Bulgarian Solitaire

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Summary

A player begins with coins arranged in piles. At each turn she rearranges the coins according to the following rule: remove the top coin from each pile, possibly eliminating piles, and form a new collected pile of coins. She repeats the process until she revisits a previously encountered arrangement, having reached a terminal cycle. Where are fixed points, if any? Are there any 2-cycles? Which states are cyclic? How can we visualize the process?

This process was dubbed "Bulgarian Solitaire" in a 1983 *Scientific American* column by Martin Gardner [11], though I first saw it in Doug West and John D'Angelo's text *Mathematical Thinking* [22]. In my sophomore level discrete mathematics class I use an activity on Bulgarian Solitaire called *The Coins Go 'Round 'n 'Round* to introduce students to partitions of integers, directed graphs, state graphs, and dynamical systems and to draw connections to the ubiquitous triangular numbers. In class I avoid using the name "Bulgarian Solitaire" since that would make it too easy for students to find answers on the Internet.

The activity takes two twenty-minute time blocks, or it can be done all as one entire class period. Each pair of students needs a dozen coins. It works best when I introduce the process prior to Part I and the graphs prior to Part II. I provide this introductory material below.

I expect students have seen the triangular numbers

$$1, 3, 6, 10, 15, 21, \ldots, t_k = \sum_{i=1}^{k} i = \frac{k(k+1)}{2}$$

earlier in the course, preferably including their representation as triangles of dots. It is helpful, but not necessary, for students to have seen the partitions of an integer. I do not expect that students have seen any graph theory prior to the activity. My students haven't and they are able to pick up on the terminology, notation, and ideas when needed.

My experience is that there are parts of the activity that challenge my strongest students, most of which I save as extra credit, but most parts are easily completed by all of my students. They really get into "playing the game," and are curious about what happens.

Notes for the instructor

I begin the activity by demonstrating the process by moving coins on an overhead projector while on the blackboard I record each state visited and connect it to the subsequent state with an arrow. Rather than piling the coins vertically, which would be difficult to see on the overhead screen, I extend them up the screen as in Figure 1. I literally remove the top coin from each pile and form the collected pile each time. For example, starting with (5,1,1,1) as on the worksheet I show students how to get (4,4), (3,3,2), (3,2,2,1), (4,2,1,1), (4,3,1), and then (3,3,2) again. Along the way I point out how the singleton piles were eliminated and that for convenience we keep the piles in increasing order by size. By the end of this introduction on the blackboard I have drawn the progression

$$(5,1,1,1) \to (4,4) \to (3,3,2) \to (3,2,2,1) \to (4,2,1,1) \to (4,3,1) \to \text{back to } (3,3,2).$$

I explain that we have repeated a state and so we're going to stop. Without my recording these states on the board, the class probably would not recognize the repeat. Students usually need to stop and think about why the process would just repeat from there and so I have them turn to their neighbor to discuss why. I also ask them to decide if we'll ever return to (5,1,1,1) again, which of course we won't. I casually introduce the terminology "4-cycle" and refer to the repeated elements as "cyclic". These terms are defined later on the worksheet, but I find students get the idea from this one example.

At this point I have students work on Exercises 1-4.

Thus far in the activity students have only drawn the progression from one state. I begin the second part by drawing the first few state graphs on the board, G_4 on four coins and G_5 on five coins as appear in Figure 2. I point out that each vertex has out-degree one since there is a unique state to which it goes, but a vertex can have in-degree more than one (or zero).

If students have not seen the partitions of an integer, I show them how to systematically list the partitions of five to make sure we have all of the vertices of G_5. Before I set them off to work on the remaining exercises, I also list out the partitions of six. A word of warning: once students have the list, some are tempted to draw the graph by first calculating where each partition on the list is sent and then trying to connect the whole kit and caboodle. In addition to taking far more time than necessary, it also results in a tangled picture of the graph. Instead, I encourage them to pick a particular partition, such as (1, 1, 1, 1, 1, 1), draw the progression from there, then pick a missing partition, and draw its progression, etc., each time connecting the new progression to the graph as soon as possible. I encourage them to work on scratch paper and later redraw the graph in an organized fashion without any crossings.

Now I have students work on Exercises 5-8.

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Worksheet on The Coins Go 'Round 'n 'Round

Part I: A never-ending process

Take some coins and arrange them into piles. You may make any number of piles and have any number of coins in each pile. The piles may be different heights and a pile may have as few as one coin in it. Now, from that starting arrangement pick up the top coin from each pile and make one new pile out of those coins. Notice that any pile that had just a single coin was eliminated. You now have a new arrangement of the coins. Repeat this process to the new arrangement of coins, i.e. pick up the top coin from each new pile and make one new pile out of those collected coins. For convenience we list the piles in increasing order by size. Keep repeating the process. Although this process never ends, we'll call it quits if we get back to a place we've already been.

For example, suppose I take eight coins and place them into one pile of five coins and three piles of one coin each, which we denote (5,1,1,1). At the first step I remove one coin from each pile, eliminating all but the tallest pile. This leaves a pile of four coins and the four collected coins, forming two piles of four coins each that we denote (4,4). At the next step I remove one coin from each pile, leaving three coins in each and two collected coins that form a new pile. The resulting arrangement is (3,3,2). These steps are illustrated in Figure 1.

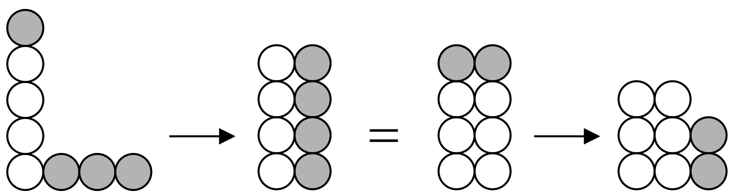

Figure 1. $(5,1,1,1) \to (4,4) \to (3,3,2)$.

When I repeat the process I get (3,2,2,1), (4,2,1,1), (4,3,1), and then (3,3,2) again, as you can check. Even though I did not get back to my original arrangement, since I have returned to an arrangement from earlier, I'll call it quits. If I were to continue, it would just repeat through those same four states over and over again. We represent this progression as

$$(5,1,1,1) \to (4,4) \to (3,3,2) \to (3,2,2,1) \to (4,2,1,1) \to (4,3,1) \to \text{back to } (3,3,2).$$

Exercises

1. Now it's your turn. Perform the process beginning with the given arrangement and record the progression until you repeat an arrangement.
   (a) (2,1,1,1)
   (b) (4)
   (c) (3,3)
   (d) (4,2,1)
   (e) (2,2,2,2)
2. Although we do not always return to our original arrangement, we will necessarily return to *some* state we have visited before. Explain why. Also explain why once we return to a previously-visited state the process repeats.

An arrangement of coins that is eventually repeated is called *cyclic* and the number of steps from a cyclic state to its next repeat is called the *length* of the cycle. For example, with eight coins the elements (3,3,2), (3,2,2,1), and (4,2,1,1), and (4,3,1) are cyclic states that form a cycle of length four, called a 4-cycle for short. An arrangement in a 1-cycle is called a *fixed point*, i.e. a fixed point is unchanged by the process.

3. Fixed points
   (a) Give an example of a fixed point from your earlier work. Give another example of a fixed point.

(b) Make a conjecture about which states are fixed points under this process.

(c) Make a conjecture about the number of coins in a fixed point state.

(d) Prove your conjectures. *Hint: Be sure to both show that these states are fixed points and that any fixed point is one of these states.*

4. 2-cycles

(a) Give an example of a 2-cycle from your earlier work. Give another example of a 2-cycle.

(b) Make a conjecture about which states are in 2-cycles under this process.

(c) ⋆ Make a conjecture about the number of coins in a 2-cycle state.

Note: The star (⋆) indicates a more difficult problem.

Part II: The state graphs

Thus far we have drawn the progression from one beginning state. For a fixed number of coins, n, we can visually represent the entire process using a *state graph* denoted G_n. The vertices (dots) are all possible arrangements of the coins, which we continue to list as a tuple ordered by size, regardless of the physical order of the piles. These labels are known as the *partitions* of the integer n. We draw an arrow from each vertex to the vertex we get by performing the process once. There is a unique arrow out of each vertex, but there may be one, several, or no arrows into a vertex. For example, the state graphs on four and five coins are shown in Figure 2.

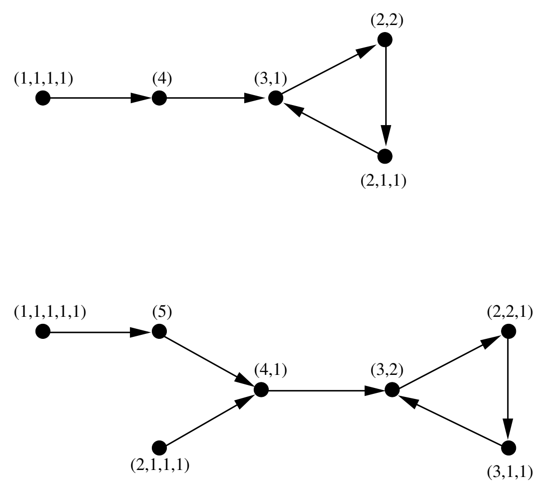

Figure 2. The state graphs G_4 and G_5.

Exercises

5. State Graphs

(a) Draw G_6, the state graph on six coins. *Hint: Be sure to include all 11 possible arrangement of six coins.*

(b) Draw G_7, the state graph on seven coins. *Hint: There are 15 vertices.*

6. Garden of Eden States

A state having no predecessors, i.e. no arrows into the vertex, is called a *Garden of Eden* state.

(a) Show that the state (4,4,3,3) on fourteen coins is *not* Garden of Eden.

(b) The state (4,3,3,2,1,1) on fourteen coins *is* Garden of Eden. Explain why.

(c) ⋆ State and prove a conjecture about which states on n coins are Garden of Eden.

7. Ten Coins
   (a) Start with an arrangement of ten coins, perform the process, and draw the progression.
   (b) Start with an arrangement of ten coins that you haven't encountered yet, perform the process, and connect this progression to your graph.
   (c) Start with another arrangement of ten coins that you haven't encountered yet, perform the process, and connect this progression to your graph.
   (d) Make a conjecture about how the process with ten coins will always end.
   (e) Generalize your conjecture.
8. Eleven Coins
   (a) List the cyclic states on eleven coins. *Hint: There is a unique end cycle.*
   (b) $\star$ Suppose that n is an integer such that there is a fixed point with n coins. Consider the process on $n + 1$ coins. Make a conjecture about which states are cyclic.

Solutions and Notes

1. The Process

   (a) $(2,1,1,1) \to (4,1) \to (3,2) \to (2,2,1) \to (3,1,1) \to$ back to $(3,2)$
   (b) $(4) \to (3,1) \to (2,2) \to (2,1,1) \to$ back to $(3,1)$
   (c) $(3,3) \to (2,2,2) \to (3,1,1,1) \to (4,2) \to (3,2,1) \to$ back to itself
   (d) $(4,2,1) \to (3,3,1) \to (3,2,2) \to (3,2,1,1) \to$ back to $(4,2,1)$
   (e) $(2,2,2,2) \to (4,1,1,1,1) \to (5,3) \to (4,2,2) \to (3,3,1,1) \to$ back to $(4,2,2)$

2. We will necessarily return to some state we have visited before because there are only finitely many states. The process will repeat from there because the process and state are the same as when we were there before.

3. Fixed Points

   (a) A fixed point from earlier work is (3,2,1). Other examples include (1) on one coin, (2,1) on three coins, (4,3,2,1) on ten coins, etc.
   (b) Conjecture: A state is fixed if and only if it is of the form $(k, k-1, \ldots, 3, 2, 1)$. *I find it necessary to remind students that a conjecture should be a complete sentence.*
   (c) Conjecture: There is a fixed point state for n coins if and only if n is a triangular number.
   (d) Proof: First we check that these states are fixed by performing the process and observing that each pile is reduced by one coin and the smallest pile is eliminated, but there are k collected coins, which reinstates the largest pile. Conversely, suppose we had a fixed state with largest pile k. Since the process reduces the size of each pile, the only way we can get a pile of k coins is as the collected pile and so there must be exactly k piles. The pile of k coins is reduced to $k-1$ coins by the process and so there must have been a pile of $k-1$ coins. Similarly that pile of $k-1$ coins is reduced to $k-2$ coins by the process and so there must have been a pile of $k-2$ coins. The argument continues until we note that the pile of two coins is reduced to one coin by the process and so there must have been a pile of one coin. This gives us k piles in total, and so there are no additional piles. We therefore have the triangle state $(k, k-1, k-2, \ldots, 3, 2, 1)$, with a triangular number of coins. *For formal proof see [22].*

4. 2-Cycles

   This exercise can easily be left as homework or omitted altogether if short on time.

   (a) A 2-cycle from earlier work is (4,2,2) $\leftrightarrow$ (3,3,1,1) on eight coins. Other examples include (2) $\leftrightarrow$ (1,1) on two coins and (6,4,4,2,2) $\leftrightarrow$ (5,5,3,3,1,1) on eighteen coins.
   (b) Conjecture: A state is in a 2-cycle if and only if it is of the form

   $$(2t, 2t-2, 2t-2, \ldots, 4, 4, 2, 2) \text{ or } (2t-1, 2t-1, 2t-3, 2t-3, \ldots, 3, 3, 1, 1)$$

   for some integer t. *My students usually guess the pattern but have difficulty stating it. This result follows from the characterization of cycles in the work of Akin and Davis [1]. While it is easy to check that states of this form are in a 2-cycle, it takes a bit more work to prove that these are the* only *2-cycles and so I do not ask my students for proof.*
   (c) ⋆ *I make this part* ***extra credit*** *since it relies on a clearly-stated conjecture in the previous part.* Adding the number of coins from the previous conjecture in the odd case we get

   $$\sum_{i=1}^{t} 2(2t-1) = 2\left[t(t+1) - t\right] = 2t^2.$$

   Conjecture: A two-cycle occurs exactly when the number of coins is twice a square. *As far as I can determine the first direct proof of this fact was given by one of my students, Augsburg College undergraduate Maria Sieve, [21].*

5. State Graphs

   (a) The state graph G_6 is shown in Figure 3.

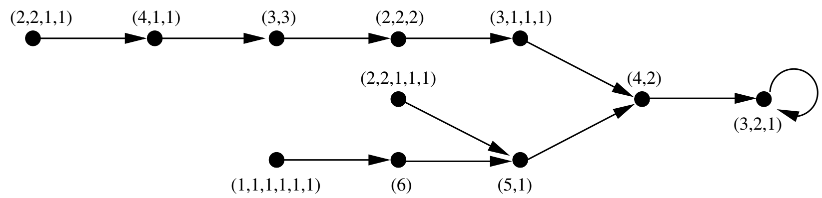

Figure 3. The state graph G_6.

   (b) The state graph G_7 is shown in Figure 4.

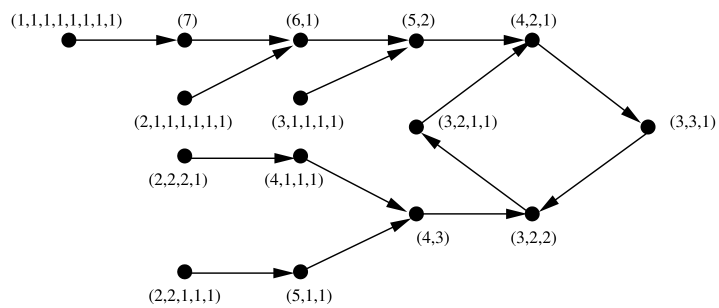

Figure 4. The state graph G_7.

The first seven state graphs are connected, but G_8 has two connected components. There are graphs with even more components [1].

6. Garden of Eden States

   (a) (4,4,3,3) is *not* a Garden of Eden state because $(5,4,4,1) \rightarrow (4,4,3,3)$.

   (b) Proof: To see why (4,3,3,2,1,1) *is* a Garden of Eden state, suppose this state had a predecessor. There are six piles of coins in (4,3,3,2,1,1), only one of which could be the collected coins from a predecessor. Therefore a predecessor would have to include at least five piles of coins. (It might have additional singleton piles.) But then there would be at least five collected coins and so our state would include a pile with at least five coins in it. Since (4,3,3,2,1,1) does not have any pile with more than four coins, the existence of a predecessor is impossible. Thus (4,3,3,2,1,1) is Garden of Eden, as claimed.

   (c) ⋆ *I make this part* ***extra credit*** *since it requires a clear understanding of the previous part.* Conjecture: A state is Garden of Eden if and only if each pile is at least two coins smaller than the number of piles. Proof: That these states are Garden of Eden follows as in the proof for (4,3,3,2,1,1). For the converse, given a state that does *not* meet the criteria, the largest pile must be no smaller than one less than the number of piles. So, we can construct a predecessor by picking up that largest pile, distributing one coin to each of the remaining piles (which we definitely have enough coins to do), and placing any remaining coins from the largest pile as new singleton piles.

7. Ten Coins

   (a-c) Any progression on ten coins leads to the fixed point (4,3,2,1). *I have sometimes done this part of the activity as an entire class, where each pair of students tries an arrangement of ten coins at once and then all report what happens.*

   (d) Conjecture: Any arrangement of ten coins progresses to the fixed point (4,3,2,1).

   (e) Conjecture: For a triangular number of coins, any arrangement progresses to the fixed point. *This result is proved in [5] and [1]. Students may try to generalize further but they should be reminded of the example of eight coins that has both a terminal 2-cycle and 4-cycle, as evidence that the outcome might depend on the initial state.*

8. Eleven coins

   (a) The cyclic states on eleven coins are (5,3,2,1), (4,4,2,1), (4,3,3,1), (4,3,2,2), (4,3,2,1,1). The easiest way to find these states is to begin with *any* partition of eleven and apply the process until you reach the 5-cycle. *Without the hint students would not know that these are the* only *cyclic elements. The pattern is that any cyclic state on eleven coins is formed from (4,3,2,1) by adding a coin to some pile or to the right of the last pile [1].*

   (b) ⋆ *I make this part* ***extra credit*** *since it is so difficult that few of my students get it. The question is stated a bit awkwardly to avoid revealing the answer to Exercise 3. You may want to restate it as "$n + 1$ coins when n is triangular".* Conjecture: For a triangular number of coins, n, a state with $n + 1$ coins is cyclic if and only if it is formed from the triangular base of n coins with the addition of a single coin. This result is proved in [1].

   For *any* number of coins, the cyclic states are formed by a triangular base with at most one coin atop *each* pile and possibly one coin to the right of the last pile. If the base triangle has k piles, and thus $t_k = \frac{k(k+1)}{2}$ coins, then there are $k + 1$ slots for possible additional coins. The cycle length will be a divisor of $k + 1$ and there are cycles of each divisor's length [1].

A somewhat different question that intrigued Gardner was the maximal number of steps needed to get from a triangular number of coins to the fixed point [11]. The answer he posed became known as the "Bulgarian Solitaire Conjecture" though it has now been settled, for example in [10].