2007) A19.
15. K. Igusa, Solution of the Bulgarian solitaire conjecture, *Math. Mag.* **58** (1985) 259–271.
16. A. Nicholson, Bulgarian Solitaire, *The Mathematics Teacher* **83** (1996) 84–86.
17. H. D. Phan and É. Thierry, Dynamics of the Picking transformation on integer partitions, *Discrete Math. Theor. Comput. Sci.* **AB(DMCS)** (2003) 43–56.
18. S. Popov, Random Bulgarian Solitaire, *Random Struct. Alg.* **27** (2005) 310–330; available at http://dx.doi.org/10.1002/rsa.20076.
19. A. Toom, Problem solution M655, *Kvant* (Quantum) 1981(7) 28–30.
20. N. Vasilyev, V. Gutenmacher, J. Rabbot, and A. Toom, *Zaochnye matematicheskie olimpiady* (Mathematical olympiads by correspondence), Nauka, Moscow, 1981.