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CHAPTER 5

the Chinese remainder theorem and knitting stitch patterns

BERIT NILSEN GIVENS

1 Introduction

1.1 Stitch Patterns in Knitting

Knitters use specialized stitch dictionaries to create distinctive cables, lacy patterns, and other interesting textures. Each stitch dictionary has many entries, with line-by-line instructions indicating the knits, purls, and other stitches that must be used to achieve the desired pattern. Knitters or knitwear designers can scan through these dictionaries when they want decorative elements or textures for a knitted object.

Each entry in a stitch dictionary usually starts with a title, a picture, and a statement of how many stitches are required. For example, a particular stitch may be named "Braided Six-Stitch Cable," followed by a standard picture of one of the cables, and then the words "Multiple of 8 plus 2." This final instruction means that for each copy of the cable pattern, you would need eight stitches, plus for the whole piece, you need an additional two stitches. To repeat the pattern once would require 10 stitches $(8+2)$, while to repeat it three times requires 26 stitches $(8\times 3+2)$. Finally, a stitch dictionary includes instructions on how to knit the stitch, either written out line-by-line using standard abbreviations for knitting instructions or in a chart. As an example, see Figure 1. Further examples can be found in [3,5,7,8].

A multiple of 6 stitches plus 1.
Row 1: *k2, yo, ssk, p2* rep from * to *, end k1.

Figure 1. A typical stitch dictionary entry.

There are many online examples as well, such as [1,4,6,9].

If a pattern designer wants to use a stitch-dictionary design to create a scarf, sweater, or hat, she can search the dictionaries for a pattern that catches her eye. Then she must adjust her stitch count to fit the pattern. A designer may have a basic hat construction that starts with 80 stitches for the brim. But to use the Braided Six-Stitch Cable, she would need either $8\times 10+2=82$ or $8\times 9+2=74$ stitches. It is usually not too much of a problem to modify the stitch count by a small number to accommodate the pattern in this way. After all, knitting is stretchy and forgiving. Most likely, the designer, knowing that the cables might pull in and make the hat a little tighter, would choose to start with 82 stitches.

However, what if a designer wants to incorporate several stitch patterns? To illustrate the problem, suppose that the designer wants to make a stole. The width of a stole can be highly variable, so maybe the designer chooses some number of stitches from 70 to 130. She has chosen a lacy design that is a multiple of 5 stitches plus 4, a second textured pattern that is a multiple of 4 stitches plus 2, and finally a cable pattern that is a multiple of 9 stitches plus 4. While the designer has a great deal of flexibility in choosing the number of stitches to start with, she does not want to change the number of stitches partway through the stole, because she would like her stole to be a proper rectangle. The problem is to find a single number that satisfies all three conditions: a multiple of 5 plus 4, a multiple of 4 plus 2, and a multiple of 9 plus 4.

The situation is still more complicated than the example above might lead one to believe. The same number of stitches in different knitted textures will not always create a rectangle, as the different fabrics may pull in or stretch out. A ribbed fabric with 100 stitches will be much narrower than a garter stitch fabric with 100 stitches. For example, Figure 2 on page 103 shows three swatches that are each 30 stitches across and 30 rows tall, but whose actual knitted sizes differ because of the stitches used. Taking into account this difference in the inherent width of different stitch patterns is the topic of Section 3.

1.2 Translating the Needlework into Mathematics

Given divisors and associated remainders, the Chinese Remainder theorem solves the problem of finding a common dividend. For example, a common dividend that gives a remainder of 1 when divided by 10 and a remainder of 3 when divided by 7 is 31. The theorem is named for Sun Tsu, and other Chinese mathematicians from the first century AD onwards, who posed puzzles that can be solved with the theorem.

Mathematicians have a different language with which to describe these concepts. To express the idea that a number x is a "multiple of 8 plus 2," mathematicians write

$$x \equiv 2 \bmod 8.$$

Formally, the *congruence* $x \equiv a \bmod m$, which is read aloud as "x is congruent to a mod m" or "x and a are congruent mod m," means that $x - a$ is divisible by m. The number m is called the *modulus* or the *mod*. The congruence can be interpreted as saying that x and a have the same remainder when divided by m. For example, $21 \equiv 13 \bmod 4$, because $21 - 13 = 8$, which is also divisible by 4, or because 21 and 13 have the same remainder when divided by 4, namely 1. If a happens to be a positive number less than m, then a itself is the remainder when x is divided by m. As an example, $21 \equiv 1 \bmod 4$ because $21 - 1$ is divisible by 4, or because 4 goes into 21 with remainder 1.

Now we can reframe the knitting designer's problem in terms of modular arithmetic. The phrase "multiple of m stitches plus a" in the knitting dictionaries is translated into modular arithmetic as $x \equiv a \bmod m$. So the designer of our stole pattern above is looking for a number x that simultaneously satisfies the three congruences:

$$\begin{aligned} x &\equiv 4 \bmod 5, \\ x &\equiv 2 \bmod 4, \\ x &\equiv 4 \bmod 9. \end{aligned} \tag{1}$$

Congruences can be treated much like equations in algebra—you can add, subtract, or multiply both sides of a congruence by the same quantity (or by congruent quantities). For example, because $37 \equiv 2 \bmod 5$ and $14 \equiv -1 \bmod 5$, then $37 \cdot 14 \equiv -2 \bmod 5$.

Linear congruences have the form $ax \equiv b \bmod m$. The process of solving a linear congruence for x is covered in any elementary number theory textbook, see [2, ch. 4], for example. Generally, when we refer to the solution (or solutions) to a linear congruence, the assumption is that we list only the positive integer solutions less than the modulus m. So while $x = 25$ is a solution to $4x \equiv 10 \bmod 18$, the default is to write this solution as $x = 7$ (note that $25 \equiv 7 \bmod 18$), because 25 is greater than 18. However, when $x = 7$ is given as a solution, the reader should understand that *any* number which is congruent to 7 mod 18 is also a solution, so $25, 43, 61, \ldots$ are all solutions, as are $-11, -29, \ldots$. If a congruence has any solution, then it has infinitely many solutions. When we say a congruence has a finite number of solutions, we will always be referring to the number of nonnegative solutions less than the modulus m.

The set $\{0, 1, 2, \ldots, m-1\}$, together with arithmetic operations mod m, is called $\mathbb{Z}_m$. Because we only need to consider possible solutions in $\mathbb{Z}_m$, one easy way to solve a congruence with a small modulus is to simply try all possible values. Consider $3x \equiv 2 \bmod 7$. Because the modulus is 7, we try the numbers $x = 0, 1, 2, \ldots 6$ for x until we find the values that make the equation true. In this case, $x = 3$ is the only solution in $\mathbb{Z}_7$, so we say this congruence has one solution.

When we have a system of congruences as in Equation (1) above, the Chinese Remainder Theorem allows us to find simultaneous solutions to systems of congruence equations. That is, by using the Chinese Remainder Theorem, we can find a number that simultaneously satisfies all three of these congruences.

1.3 A Knitting-inspired Extension of the Mathematics

All of the above discussion presupposes that our knitting designer wants the same number of stitches on each section of stitch pattern. However, the fact is that knitting stitch patterns often distort the knitted fabric, so that a number of stitches in, say, ribbing, will be much narrower than the same number of stitches in a lace pattern. As an example, consider the three swatches in Figure 2.

All three swatches are 30 stitches wide and 30 rows tall, knit with the same yarn on the same size needles, but their finished measurements are clearly different. On the far left of Figure 2, the garter stitch is nine inches wide. The 1×1 ribbing on the right is only five-and-one-half inches wide. The most common way to measure gauge is in number of stitches per four inches (or 10 centimeters) or simply in stitches per inch (spi). Thus, in this yarn on these needles, the garter stitch has $u = \frac{10}{3}$ spi and the ribbing has $v = \frac{60}{11}$ spi. If a knitwear designer was going to start with the garter stitch and wanted to create a rectangular item by following it with the 1×1 ribbing on the right, she would want to increase the number of stitches by the factor $\frac{v}{u} = \frac{60}{11} \cdot \frac{3}{10} = \frac{18}{11}$. Because the ribbing pattern pulls in to become narrower, the number of stitches needed in the ribbing pattern to make the same width as the garter stitch is greater.

More generally, suppose stitch pattern A, at u spi, uses a multiple of m stitches plus a, while pattern B, at v spi, uses a multiple of n stitches plus b. Further assume that pattern B is inherently narrower than pattern A, which means that $v > u$. Now $\frac{v}{u}$ is the factor that adjusts for the difference between the widths of of the two patterns.

Figure 3 was knit by first casting on 30 stitches and knitting 30 rows in garter stitch, seen at the bottom of the swatch. Then, without changing the stitch counts, 30 rows of 2×2 ribbing followed. Since ribbing has a higher spi than garter, the resulting fabric is narrower and pulls in. Above the narrower ribbing, there are another 30 rows of garter, and the fabric is wider again.

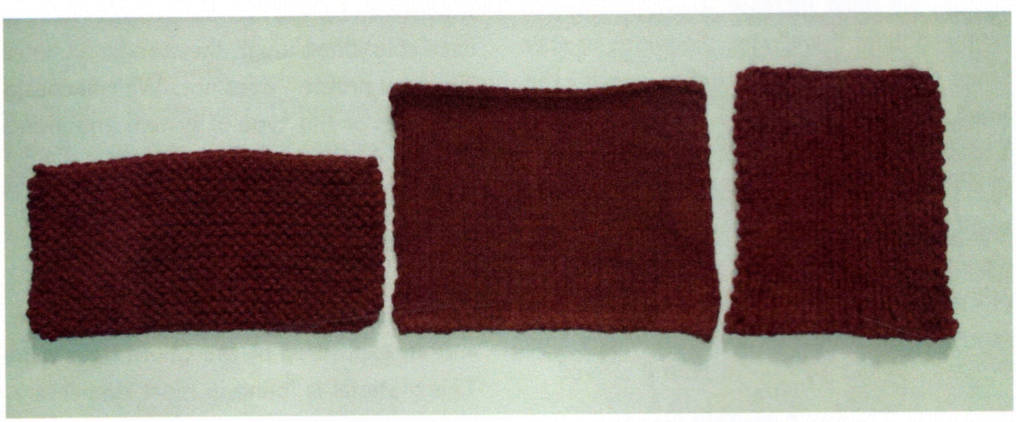

Figure 2. Different knitting stitch patterns with the same number of stitches give different widths.

Figure 3. Keeping the stitch count constant will not necessarily keep the width constant. Starting at the bottom, the first three stitch patterns are 30 stitches wide, but the width in inches varies. The topmost portion is 37 stitches wide and roughly the same width in inches as the garter stitch just below it.

Finally, on the uppermost portion of the swatch, we increased the total number of stitches to compensate for the difference in gauge and maintain a constant width and began the 2×2 ribbing again. In Figure 3 we have $u = \frac{17}{4}$ for the garter stitch and $v = \frac{21}{4}$ for the ribbing, giving $\frac{v}{u} = \frac{21}{17}$. If the garter stitch portion is M stitches across, we must increase to $M(\frac{v}{u})$ stitches when starting the ribbing if we want the width to stay constant. Since the garter is 30 stitches across, the ribbing required roughly $30(\frac{21}{17}) = 37.06$ stitches. We rounded this to 37, so that we needed to increase seven stitches total. This was done by first noticing that seven new stitches will split the 30 existing stitches into eight groups. The quotient of 30 by 8 is 3 with remainder 6. That means six groups of 4 stitches each and two groups of 3 stitches each. In the usual knitting terminology, this becomes

k3, *M1, k4*, repeat from * to * until the last 3 stitches, M1, k3.

The upper two sections in Figure 3 are roughly the same width, even with no blocking or stretching.

In general, it is reasonable to assume that u and v are rational numbers, so $\frac{v}{u}$ can be reduced to a fraction $\frac{p}{q} > 1$ in lowest terms. In other words, p and q are *relatively prime*, meaning that they satisfy $\gcd(p, q) = 1$. Then, in order to make a rectangle that consists of pattern A followed by pattern B, our designer needs to solve the following system of equations:

$$x \equiv a \bmod m,$$
$$\frac{p}{q}x \equiv b \bmod n.$$

Unfortunately, $\frac{p}{q}x$ might not be an integer, so we must round off to an integer somehow. We could round up, round down, or round to the nearest integer. For simplicity here, we will always round *down* to the nearest integer by using the floor function $\left\lfloor \frac{p}{q}x \right\rfloor$. Then we must solve the system

$$x \equiv a \bmod m,$$
$$\left\lfloor \frac{p}{q}x \right\rfloor \equiv b \bmod n.$$

We show that when m and n are relatively prime, this system can be rewritten so that we may apply the Chinese Remainder Theorem as before. However, even when m and n are not relatively prime, the system may have solutions though the standard Chinese Remainder Theorem problem does not. We will consider some special cases of this type of system and investigate the existence of solutions.

2 Mathematical Work

We begin with some background on linear congruences and an exposition of the Chinese Remainder Theorem. This material is found in most elementary number theory textbooks, see Burton [2, ch. 4], for example. The Chinese Remainder Theorem is Theorem 4.8 in [2].

2.1 Background

As discussed above, congruences can be manipulated in much the same way as equations in algebra. One can add, subtract, or multiply congruent quantities to both sides of a congruence, for example. The exception is that division of both sides of a congruence is generally not valid. Division by an integer a can be thought of as multiplying by the number's inverse $\frac{1}{a}$, but $\frac{1}{a}$ is not defined in $\mathbb{Z}_m$. However, we shall see below that when $\gcd(a, m) = 1$, Proposition 1 gives us an interpretation of the symbol a^{-1} in $\mathbb{Z}_m$ that serves the same role as $\frac{1}{a}$ in the integers, namely that $a \cdot a^{-1} \equiv 1 \bmod m$.

The notation $a|b$ means that a *divides* b; that is, $b = ak$ for some integer k. The *greatest common divisor*, $\gcd(a, b)$, of integers a and b is the largest integer that divides both a and b. Proposition 1 [2, p. 76] formalizes the above discussion of division and inverses.

Proposition 1 The congruence $ax \equiv b \bmod m$ has integer solutions if and only if $\gcd(a, m)|b$. If the congruence has any solutions, then it has exactly $\gcd(a, m)$ solutions.

In particular, if $\gcd(a, m) = 1$, then $ax \equiv 1 \bmod m$ has integer solutions. In that case, there is exactly one solution for x, which we can define as the *inverse* of a mod m. For example, the congruence $4x \equiv 1 \bmod 9$ has solution $x = 7$, so the inverse of 4 in $\mathbb{Z}_9$ is 7. Whenever we wish to divide by 4, we can instead multiply by 7.

Theorem 1 (Chinese Remainder Theorem) Suppose $m_1, m_2, \ldots, m_k$ are pairwise relatively prime. Let $M = m_1 m_2 \cdots m_n$. Then the system of congruences

$$\begin{aligned} x &\equiv a_1 \bmod m_1, \\ x &\equiv a_2 \bmod m_2, \\ &\vdots \\ x &\equiv a_k \bmod m_k. \end{aligned}$$

has a unique solution between 0 and $M - 1$.

Furthermore, if $n_i = M/m_i$ and y_i satisfies $n_i y_i \equiv a_i \bmod m_i$, then a solution is given by

$$x \equiv n_1 y_1 + n_2 y_2 + \cdots n_k y_k \bmod M.$$

Proof: First we prove existence by showing that the given formula for x is indeed a solution. We have $n_j = M/m_j = m_1 m_2 \cdots m_i \cdots m_{j-1} m_{j+1} \cdots m_k$. Therefore, m_i and n_i are relatively prime, guaranteeing the existence of y_i by Proposition 1. For $i \neq j$, $m_i | n_j$ and so $n_j \equiv 0 \bmod m_i$. As a result, $n_i y_i \equiv a_i \bmod m_i$, while for $j \neq i$, we have $n_j y_j \equiv 0 \bmod m_i$.

On the other hand, suppose x_1 and x_2 are two solutions in $\mathbb{Z}_M$, so that $0 \leq x_1 \leq x_2 < M$. Then for each i, we have $m_i | x_2 - x_1$. Because the moduli are relatively prime, we can conclude that their product divides $x_2 - x_1$. Thus M divides $x_2 - x_1$, which is a number between 0 and $M - 1$. This is only possible if $x_2 - x_1 = 0$, or $x_2 = x_1$. □

As an example, let's go back to our knitwear designer and her stole. The system of congruence equations she needs to solve is

$$\begin{aligned} x &\equiv 4 \bmod 5, \\ x &\equiv 2 \bmod 4, \\ x &\equiv 4 \bmod 9. \end{aligned}$$

We follow these steps:

1. $M = 5 \cdot 4 \cdot 9 = 180$. Because the moduli 4, 5, and 9 are relatively prime, we know that there is an answer less than or equal to 180.

2. Next, $n_1 = M/m_1 = 180/5 = 36$, $n_2 = M/m_2 = 180/4 = 45$, and $n_3 = M/m_3 = 180/9 = 20$.

3. This is the hardest step. We have to solve three different congruence equations. The first equation is

$$\begin{aligned} n_1 y_1 &\equiv a_1 \bmod m_1, \\ 36 y_1 &\equiv 4 \bmod 5, \\ 1 y_1 &\equiv 4 \bmod 5, \text{ as 36 is congruent to 1 mod 5} \\ y_1 &= 4. \end{aligned}$$

Similarly, the second equation is $45y_2 \equiv 2 \bmod 4$ with solution $y_2 = 2$. The third equation is $20y_3 \equiv 4 \bmod 9$, resulting in $y_3 = 2$.

4. Finally, we find the number x by computing

$$x \equiv n_1y_1 + n_2y_2 + n_3y_3 \bmod M$$
$$x \equiv 36(4) + 45(2) + 20(2) \bmod 180$$
$$x \equiv 144 + 90 + 40 \bmod 180$$
$$x \equiv 274 \bmod 180$$

Then divide 274 by 180, and find that the remainder is 94.

To conclude, the number that our designer is looking for is 94. If she wants a smaller number, she will have to change her plans; this is the smallest positive number that will work with these conditions. If she wants a bigger number, she can add the number M as many times as necessary to get her desired amount to cast on. Because $M = 180$, the next smallest possible number of stitches is 274.

The Chinese Remainder Theorem will work for any number of congruence equations, as long as the moduli $m_1, m_2, \ldots m_k$ are all relatively prime. However, if the moduli are not relatively prime, a few things can go wrong. First of all, the formula in the steps above will not give us the correct answer. Even worse, there may not even be an answer. Consider as an example the problem of two stitch patterns, one of which is a multiple of 5 stitches plus 2, while the other is a multiple of 5 stitches plus 3. This would mean finding a number that when divided by 5 has a remainder of 2 and also a remainder of 3, which is a logical contradiction.

A solution may fail to exist in less obvious examples as well. We cannot solve $x \equiv 2 \bmod 6$ and $x \equiv 3 \bmod 4$. The Chinese Remainder Theorem formula cannot be used, because the moduli 6 and 4 share a common factor 2, so that the moduli are not relatively prime. Furthermore, if $x \equiv 2 \bmod 6$, that means that x is a multiple of 6 stitches plus 2. A moment's thought will reveal that x must be even. On the other hand, $x \equiv 3 \bmod 4$ means that x is a multiple of 4 plus 3, which implies that x is odd. Our desired solution x would have to be both even and odd, which is impossible.

Even if the moduli are not relatively prime, there may be a solution, but we cannot use the Chinese Remainder Theorem to find it. We can get around this difficulty through a couple of tricks. First, a congruence equation with a modulus that has more than one prime factor can be decomposed into several congruence equations with smaller moduli. Second, when several congruence equations are redundant, we can eliminate some of them, thereby simplifying our system.

Proposition 2 Let $m, n \geq 2$ and let a and b be integers.

1. (Decomposition) Let $m = m_1m_2$, where $m_1, m_2 \geq 2$ and $\gcd(m_1, m_2) = 1$. Then $x \equiv a \bmod m$ if and only if $x \equiv a \bmod m_1$ and $x \equiv a \bmod m_2$.

2. (Redundancy) Suppose $n|m$ and $n|a - b$. Then $x \equiv a \bmod m$ implies that $x \equiv b \bmod n$.

The proofs of these two statements are typical of the kind of arguments made in any introduction to elementary number theory.

Proof: The forward direction of the decomposition step follows immediately from the statements that m divides $x - a$ and m_i divides m. By the transitivity of divisibility, m_i divides $x - a$ for $i = 1, 2$. For the converse, suppose $x \equiv a \bmod m_1$ and $x \equiv a \bmod m_2$, so that $x - a$ is divisible by both m_1 and m_2. We can conclude that m_1m_2 divides $x - a$ because m_1 and m_2 have no common factors.

Now suppose $n|m$, $n|a-b$, and $x \equiv a \bmod m$. Then $x - a$ is divisible by m. Thus we have integers k, ℓ, and j such that $m = nk$, $a - b = n\ell$, and $x - a = mj$. A little algebra yields $x - b = (x - a) + (a - b) = mj + n\ell =$

$nkj + n\ell$, showing that n divides $x - b$. Then we have $x \equiv b \bmod n$, completing the proof of redundancy. □

The corollary to the redundancy statement is that we can find a solution to the system $x \equiv a \bmod m$ and $x \equiv b \bmod n$ by solving only $x \equiv a \bmod m$. The congruence $x \equiv b \bmod n$ is redundant.

As an example, consider the system $x \equiv 5 \bmod 6$ and $x \equiv 3 \bmod 4$. The congruence equation $x \equiv 5 \bmod 6$ can be decomposed into the two equations $x \equiv 5 \bmod 2$ and $x \equiv 5 \bmod 3$. Both equations can be simplified, giving us the two equations $x \equiv 1 \bmod 2$ and $x \equiv 2 \bmod 3$. The congruence equation $x \equiv 3 \bmod 4$ cannot be further decomposed. But we can observe that $x \equiv 1 \bmod 2$ is implied by $x \equiv 3 \bmod 4$. Hence, if we found a solution to $x \equiv 3 \bmod 4$, we would automatically also have a solution to $x \equiv 1 \bmod 2$, so we may safely omit the equation $x \equiv 1 \bmod 2$. Our original problem has been reduced to the problem of solving $x \equiv 2 \bmod 3$ and $x \equiv 3 \bmod 4$. The Chinese Remainder Theorem can now be used to guarantee that there is a solution less than or equal to 12. In fact, the solution turns out to be 11. Any number that is congruent to 11 mod 12 will work, so the number of stitches could be 11, 23, 35, 47, and so on.

This next variation of the Chinese Remainder Theorem gives a necessary and sufficient condition under which there is at least one solution of a system of two congruences with moduli that are not relatively prime.

Theorem 2 The system of congruences

$$x \equiv a \bmod m$$
$$x \equiv b \bmod n$$

has a solution if and only if $\gcd(m, n) | a - b$.

Proof: Let $d = \gcd(m, n)$ and assume that $d | a - b$. Let j be an integer such that $a - b = dj$. By Proposition 1, there is an integer y such that $ny \cong d \bmod m$. Define $x = jny + b$. In order to check that x is a solution to the given system, we immediately observe that $x = jny + b \equiv b \bmod n$. Next notice that $x = jny + b \equiv jd + b \bmod m$. Substituting for dj, we find that $x \equiv a \bmod m$.

On the other hand, if $x \equiv a \bmod m$ and $x \equiv b \bmod n$, then there are integers k and ℓ satisfying $x - a = mk$ and $x - b = n\ell$. Then $a - b = (x - b) - (x - a) = n\ell - mk$. Clearly, d divides both n and m, so d divides $n\ell - mk = a - b$. □

2.2 Some Examples Applied to Knitting

Let's consider a couple of examples to demonstrate how a knitwear designer could use the Chinese Remainder Theorem.

Example 1 (A hat.) A knitwear designer has some colorwork designs for a hat. One is a large 18-stitch panel of a star that she wants featured on the front of the hat. The second design, which requires a multiple of 11 stitches plus 1, will fill in the sides and back with decorative stripes of small stars. This means that the number of stitches must have the form of $y + 18$, where $y \equiv 1 \bmod 11$. This means that $y = 11k + 1$ for some integer k and so $y + 18 = 11k + 1 + 18 = 11k + 19$. In other words, the total number of stitches needed is congruent to 19 modulo 11.

In addition to the colorwork, she has some structural requirements for the hat. She wants to start with some 2-by-2 ribbing at the brim in a solid color. Then she will change into plain knitting (stockinette stitch), work the large 18-stitch pattern for the front, and repeat the 11-stitch colorwork design around the sides and back. Finally, she has a standard system for making the crown decreases, which requires a multiple of 6 stitches, so that the crown decreases can be spaced at six even intervals around the hat. She needs to solve the three congruence equations

$$x \equiv 0 \bmod 4 \qquad \text{for the ribbing,}$$
$$x \equiv 19 \bmod 11 \qquad \text{for the colorwork,}$$
$$x \equiv 0 \bmod 6 \qquad \text{for the crown decreases.}$$

Because 4 and 6 share a common factor, she first must decompose the third equation by factoring 6. At the same time, the second equation can be simplified by realizing that $19 \equiv 8 \bmod 11$. Then she gets

$$x \equiv 0 \bmod 4,$$
$$x \equiv 8 \bmod 11,$$
$$x \equiv 0 \bmod 2,$$
$$x \equiv 0 \bmod 3.$$

The first equation implies the third equation, so she can safely omit the third equation. She applies the steps of the Chinese Remainder Theorem to find that $x \equiv 96 \bmod 132$ is a solution. If she casts on 96 stitches for her hat, she will be able to use all three stitch patterns without modifications.

Example 2 (A scarf.) For this scarf, our hypothetical designer would like to use three lace patterns from stitch dictionaries. She wants to start with the Flower Eyelet [5, p. 54], which uses a multiple of 16 plus 8, followed by Tulip Lace [5, p. 54], a multiple of 8, and end with All-over Eyelets [3, p.54], which uses a multiple of 10 plus 1. This gives us the three equations

$$x \equiv 8 \bmod 16,$$
$$x \equiv 0 \bmod 8,$$
$$x \equiv 1 \bmod 10.$$

Right away, she can see that the first two equations require an even number, while the third requires an odd number. She decides to add a single extra plain knit stitch on the right side of the All-over Eyelets pattern, so that this equation becomes $x \equiv 2 \bmod 10$. Because 10 has the two distinct prime factors 2 and 5, she further decomposes this equation into $x \equiv 2 \bmod 2$ and $x \equiv 2 \bmod 5$. Then she has

$$x \equiv 8 \bmod 16,$$
$$x \equiv 0 \bmod 8,$$
$$x \equiv 0 \bmod 2,$$
$$x \equiv 2 \bmod 5.$$

Further, she notices that the first equation implies both the second and third, so she can omit those. This leaves her with

$$x \equiv 8 \bmod 16,$$
$$x \equiv 2 \bmod 5.$$

Now she has just two equations with relatively prime moduli, so she uses the Chinese Remainder Theorem formula to find $x \equiv 72 \bmod 80$. She decides to use $x = 72$, as that seems like a good amount for a wide scarf. If she wanted a wider stole or wrap, she could add any multiple of 80: 152, 232, or 312, etc.

3 Rescaling for Different Gauges

As explained in Section 1.3, our knitwear designer might need to rescale her stitch count if the stitch patterns have different gauges. To compensate for the different gauges, we apply a scaling factor p/q, and then solve a system of congruences of the form

$$\begin{aligned} x &\equiv a \bmod m, \\ \left\lfloor \frac{p}{q} x \right\rfloor &\equiv b \bmod n. \end{aligned} \tag{2}$$

We may assume that p/q is a rational number in lowest terms, so that p and q are relatively prime. Furthermore, we choose m to be the modulus corresponding to the wider stitch pattern, so that $p/q > 1$.

In order to simplify this system, we write $x = qk + r$, where $0 \le r < q$, and $\bar{r} = \lfloor pr/q \rfloor$. This yields

$$\begin{aligned} qk &\equiv a - r \bmod m, \\ pk &\equiv b - \bar{r} \bmod n, \\ x &= qk + r. \end{aligned} \tag{3}$$

As an example, consider the system

$$\begin{aligned} x &\equiv a \bmod 6 \\ \left\lfloor \frac{15}{4} x \right\rfloor &\equiv b \bmod 10. \end{aligned}$$

By making the substitution $x = 4k + r$, we obtain

$$\begin{aligned} 4k &\equiv a - r \bmod 6 \\ 15k &\equiv b - \bar{r} \bmod 10. \end{aligned}$$

In a given problem, we know a, and r comes from the division algorithm, so we need only consider $0 \le r < 4$. However the corresponding values of $\bar{r}$ do not increase by a set amount for each unit increase in r. The values of $\bar{r}$ as r increases in the example above are shown in Table 1. In particular, if we look at the values of $\bar{r}$ modulo 4, we observe something alarming: two values of r that are congruent modulo 4 result in values of $\bar{r}$ that are different modulo 4! For example, $r_1 = 1$ and $r_2 = 5$ are congruent mod 4, but the corresponding values of $\bar{r}$ are $\bar{r}_1 = 3$ and $\bar{r}_2 = 18$, which are *not* congruent mod 4. However, if we consider the values of $\bar{r}$ mod 15, we see that $\bar{r}_1$ and $\bar{r}_2$ are equivalent mod 15. In fact, this is generally true.

r	0	1	2	3	4	5	6	7
$\bar{r}$	0	3	7	11	15	18	22	26

Table 1. The values of $\bar{r} = \lfloor 15r/4 \rfloor$ for $p = 15$, $q = 4$.

Lemma 1 If $r_1 \equiv r_2 \bmod q$, then $\bar{r}_1 \equiv \bar{r}_2 \bmod p$.

Proof: The congruence $r_1 \equiv r_2 \bmod q$ implies that $r_2 = qk + r_1$ for some integer k. Then $\bar{r}_1 - \bar{r}_2 = \lfloor pr_1/q \rfloor - \lfloor pr_2/q \rfloor = \lfloor pr_1/q \rfloor - \lfloor p(qk + r_1)/q \rfloor = \lfloor pr_1/q \rfloor - \lfloor pk + pr_1/q \rfloor = \lfloor pr_1/q \rfloor - pk - \lfloor pr_1/q \rfloor = -pk$. Thus $p | \bar{r}_1 - \bar{r}_2$. □

In other words, the map $r \mapsto \left\lfloor \frac{p}{q} r \right\rfloor$ is well-defined as a function from $\mathbb{Z}_q$ to $\mathbb{Z}_p$. Therefore, a full set of solutions to the system of equations (3) can be obtained by considering only values of r with $0 \le r < q$.

3.1 Other Versions of the Standard Chinese Remainder Theorem

To investigate whether or not the general system (2) has solutions, we return to the standard Chinese Remainder Theorem with two equations and consider what happens when we weaken the hypotheses to allow coefficients on x and consider the case when m and n are not relatively prime. We consider general systems of the form

$$\begin{aligned} cx &\equiv a \bmod m, \\ dx &\equiv b \bmod n. \end{aligned} \tag{4}$$

The next few results on this general system are certainly already known, but we will sketch the proofs here.

Theorem 3 Suppose $\gcd(c, m) = \gcd(d, n) = 1$. The system of congruences (4) has a solution if and only if $\gcd(m, n)$ divides $bc - ad$.

Proof: Let $g = \gcd(m, n)$ and suppose that g divides $bc - ad$. Because $\gcd(c, m) = \gcd(d, n) = 1$, there exist integers a' and b' such that $ca' \equiv a \bmod m$ and $db' \equiv b \bmod n$. This means that for some integers k and j, we have $ca' = mk + a$ and $db' = nj + b$. Multiplying the first equation by d, the second by c, and subtracting yields

$$cd(a' - b') = mdk - ncj + (bc - ad). \tag{5}$$

Because g divides the right-hand side of this equation, then $g | cd(a' - b')$. However, g is relatively prime to both c and d, so that g must divide $a' - b'$. By Theorem 2, the system $x \equiv a' \bmod m$ and $x \equiv b' \bmod n$ has a solution, and such a solution is also a solution to (4).

On the other hand, suppose that the given system has a solution x_0. Then there are integers k and j such that $cx_0 = mk + a$ and $dx_0 = nj + b$. As in the previous paragraph, multiply the first equation by d and the second by c, and then subtract to get $0 = mkd - njc - (bc - ad)$. Knowing that g divides both m and n allows us to conclude that g also divides $bc - ad$. □

Next, what happens if $\gcd(c, m)$ or $\gcd(d, n)$ is not 1? We now have three different greatest common divisors to keep track of:

$$g = \gcd(m, n),$$
$$g_1 = \gcd(c, m),$$
$$g_2 = \gcd(d, n).$$

Luckily the theorem above can be used to handle the cases when g_1 and g_2 are any positive integers.

Corollary 1 The system of congruences in (4) has a solution if and only if

1. $\gcd(c, m)$ divides a;
2. $\gcd(d, n)$ divides b; and
3. $\gcd(c, m) \cdot \gcd(d, n) \cdot \gcd\left(\frac{m}{\gcd(c, m)}, \frac{n}{\gcd(d, n)}\right)$ divides $bc - ad$.

Proof: As above, let $g_1 = \gcd(c, m)$, $g_2 = \gcd(d, n)$, and $g = \gcd(m, n)$. According to Proposition 1, each congruence individually has a solution if and only if g_1 divides a and g_2 divides b. Then the given system of congruences is equivalent to

$$\frac{c}{g_1}x \equiv \frac{a}{g_1} \bmod \frac{m}{g_1},$$
$$\frac{d}{g_2}x \equiv \frac{b}{g_2} \bmod \frac{n}{g_2}.$$

By the previous theorem, this system has a solution if and only if condition 3 holds. □

Lemma 2 Suppose $g_1 | m$ and $g_2 | n$. Then $\gcd(m, n)$ divides $g_1 g_2 \gcd\left(\frac{m}{g_1}, \frac{n}{g_2}\right)$.

Proof: The equation $mx + ny = k$ has integer solutions if and only if $\gcd(m, n)$ divides k. Thus there are integers x and y for which

$$\frac{m}{g_1}x + \frac{n}{g_2}y = \gcd\left(\frac{m}{g_1}, \frac{n}{g_2}\right).$$

Now we multiply both sides of this equation by $g_1 g_2$ to get

$$m(g_2 x) + n(g_1 y) = g_1 g_2 \gcd\left(\frac{m}{g_1}, \frac{n}{g_2}\right),$$

implying that $\gcd(m, n)$ divides the right-hand side. □

Combining the last corollary and lemma and then taking the contrapositive gives our final result.

Corollary 2 If $\gcd(m, n)$ does not divide $bc - ad$, then the system (4) has no solutions.

3.2 Results on the Knitting-inspired System

Now we return to our knitting-inspired system (3), restated here:

$$qk \equiv a - r \bmod m$$
$$pk \equiv b - \bar{r} \bmod n, \qquad \text{where } \bar{r} = \left\lfloor \frac{pr}{q} \right\rfloor$$
$$x = qk + r, \qquad 0 \le r < q.$$

If this system has a solution, then the Chinese Remainder Theorem guarantees at least one solution with $k < \operatorname{lcm}(m, n)$. Naïvely, we can estimate that if there is a solution, then there is at least one solution with $x < qk + r < q\operatorname{lcm}(m, n) + q$. However, we can improve this bound by taking advantage of the fact that for integers, $x < n$ is equivalent to $x \le n - 1$. So if there is a solution, there is at least one solution with $x \le$

$qk + r \leq q(\text{lcm}(m,n) - 1) + (q-1) = q\text{lcm}(m,n) - 1$. In other words, there is a solution $x < q\text{lcm}(m,n)$.

We can simplify the results of Corollary 1(3) by defining $s \equiv pr \bmod q$ with $0 \leq s < q$, so that $pr - s = q\bar{r}$. The quantity $bc - ad$ in Corollary 1 is computed to be $p(a-r) - q(b-\bar{r}) = ap - pr - qb + q\bar{r} = ap - pr - qb + pr - s = ap - bq - s$.

Let $g_1 = \gcd(q,m), g_2 = \gcd(p,n)$ and $d = \gcd\left(\frac{m}{g_1}, \frac{n}{g_2}\right)$. To summarize, applying Corollary 1 yields the fact that solutions to the system (3) exist if and only if

$$\begin{aligned} g_1 &\text{ divides } a - r, \\ g_2 &\text{ divides } b - \bar{r}, \\ g_1 g_2 d &\text{ divides } ap - bq - s. \end{aligned} \tag{6}$$

Because s varies as r ranges over $\mathbb{Z}_q$, it is entirely possible for the gauge-corrected knitting-inspired system to have solutions when a standard system with fewer parameters would not. For given values of p, q, m, and n, the ability to select different values of r between 0 and $q - 1$ allows for a greater variety of possible solutions, as shown in the example below.

Example 3 Consider again the system in (2) with $m = 6, n = 10$, and $p/q = 15/4$. Such a system would arise if a knitter had two stitch patterns, A and B, where pattern A uses a multiple of 6 plus some number, a, and pattern B uses a multiple of 10 plus some number, b. Furthermore, 4 stitches in pattern A is the same width as 15 stitches in pattern B. (Admittedly, the large difference between p and q is unlikely in a real knitting scenario, but for ease of computation in this example, we will use these numbers.)

With these values for m, n, p and q, we get $g_1 = 2, g_2 = 5$ and $d = \gcd(3,2) = 1$. With the substitution $x = 4k + r$, the system becomes

$$\begin{aligned} 4k &\equiv a - r \bmod 6, \\ 15k &\equiv b - \bar{r} \bmod 10. \end{aligned} \tag{7}$$

Solutions exist if and only if

$$\begin{aligned} 2&|a - r, \\ 5&|b - \bar{r}, \\ 10&|15a - 4b - s. \end{aligned}$$

On the other hand, if $2|a - r$ and $5|b - \bar{r}$, we may write $a = 2\ell + r$ and $b = 5j + \bar{r}$. Then $15a - 4b - s = 15(2\ell + r) - 4(5j + \bar{r}) - s = 30\ell - 20j + 15r - 4\bar{r} - s$. Recall that $s = 15r - 4\bar{r}$, so we have $30\ell - 20j$, which is clearly divisible by 10. In this example, the third condition is always satisfied; solutions will exist precisely when $2|a - r$ and $5|b - \bar{r}$. Now it is a simple matter to compute $\bar{r}$ for each r, as in Table 1, and then determine which values of a and b satisfy the necessary conditions. Table 2 summarizes the computations.

From Table 2, we see that if $a = 5$, for example, then r can be 1 or 3 and b can be 1, 3, 6, or 8. If pattern A were a multiple of 6 stitches plus 5, then the knitter needs pattern B to be a multiple of 10 stitches plus b, where $b = 1, 3, 6$, or 8. If b does not equal one of these numbers, then she will have to change her plans or adapt one of the stitch patterns to fit her requirements.

r	$\bar{r}$	a	b
0	0	0, 2, 4	0, 5
1	3	1, 3, 5	3, 8
2	7	0, 2, 4	2, 7
3	11	1, 3, 5	1, 6

Table 2. A summary of the calculations for the example $m = 6, n = 10, p/q = 15/4$.

This example shows two things: First, the degree of freedom granted to us by allowing the remainder r to range from 0 to $q - 1$ allows solutions to more systems than can be solved in the traditional Chinese Remainder Theorem systems (as in Theorem 3). For example, system (7) would have solutions only for $b = 0$ and 5 if we were forced to have $r = \bar{r} = 0$, but allowing r to vary lets us solve it for other values of b. Second, even with that freedom, there are still examples of systems with no solutions. In (7), there are no solutions when $b = 4$ or 9.

For the first point, consider only the equation $qk \equiv a - r \bmod m$. This has solutions if and only if $\gcd(q, m)$ divides $a - r$, so it is conceivable that for some choices of a there might be no solution. However, $a - r$ ranges over q consecutive integers, meaning that exactly $\frac{1}{\gcd(q,m)}$ of the q choices of r lead to an equation which is solvable. Each r for which $qk \equiv a - r \bmod m$ has solutions yields exactly $\gcd(q, m)$ solutions for k. If $m > q$ then the values of $a - r$ are distinct. Thus, there are $\frac{1}{\gcd(q,m)} \cdot q \cdot \gcd(q, m) = q$ values of k that solve the equation for some value of r. On the other hand, if $m \leq q$, then a similar argument establishes that there are m such values of k. We have now shown the following:

Lemma 3 There are exactly $\min(m, q)$ distinct values of k such that

1. $0 \leq k < m$;
2. $qk \equiv a - r \bmod m$, for some $0 \leq r < q$.

In particular, if $m \leq q$, then all values of k between 0 and $m - 1$ satisfy the second condition.

Lemma 3 tells us that the first equation in (3) *always* has solutions for some r. Now, fix one of those solutions, say $k \equiv k' \bmod m$, and fix the corresponding r. Then we can count the number of solutions to

$$\begin{aligned} k &\equiv k' \bmod m, \\ pk &\equiv b - \bar{r} \bmod n. \end{aligned} \tag{8}$$

This has solutions when

1. $\gcd(p, n)$ divides $b - \bar{r}$ and
2. $\gcd(p, n) \cdot \gcd\left(m, \frac{n}{\gcd(p,n)}\right)$ divides $pk' - b + \bar{r}$.

Unfortunately, the values of $b - \bar{r}$ may not be consecutive integers, so even as r ranges from 0 to $q - 1$, the values of $\bar{r}$ may erratically skip, repeat (modulo n), or otherwise be badly behaved. In the example above, when $r = 0, 1, 2, 3$, then $\bar{r} = 0, 3, 7, 11$.

However, there is one special case in which $\bar{r}$ is well-behaved—when $p = q + 1$. In practice, this special case is arguably the most common situation that a knitter might face. Differences in gauge between different stitch patterns are generally not large. For example, the scaling factor for the differences in gauge between the samples of garter stitch and the 1×1 ribbing in Figure 2 is about $3/2$, and these two patterns were chosen because garter is known to be especially wide and ribbing to be especially narrow. Generally, the scaling factor between any two stitch patterns is usually between 1 and 1.5. Thus the scaling factor often can be approximated by $\alpha = \frac{q+1}{q}$. As another example, in Figure 3, we have $p/q = 21/17$. Since $21/17$ is between $6/5$ and $5/4$, we could attempt to use either of those quotients as our approximation, giving us a ratio with $p = q + 1$. Thus, we now conclude this chapter with a brief investigation of the special case $p = q + 1$.

If $p = q + 1$, then $\lfloor pr/q \rfloor = \lfloor (q+1)r/q \rfloor = \lfloor r + r/q \rfloor = r$, because $0 \leq r/q < 1$. Thus $s = \bar{r} = r$. With $g_1 = \gcd(q, m)$, $g_2 = \gcd(q+1, n)$ and $d = \gcd(\frac{m}{g_1}, \frac{n}{g_2})$, we see that (6) becomes

$$\begin{aligned} g_1 &\text{ divides } a - r, \\ g_2 &\text{ divides } b - r, \\ g_1 g_2 d &\text{ divides } ap - bq - r. \end{aligned} \tag{9}$$

These statements can be thought of as congruences in the variable r:

$$\begin{aligned} r &\equiv a \bmod g_1, \\ r &\equiv b \bmod g_2, \\ r &\equiv ap - bq \bmod g_1 g_2 d. \end{aligned}$$

However, if r is a solution to the third equation, then r is congruent to $ap - bq$ modulo both g_1 and

g_2. Also, g_1 divides q, so $bq \equiv 0 \bmod g_1$. Consequently, $r \equiv ap \bmod g_1$. Remember that $p = q + 1$ here, so $r \equiv a \bmod g_1$. Similarly it can be shown that $r \equiv b \bmod g_2$. Thus in this special case, we only need to select r so that $r \equiv ap - bq \bmod g_1 g_2 d$.

Example 4 Let's consider a last example. The knitwear designer has two stitch patterns, pattern A and pattern B. Stitch pattern A has a gauge of 20 stitches per 4 inches, while pattern B has 24 stitches per 4 inches. This means that to get the same width of fabric in the two stitches, the scaling factor is $\frac{24}{20} = \frac{6}{5}$. Thus we let $p = 6$ and $q = 5$. Also, pattern A uses a multiple of 12 stitches plus 2, while pattern B uses a multiple of 9 stitches plus 4. To summarize, we have $m = 12, n = 9, a = 2$, and $b = 4$. Now we compute the various greatest common divisors: $g_1 = \gcd(q, m) = 1, g_2 = \gcd(p, n) = 3$ and $d = \gcd(\frac{m}{g_1}, \frac{n}{g_2}) = \gcd(12, 3) = 3$. Thus we need r to be congruent to $ap - bq \bmod 9$, which gives $r \equiv -8 \bmod 9$. We use $r = 1$ and the two equations to solve are:

$$5k \equiv 2 - 1 \bmod 12$$
$$6k \equiv 4 - 1 \bmod 9$$

These can be solved and simpified either by using techniques from number theory or by simply trying values between 0 and the modulus for k. The unique solution for the congruence modulo 12 is $k \equiv 5 \bmod 12$. Using Proposition 2, this is equivalent by decomposition to $k \equiv 1 \bmod 4$ and $k \equiv 2 \bmod 3$. The second congruence equation has the three solutions $k \equiv 2$, 5, or 8 mod 9. These three together are equivalent to $k \equiv 2 \bmod 3$. Putting it all together, the system to be solved is

$$k \equiv 1 \bmod 4,$$
$$k \equiv 2 \bmod 3.$$

Finally, the Chinese Remainder Theorem is applied to find that $k = 5 \bmod 12$ is a solution, meaning that k could equal 5, 17, 29, and so on. Then $x = qk + r$, so $x = 26, 86, 146$, and so on.

3.3 Conclusion

This exploration of an interesting variation of the Chinese Remainder Theorem was inspired by realizing the connection the theorem has to knitting stitch patterns. It has been, and will continue to be, true that professional knitwear designers probably solve these problems in an ad hoc way, not using the Chinese Remainder Theorem. The actual numbers involved tend to be fairly small and amenable to an approach of just "puzzling it out" with paper and pencil. Nevertheless, the investigation itself is of interest and not something that would have occurred to me without the knitting inspiration. Thinking about the scaling factor for the stitch patterns led me to a rich investigation of a topic that, as far as I know, has never been considered before. It is a reminder that mathematics can be found in, or inspired by, a great variety of topics. Being open to the connections between mathematics and needlework can lead to new and fun mathematical ideas.

4 The Needlework

4.1 Using the CRT to Design a Cowl

We now present a sample pattern for a cowl. The cowl, modeled by the author in Figure 4, is a study in striping patterns and works up quickly in worsted weight yarn in two colors. Experiment with high-contrast colors, or with more subtle blendings of color. You can even try making it with a variety of scraps. The changing stitch patterns keep interest high as you observe the different color combinations.

The first pattern consists of the bold zigzag stripes of the Garter Stitch Chevron from *The Harmony Guide to Knitting Stitches* [3, p. 35]. The second striped pattern is the Little Tents from Barbara Walker's *A Treasury of Knitting Patterns* [7, p. 103], adapted for use with two colors. The Garter Stitch Chevron uses a multiple of 11 stitches. The Little Tents pattern uses a multiple of 8 stitches plus 1. Thus we need to solve

$$x \equiv 1 \bmod 8,$$
$$x \equiv 0 \bmod 11.$$

Using the Chinese Remainder Theorem, we get $x = 33 + 88k$ for any integer k. To make a relatively small cowl in a light worsted weight, 33 stitches is a good width. For a wide stole, one could use $33 + 88 = 121$ stitches.

After I began knitting with the two patterns above, I decided that one more pattern was needed to balance the colors. Luckily, I hit upon the Two-Color Star Stitch [8, p. 92], which happened to be a multiple of 3. As 33 is already a multiple of 3, it was easy to incorporate this pattern as well. Finally, I added 3 stitches on each side for a garter-stitch border, increasing the cast-on number of stitches to 39 and modifying the stitch instructions to include these border stitches.

4.2 The Chinese Remainder Theorem Cowl—A Striping Stitch Sampler

Notes

- ⋆ The pattern is written using a provisional cast-on, from which stitches are picked up and then joined to the other end of cowl using a 3-needle bind-off. Alternatively, the stitches could be kitchenered instead of the 3-needle bind-off. Finally, if a long-tail or other standard cast-on is used, the cowl can be bound off normally and then seamed with a darning needle.
- ⋆ Which color is considered the main color and which is the contrast color switches between color A and color B as the pattern progresses.
- ⋆ The Little Tents pattern requires the use of a circular needle or a double-pointed needle, because the contrasting color is worked in only one row at a time. Sometimes you may have to turn the work and slide the stitches to the other end of the circular needle in order to pick up the contrasting color and knit.
- ⋆ The instructions for the three-stitch garter border are integrated into the row-by-row instructions.

Figure 4. The author modeling her Chinese Reminder Theorem Cowl.

Gauge

18 stitches per 4 inches on size 7 needles in stockinette stitch (gauge not crucial for this project).

Supplies

- ⋆ Worsted or light worsted weight yarn in two colors, one skein in color A and one skein in color B. Purple and green sample shown was knit in Berroco Vintage, using less than 50 g of each color, with purple for A and green for B. The teal and mustard sample is made from Cascade 220.
- ⋆ US 7 circular needles, or size needed to obtain gauge.
- ⋆ Darning needle, for seaming or grafting.

Garter Chevron | Two-Color Star Stitch | Little Tents

Figure 5. The three stitch patterns in the Chinese Remainder Theorem cowl, knit in Cascade 220.

Abbreviations

k2tog	knit 2 together (right-leaning decrease)
ssk	slip 2 stitches knitwise to right hand needle. Then slip these two stitches back to left-hand needle and knit 2 together through the back loop (left-leaning decrease)
sl1	at beginning of row, slip one stitch knitwise with yarn in back.
sl5 pwise wyib	hold working yarn in back, slip 5 stitches purlwise.
sl5 pwise wyif	hold working yarn in front, slip 5 stitches purlwise.
kfb	knit into the front and then the back of the next stitch (one stitch increased)
yo	yarn over needle
Pull up stitch	On RS, put tip of right-hand needle under slipped strand from 2 rows below. Knit the next stitch, pulling new stitch under the slipped strand as well. On WS, put tip of right-hand needle under slipped strand from 2 rows below. Lift this strand onto left-hand needle, then purl together with next stitch on left-hand needle.
MC	Main color
CC	Contrasting color

Stitch Patterns

⋆ *Garter Stitch Chevrons (starting on WS).* A multiple of 11 stitches plus 6.

Rows 1-5: (in CC) Sl1, Knit across.

Rows 6, 8, 10: (in MC) Sl1, k2, *K2tog, k2, kfb, kfb, k3, ssk*. Repeat from * to * across to last three stitches, k3.

Rows 7, 9, 11: (in MC) Sl1, k2, purl to last three stitches, k3.

Row 12: (in CC) Sl1, k2, *K2tog, k2, kfb, kfb, k3, ssk*. Repeat from * to * across to last three stitches, k3.

⋆ *Little Tents (starting on WS, worked over 39 stitches).* A multiple of 8 stitches plus 7. This pattern alternates 3 rows of MC with 1 row of CC. Because these are odd numbers, the needed strand of yarn will sometimes be on the left and sometimes on the right side of the work when starting a particular row. You must use a circular needle so that you can slide the stitches to the other side and work the appropriate row. (If you don't have a circular needle you can get a different look by instead knitting 2 rows of MC and 2 rows of CC. One way to do this is to knit rows 1 and 6 in MC, then rows 7 and 8 in CC. Repeat these four rows.)

Row 1: (WS, MC) Sl1, k4, *p5, k3*. Repeat from * to * across to last five stitches, k5.

Row 2: (RS, CC) Sl1, k4, *sl5 pwise wyif, k3*. Repeat from * to * across to last five stitches, k5.

Row 3: (RS, MC) Slide stitches to other side of circular needle and pick up strand in MC. Sl1, p4, *k5, p3*. Repeat from * to * across to last five stitches, p5.

Row 4: (WS, MC) Sl1, p6, *pull up stitch, p7*. Repeat from * to * across to last seven stitches, p7.

Row 5: (RS, MC): Sl1, p4, *k5, p3*. Repeat from * to * across to last five stitches, p5.

Row 6: (WS, CC) Sl1, p4, *sl5 pwise wyib, p3*. Repeat from * to * across to last five stitches, p5.

Row 7: (WS, MC) Slide stitches to other side of circular needle and pick up strand in MC. Sl1, k4, *p5, k3*. Repeat from * to * across to last five stitches, k5.

Row 8: (RS, MC) Sl1, k6, *pull up stitch, k7*. Repeat from * to * across to last seven stitches, k7.

⋆ *Two-color Star Stitch (starting on WS).* A multiple of 3 stitches.

Row 1: (WS,MC) Sl1, k2, purl to last 3 stitches, k3.

Row 2: (RS,MC) Sl1, K3, *k3, pull the first of these three over the other two and off the right hand needle, yo.* Repeat from * to * across to last 5 stitches, k5.

Row 3: (WS,CC) Sl1, k2, purl to last 3 stitches, k3.

Row 4: (RS,CC) Sl1, K4, *yo, k3, pull the first of these three over the other two and off the right hand needle.* Repeat from * to * across to last 4 stitches, k4.

Instructions

Using color A and waste yarn, provisionally cast on 39 stitches (or use a long-tail cast-on, see Notes).

Every row includes a three-stitch garter-stitch border and a slipped stitch at the beginning. These stitches are included in the stitch pattern directions. You may wish to use stitch markers to remind yourself of these selvage stitches and the location of the beginning and ending of stitch repeats, as needed.

Row	MC	CC	Instructions
1–41	B	A	Working three repeats across, work rows 1–12 of Garter Stitch Chevron three times. Then work rows 1–5 again.
42			Knit one row in color A.
43–70	A	B	Working five repeats across, work rows 1–8 of Little Tents three times. Then work rows 1–4 again.
71–74			Purl 3 rows in color A.
75–118	B	A	Work rows 1–4 of Two-color Star Stitch 11 times. If you wish to increase the length of the cowl, repeat as many times, *k*, as desired and see below for modifications.
119			Purl one row in color A.
120–121			Slide stitches to other side of circular needle and prepare to work with color B. Purl two rows with color B.
122–162	A	B	Working three repeats across, work rows 1–12 of Garter Stitch Chevron three times. Then work rows 1–5 again.
163			Knit one row in color B.
164–191	B	A	Working five repeats across, work rows 1–8 of Little Tents three times. Then work rows 1–4 again.
192–197			Knit 6 rows in color B.

An easy way to change the circumference of the cowl is to increase the number of repeats of the Two-color Star Stitch starting at Row 75. If the Star Stitch is worked k times, let $\alpha = 75 + 4k$. Then the modified row numbers are as follows.

Original Row Number	Modified Row Number
118	α
119	$\alpha + 1$
120–121	$\alpha + 2$ to $\alpha + 3$
122–162	$\alpha + 4$ to $\alpha + 44$
163	$\alpha + 45$
164–191	$\alpha + 46$ to $\alpha + 73$
192–197	$\alpha + 74$ to $\alpha + 79$

Undo the provisional cast on and place live stitches from cast-on edge to the second needle. Using a third needle, bind off using a 3-needle bind-off to seam the cast-on and bind-off edges together. (Alternatively, if a standard cast-on like the long-tail cast-on was used, bind off and seam the cast-on edge to the bind-off edge. See Notes.)

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