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Dials and Levers and Glyphs, Oh My! Linear Algebra Solutions to Computer Game Puzzles

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Let the games begin!

I am a gaming geek. Not necessarily the kind of gaming geek you might expect a mathematician to be: I get beaten at chess by 12-year-olds, and I refuse to even go near the game of Go. The few times I've tried role-playing games, my mismanagement of my posse invariably caused my men to starve to death before they even encountered a monster. No, my games of choice are computer adventure games. *Myst*, *Riven*, *The Longest Journey*: you know the type. Lush visuals. Extravagant music. Marvelous machines, strange creatures, often haunting storylines, and the best part: puzzles. Ah, the glory of cracking codes, of damming rivers, of finally getting that hovercraft to work!

So you can imagine my excitement when during a cold and gloomy winter vacation, as I sat curled up on the couch with one of my beloved games, I got the following email from a student I'd taught in the fall. Chris wrote: "I was playing [*Myst*] and it had a riddle. Suppose you have 3 rows of numbers... Also, suppose there are 2 levers... Anyway, the game asked me to give a certain combination of numbers... Rather than just play the game, I wanted to play math."

My heart started beating faster.

He continued: "This reminded me of... our dihedral groups, but not exactly. Just messing around, I did find that every combination of levers had an inverse." Keep in mind that the puzzle Chris was describing was not obviously mathematical (at least, to a non-mathematician). It wasn't solvable by elementary arithmetic or geometry: its mathematical solution required the use of linear and abstract algebra. Upon reading this email, I glowed with pride: here was my student, from my very first abstract algebra class, recognizing that the puzzle's levers could be thought of as elements of a group. I immediately flashed back to a similar puzzle my friend, Vanessa, and I had encountered in a game. In that case, Vanessa was the one to interpret the problem mathematically. All of a sudden she and I were madly solving a system of equations and—ta da!—our puzzle was solved.

Mulling over these two puzzles, I began to wonder: were there other puzzles like that out there? The answer was a resounding **yes**. In this paper, I'll explore merely a few of the myriad linear and abstract algebra problems that masquerade as lever, dial, and slide puzzles in computer games. I urge you to play along at home as you read. My descriptions of situations will likely be more understandable that way; plus, hearing chimes that indicate you've solved a puzzle can be much more exciting than just reading about such phenomena.

Clock arithmetic

Let's start by discussing the puzzle in *Myst* about which Chris was writing. Near the beginning of the game, you encounter a contraption in a clock tower, which focuses mightily on the number 3. The contraption consists, in part, of three levers; from left to right, we'll call these levers A, B, and C. The first time we encounter the contraption, we also see three number 3s facing us, vertically aligned. Basic experimentation with the levers shows us that, in fact, each number 3 marks a face of a "dial" which has exactly two other faces, marked with the numbers 1 and 2, respectively. So we have three dials, each with three faces, marked with the numbers 1, 2, and 3. Here is where a math-friendly person might get excited: regarding these numbers as integers modulo 3, the set of numbers on each dial can be identified with the set $\mathbb{Z}_3 = \{0, 1, 2\}$ (where a dial's number 3 is identified with 0 in $\mathbb{Z}_3$). Since there are three dials, what we have represented here is the set $\mathbb{Z}_3^3$. Let us identify the numbers facing us at any moment on the top, middle, and bottom dials with, respectively, the first, second, and third coordinates of an element in $\mathbb{Z}_3^3$: so, for instance, if the numbers facing us at a certain time read 2, 1, 3 from top to bottom, we'll identify this situation with the vector $(2, 1, 0)$.

Now, if we are clever, we can deduce from facts learned elsewhere in the game that we must somehow rotate these dials so that the numbers facing us, from top to bottom, are 2, 2, and 1 (the proof of this is non-mathematical, and is left to the reader). The mathematical question is: how can this (efficiently) be done?

Well, first, more careful experimentation is needed. Let's say we start with the dials in the positions associated with vector (x_1, x_2, x_3) in $\mathbb{Z}_3^3$. It is not hard to discover that pulling lever A leaves the top dial alone, while rotating the middle and bottom dials so that their resulting visible faces show numbers that are 1 more (modulo 3) than the numbers they previously showed. That is, the resulting situation will be associated with vector $(x_1, x_2 + 1, x_3 + 1)$, where addition is done modulo 3. Lever B, on the other hand, leaves the bottom dial alone, while rotating the top two dials so that (x_1, x_2, x_3) becomes $(x_1 + 1, x_2 + 1, x_3)$. Finally, Lever C merely resets all the dials to their initial position, with three 3s facing us. Lever C is merely an aid so that struggling puzzle-solvers can start from scratch, and thus we may essentially ignore it for the rest of this discussion.

We are now at a point where we can translate this problem entirely into mathematical terms. What we in fact have going on here is a group action: in particular, the action of a subgroup of $\mathbb{Z}_3^3$ on $\mathbb{Z}_3^3$ via left translation. Pulling lever A adds $(0, 1, 1)$ to any element of $\mathbb{Z}_3^3$, while pulling lever B adds $(1, 1, 0)$ to any element. We begin with element $(0, 0, 0)$ in $\mathbb{Z}_3^3$, and want to pull each of levers A and B a particular number of times so that we obtain the element $(2, 2, 1)$: that is, so that we add $(2, 2, 1)$ to $(0, 0, 0)$. This corresponds to writing $(2, 2, 1)$ as a linear combination

$$(2, 2, 1) = \lambda_1(0, 1, 1) + \lambda_2(1, 1, 0),$$

where $\lambda_1, \lambda_2 \in \mathbb{Z}^+$, and where our addition takes place in $\mathbb{Z}_3^3$; we can then pull lever A repeatedly λ_1 times and lever B repeatedly λ_2 times to solve the puzzle. (Notice that it does not matter in which order we pull the levers, as $\mathbb{Z}_3^3$ is an abelian group.) Thus, our immediate goal is to find positive integral solutions to this system of congruences modulo 3:

$$\begin{aligned} \lambda_2 &\equiv 2 \pmod 3 \\ \lambda_1 + \lambda_2 &\equiv 2 \pmod 3 \\ \lambda_1 &\equiv 1 \pmod 3. \end{aligned}$$

Which is all well and good, except that this system clearly has no solutions, since if $\lambda_1 \equiv 1 \pmod 3$ and $\lambda_2 \equiv 2 \pmod 3$, then $\lambda_1 + \lambda_2$ must of necessity be congruent to 0 (mod 3).

So we return to the puzzle and mess with the levers some more; or, if we are impatient, we consult a walk-through of the game. Either way, one will discover the following sneaky fact: holding down lever A or lever B for a beat adds (in $\mathbb{Z}_3^3$) the vector (0, 1, 0) to the currently represented vector; furthermore, copies of (0, 1, 0) are continually added as long as the lever is held down. Thus, by holding lever A or B down for an appropriate amount of time, one can add the vector $(0, \lambda_3, 0)$ (for any $\lambda_3 \in \mathbb{Z}^+$) to the currently represented vector. Thus, our revised mathematical goal is to find positive integers λ_1, λ_2 and λ_3 so that in $\mathbb{Z}_3^3$ we have

$$(2, 2, 1) = \lambda_1(0, 1, 1) + \lambda_2(1, 1, 0) + \lambda_3(0, 1, 0).$$

So the system of congruences for which we really need to find positive integral solutions is

$$\begin{aligned} \lambda_2 &\equiv 2 \pmod{3} \\ \lambda_1 + \lambda_2 + \lambda_3 &\equiv 2 \pmod{3} \\ \lambda_1 &\equiv 1 \pmod{3}. \end{aligned}$$

It is easy to see that this system has solution

$$\lambda_1 = 1, \quad \lambda_2 = 2, \quad \text{and} \quad \lambda_3 = 2$$

(where these solution values are unique modulo 3). Turning to the game, we first make sure the dials are in their initial position (pulling lever C to put them there, if need be). Then we can solve the puzzle by pulling lever A once and lever B twice, and holding B down after its second pull exactly long enough to add the vector (0, 1, 0) twice. And voilà, we hear a grinding noise and the gear at the bottom of the contraption opens. Moreover, we have now gained access to a book that will allow us to travel a different world. Nice, huh?

A stasis gun and skulls

We next turn to one of my favorite games, *Timelapse*. A number of puzzles in this game can be interpreted mathematically; we focus on two such puzzles. The first puzzle that we'll discuss is the second of those two that we encounter in the game: it is the stasis tube gun puzzle. (You should likely save your game before exploring the mechanisms of this puzzle, as if you don't solve it quickly enough you will be conquered by a robot and lose the game: not good.) The puzzle consists of six tricolored circles and six yellow triangles. (See Figure 1.) In the center of the puzzle, there is a small, red hexagonal button. At first, clicking on the button merely makes a small noise; the button appears to be currently inactive. Let C_1 be the circle in the upper right-hand corner of the puzzle, and let $C_2, C_3, \ldots, C_6$ be the puzzle's other circles, in clockwise order from C_1. Next, for $i = 1, 2, \ldots, 6$, let T_i be the triangle nearest to C_i. Each circle is divided into red, green and blue sectors, of equal size, in clockwise order in the circle. We'll say that a circle is in state 0 (respectively, states 1 or 2) if its red (respectively, blue or green) sector faces the center of the puzzle. We will soon see that clicking on any of the triangles rotates several of the circles clockwise by multiples of 120°; the set of all orientations of the circles can therefore be identified with a subset of $\mathbb{Z}_3^6$, where the ith entry of an element $x \in \mathbb{Z}_3^6$ is the state of C_i in orientation x of the circles. When you first encounter the puzzle, the orientation of the circles corresponds to the vector $(1, 0, 2, 1, 0, 2) \in \mathbb{Z}_3^6$.

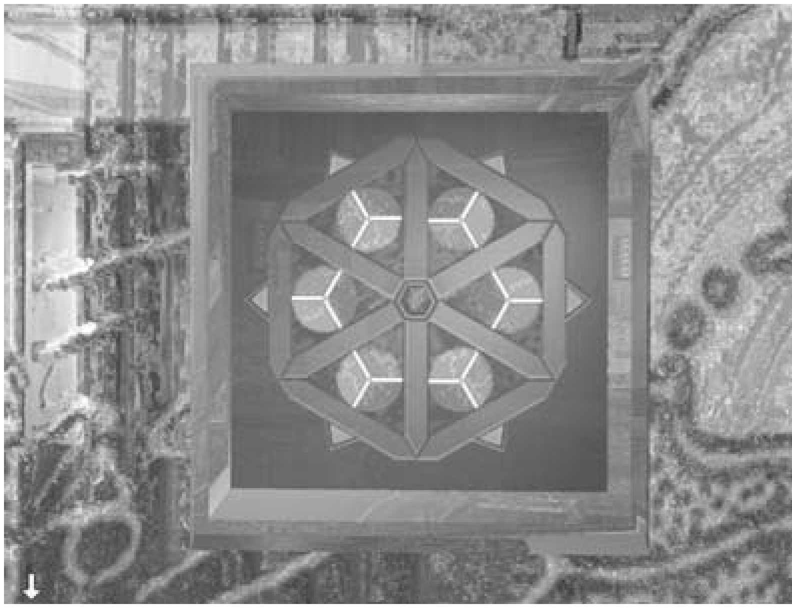

Figure 1 The stasis tube gun puzzle in *Timelapse*

At this point, one might begin to wonder if this puzzle's solution is mathematically similar to that of the clock tower puzzle; one will discover that, indeed, it is. Experimentation shows that clicking on any given yellow triangle rotates exactly three of the circles clockwise by either 120° or 240°, while leaving the other circles alone. For instance, clicking on T_1 rotates C_1 and C_6 clockwise by 120° and rotates C_2 clockwise by 240°, but does nothing to the other circles, while clicking on T_4 rotates C_3 and C_5 clockwise by 240° and rotates C_2 clockwise by 120°, while leaving the other circles alone. Thus, clicking on a triangle causes the states of exactly three circles to change. For instance, clicking on T_1 causes the states of C_1 and C_6 to increase by 1 (mod 3) and the state of C_2 to increase by 2 (mod 3). Mathematically, this corresponds to adding a certain vector to the vector associated with an orientation of the circles. Specifically, for each $i = 1, 2 \dots, 6$, clicking on triangle T_i adds v_i to the puzzle's current orientation vector, where

$$v_1 = (1, 2, 0, 0, 0, 1), \quad v_2 = (2, 2, 1, 0, 0, 0), \quad v_3 = (0, 2, 1, 1, 0, 0)$$
$$v_4 = (0, 0, 2, 1, 2, 0), \quad v_5 = (0, 0, 0, 2, 1, 1), \quad \text{and } v_6 = (1, 0, 0, 0, 2, 1).$$

Now, what's our goal for this puzzle? Well, recall the inactive red hexagonal button in the center of the puzzle; chances are we want to activate it. Moreover, in adventure-game language, its shape and color suggest that we will be able to do this by rotating the circles so that each circle's red sector faces the puzzle's center: that is, so that each circle is in state 0. Thus, mathematically, we begin with the element $(1, 0, 2, 1, 0, 2)$ in $\mathbb{Z}_3^6$, and want to obtain the element $(0, 0, 0, 0, 0, 0)$: that is, we want to add $(2, 0, 1, 2, 0, 1)$ to $(1, 0, 2, 1, 0, 2)$. Since what we can do is limited to the actions performed by the T_i, we want to find positive integers $\lambda_1, \lambda_2, \dots, \lambda_6$ so that

$$(2, 0, 1, 2, 0, 1) = \sum_{i=1}^{6} \lambda_i v_i.$$

Looks familiar, huh? Only this time, this corresponds to solving a system of six congruences in six unknowns, which is not so nice to do by hand. An alternate way of solving this system is to use slightly more sophisticated linear algebra. Specifically, we can represent this system by the matrix equation

$$A\mathbf{x} = \mathbf{b},$$

where A is the matrix whose ith column is v_i, $\mathbf{x}$ is the column vector whose ith entry is λ_i, and $\mathbf{b}$ is the transpose of $(2, 0, 1, 2, 0, 1)$. Note that we are thinking of all these matrices as being over $\mathbb{Z}_3$. Solving for $\mathbf{x}$, we obtain

$$\mathbf{x} = A^{-1}\mathbf{b};$$

so we're done if we can invert A. If we don't wish to invert a 6×6 matrix over $\mathbb{Z}_3$ by hand, we can do the inversion using, for instance, the GAP system for computational discrete algebra. We can also then multiply A^{-1} by $\mathbf{b}$ using that system, and obtain the solution

$$\mathbf{x}^T = (A^{-1}\mathbf{b})^T = (2, 1, 0, 0, 1, 1).$$

(Again, these solution values are unique modulo 3.) Thus, to solve the puzzle it should suffice to click on T_1 twice, and each of T_2, T_5 and T_6 once. Sure enough, this works! Nothing happens immediately, but now clicking on the red button in the puzzle's center causes the button to disappear as the red sectors of the circles come together to form a hexagon. Further, we now have access to a gun, and can shoot the robot (assuming we have sufficiently good non-intuitive aim)!

We now turn to another *Timelapse* puzzle. A computer adventure game without a Mayan world seems to be almost as rare as an even prime number; in *Timelapse* you encounter a Mayan calendar puzzle. (Before you experiment with this puzzle, you may want to save your game: one of our examples will assume that we start with the puzzle in its original state, and unlike many other puzzles, this puzzle does not reset to its original state when you back away from it and then return.)

The puzzle contains three rings: two on its left side, and one on its right. We'll call the inner left ring R_1, the outer left ring R_2, and the right ring R_3. Each of these rings displays symbols. At any given time, three symbols, one from each of the three rings, are aligned; basic experimentation yields that you can change the symbols that are aligned by turning the rings (we will later discuss the turning of the rings in more detail). You will need to use this puzzle to obtain access to four temples (respectively associated with skulls, jaguars, monkeys, and lizards); each temple's access requires a different combination of symbols be aligned. It is straightforward to discover that R_1, R_2, and R_3 display 8, 12, and 16 distinct symbols, respectively. The original aligned symbols are shown in FIGURE 2; we'll identify each of these symbols with the number 0 (on their respective rings). We number the remaining symbols clockwise on each ring. Clearly, then, we can identify the set of all possible alignments of symbols with the set $\mathbb{Z}_8 \times \mathbb{Z}_{12} \times \mathbb{Z}_{16}$.

Now it's time to explore more carefully the movements of the rings. We can rotate the rings in this puzzle either clockwise or counterclockwise. (This does not constitute a fundamental difference between this puzzle and the other puzzles we've discussed, but does allow us slightly more flexibility in executing our solution, as in this case our solution can involve negative integers.) Now, suppose the alignment of symbols at a certain time is represented by an element x in $\mathbb{Z}_8 \times \mathbb{Z}_{12} \times \mathbb{Z}_{16}$. If we rotate R_1 one position counterclockwise, this also turns R_2 one position counterclockwise while leaving R_3 unchanged, so we've added $v_1 = (1, 1, 0)$ to x. If we instead rotate R_2 one

Figure 2 The Mayan Calendar in *Timelapse*

position counterclockwise, this also turns R_1 one position counterclockwise and R_3 one position clockwise, so we've added $v_2 = (1, 1, 15)$ to x. Finally, if we rotate R_3 one position counterclockwise, this also rotates R_1 one position counterclockwise and doesn't move R_2, so we've added $v_3 = (1, 0, 1)$ to x.

Since we must obtain four different alignments of symbols to gain access to every temple, and since this puzzle does not reset itself to its original state when we leave it and come back, solving this puzzle efficiently requires we treat it a little differently than we did the other puzzles. To solve our previous puzzles, we merely had to write one element of our set as a linear combination of specific vectors; here, we need to write at least four elements of our set as linear combinations of the v_i (we may need to write more than four this way if we mess up at some point, since the puzzle doesn't reset itself). Thus, we will try to answer the question: given any alignment x and any element (a, b, c) of $\mathbb{Z}_8 \times \mathbb{Z}_{12} \times \mathbb{Z}_{16}$, how can we add (a, b, c) to x by turning the rings? Well, let $(a, b, c) \in \mathbb{Z}_8 \times \mathbb{Z}_{12} \times \mathbb{Z}_{16}$. It suffices to find $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{Z}$ so that in $\mathbb{Z}_8 \times \mathbb{Z}_{12} \times \mathbb{Z}_{16}$ we have

$$(a, b, c) = \sum_{i=1}^{3} \lambda_i v_i;$$

that is, we want to find λ_i that solve the system of congruences

$$\begin{aligned} \lambda_1 + \lambda_2 + \lambda_3 &\equiv a \pmod{8} \\ \lambda_1 + \lambda_2 &\equiv b \pmod{12} \\ 15\lambda_2 + \lambda_3 &\equiv c \pmod{16}. \end{aligned}$$

Note that, for this puzzle, the λ_i can be negative: if a λ_i is negative, this simply means we should rotate the corresponding ring by $|\lambda_i|$ positions in the *clockwise*, rather than the counterclockwise, direction. It's straightforward to show that if a_1, b_1 and c_1 are any integers with $a_1 \equiv a \pmod{8}$, $b_1 \equiv b \pmod{12}$, and $c_1 \equiv c \pmod{16}$, then a solution to our system is given by

$$\lambda_1 = 2b_1 + c_1 - a_1, \quad \lambda_2 = a_1 - b_1 - c_1, \quad \text{and} \quad \lambda_3 = a_1 - b_1.$$

Let's now put our work to the test. Starting in our original state $(0, 0, 0)$, we'll attempt to open one of the temples. (Before we start, it's important to know that in order to be able to open a particular temple using the puzzle, the crystal ball across from the puzzle must be in a specific position; we will not discuss this further here, but it's something of which to be aware when playing the game.) The order in which we open the temples doesn't matter (though it will affect our computations). Let's open the Skull Temple first. We learn elsewhere in the game that this requires the attainment of state $(4, 2, 4)$. This means we want to add $(4, 2, 4)$ to our current state, so, in this case, $a = 4$, $b = 2$, and $c = 4$. Thus, one solution to our puzzle is given by $\lambda_1 = 4$, $\lambda_2 = -2$, and $\lambda_3 = 2$. So we turn R_1 4 positions counterclockwise, R_2 2 positions clockwise, and R_3 2 positions counterclockwise. Ah, success—we hear a lovely chime, and future exploration will show us that the Skull Temple is now unlocked.

But we're not yet done with the puzzle: we need to obtain three more elements of $\mathbb{Z}_8 \times \mathbb{Z}_{12} \times \mathbb{Z}_{16}$. Let's open the Jaguar Temple next. In order to open this temple, it turns out that we must obtain state $(2, 0, 2)$. Were we starting in our original state, $(0, 0, 0)$, a solution to our puzzle could be obtained using $a = 2$, $b = 0$, and $c = 2$: thus, a solution would be $\lambda_1 = 0$, $\lambda_2 = 0$, and $\lambda_3 = 2$. But since we are currently in state $(4, 2, 4)$ (the state which opened the Skull Temple), this solution won't work: we instead wish to add $(6, 10, 14)$ to our current state. In this case, then, $a = 6$, $b = 10$, and $c = 14$. Using these values would yield relatively large absolute values for the λ_i, requiring lots of turning of rings (for instance, we'd have to turn R_1 28 positions counterclockwise); instead, using the valid values $a_1 = b_1 = c_1 = -2$, we obtain solution $\lambda_1 = -4$, $\lambda_2 = 2$, and $\lambda_3 = 0$. Hence, we can open the Jaguar Temple simply by turning R_1 four positions clockwise and R_2 two positions counterclockwise. We leave the opening of the remaining two temples to the reader.

We end this section by pointing out another way in which this puzzle differs from our other puzzles: our solutions for this puzzle are not unique modulo the moduli in our system. For instance, we mentioned above that starting in initial state $(0, 0, 0)$, to open the Jaguar Temple we must obtain state $(2, 0, 2)$, and can do that using $\lambda_1 = 0$, $\lambda_2 = 0$, and $\lambda_3 = 2$. But we can also do it using $\lambda_1 = 8$, $\lambda_2 = 16$, and $\lambda_3 = 2$; note that our new λ_1 and λ_2 values are not congruent to their previous values modulo 12, one of the moduli in our system. So for this puzzle, as a result of our equations involving different moduli, we do not have the same kind of uniqueness of solutions as we had for our previous puzzles.

Reflection

Of course, these are all variations on the same puzzle. In fact, when one is attuned to this type of thing, one notices this Puzzle everywhere: I was watching a friend play *SpongeBob SquarePants: Battle for Bikini Bottom*, when he encountered the Puzzle (I muttered, "Ah, this time it's $\mathbb{Z}_2^8$. . . "). But don't let this distract you from the other math puzzles that are out there. Computer games contain both obvious and subtle mathematical puzzles: ones where you need only translate a foreign number system's numerals and apply basic arithmetic, and ones, such as the Puzzle, that may not originally appear to be mathematical at all.

And one of the most beautiful things about computer games is that they allow you to travel to worlds to which we can't physically go in real life: including mathematical worlds. I'll end this discussion with a teaser. At one point in the old text adventure game *Trinity*, you have a gnomon you must screw into a hole—but the threads in the hole are going the wrong way. Wandering about, you encounter an "abstract sculpture," inscribed with the words *Felix Klein 1849–1925*. Nearby are strange leafy tunnels. When playing this game, my friend Jen and I suddenly grabbed each other in excitement, as we realized what was going to happen, and how that would solve the puzzle. And with that, I leave you to enjoy the mathematical labyrinths of games.

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A Tree That's Not a Tree

A graph is a tree if it is connected and has no cycles; so this is not a tree.

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