How to change coins, M&M's, or chicken nuggets:
The linear Diophantine problem of Frobenius

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Summary

Let's imagine that we introduce a new coin system. Instead of using pennies, nickels, dimes, and quarters, let's say we agree on using 4-cent, 7-cent, 9-cent, and 34-cent coins. The reader might point out the following flaw of this new system: certain amounts cannot be exchanged, for example, 1, 2, or 5 cents. On the other hand, this deficiency makes our new coin system more interesting than the old one, because we can ask the question: "which amounts can be changed?" In the next section, we will prove that there are only finitely many integer amounts that *cannot* be exchanged using our new coin system. A natural question, first tackled by Ferdinand Georg Frobenius and James Joseph Sylvester in the 19th century, is: "what is the *largest* amount that cannot be exchanged?" As mathematicians, we like to keep questions as general as possible, and so we ask: given coins of denominations $a_1, a_2, \ldots, a_d$, which are positive integers without any common factor, can you give a formula for the largest amount that cannot be exchanged using the coins $a_1, a_2, \ldots, a_d$? This problem is known as the *Frobenius coin-exchange problem*. One of the appeals of this famous problem is that it can be stated in every-day language and in many disguises, as the title of these notes suggests. To be precise, suppose we're given a set of positive integers

$$A = \{a_1, a_2, \ldots, a_d\}$$

with $\gcd(a_1, a_2, \ldots, a_d) = 1$ and we call an integer k *representable* (in terms of A) if there exist nonnegative integers $m_1, m_2, \ldots, m_d$ such that

$$k = m_1 a_1 + \cdots + m_d a_d \ .$$

In the language of coins, this means that we can exchange the amount k using the coins $a_1, a_2, \ldots, a_d$. The Frobenius problem (often called the *linear Diophantine problem of Frobenius*) asks us to find the largest integer that is not representable. We call this largest integer the *Frobenius number* and denote it by $g(a_1, \ldots, a_d)$. In the worksheet questions we will outline a proof for the folklore result for $d = 2$:

$$g(a, b) = ab - a - b \ .$$

This simple-looking formula for $g(a, b)$ inspired a great deal of research into formulas for the Frobenius number $g(a_1, a_2, \ldots, a_d)$, with limited success: While it is safe to assume that the $d = 2$ solution has been known for more than a century, no analogous formula exists for $d \geq 3$. The case $d = 3$ is solved algorithmically, i.e., there are efficient algorithms to compute $g(a, b, c)$ [7, 9, 10], and in form of a semi-explicit formula [8, 14]. The Frobenius problem for fixed $d \geq 4$ has been proved to be computationally feasible [1, 11], but not even an efficient practical algorithm for $d = 4$ is known.

A second classic theorem for the case $d = 2$, which Sylvester posted as a math problem in the *Educational Times* [18], concerns the *number* of non-representable integers. Sylvester proved that exactly half of the integers between 1 and $(a - 1)(b - 1)$ are representable (in terms of a and b). In other words, there are exactly $\frac{1}{2}(a - 1)(b - 1)$ non-representable integers. We will also outline a proof of Sylvester's Theorem.

Notes to the instructor

The first nine worksheet questions are suitable for any course in which the students discuss gcd's and the Euclidean algorithm. The next few questions assume some basic number theory, in particular, knowledge about the greatest-integer function and inverses in $\mathbb{Z}_n$. The different projects naturally vary in depth. Most problems in the Euclidean algorithm section are elementary; the slightly more complicated ones have a hint attached to them. The problems in the counting function section are a bit more advanced but should be doable in, e.g., an elementary number theory class. Further extensions and student research projects are discussed below.

The idea of the proofs hidden in the projects of the Euclidean algorithm section first appeared in [12], to the best of my knowledge. The questions in the counting function section are from [4].

Extensions: Beyond $d = 2$

This section includes an outline of what is known for the general Frobenius problem and some open problems, many of which are suitable for computational exploration and undergraduate research projects. One such extension was already mentioned in Question 11. For more, we refer to the research monograph [15]; it includes more than 400 references to articles written about the Frobenius problem.

To give the state of the art for the case $d = 3$ and beyond, we define the *generating function* of all representable integers, given some fixed parameters $a_1, a_2, \ldots, a_d$ with no common factor, as $F(x) := \sum_{k \text{ representable}} x^k$. One can prove that this generating function can always be written as a rational function of the form

$$F(x) = \sum_{k \text{ representable}} x^k = \frac{p(x)}{(1-x^{a_1})\,(1-x^{a_2})\cdots(1-x^{a_d})}\,.$$

Furthermore, in the case $d = 2$ one can show that $F(x) = 1 - x^{a_1 a_2}$. Denham [8] recently discovered the remarkable fact that for $d = 3$, the polynomial p in the numerator has either 4 or 6 terms. He gave semi-explicit formulas for p, from which one can deduce a semi-explicit formula for the Frobenius number $g(a, b, c)$. This formula was independently found by Ramírez-Alfonsín [14]. As we already remarked in the introduction, there is no "easy" formula for $d = 3$ that would parallel Theorem 2. However, Denham's theorem implies that the Frobenius number in the case $d = 3$ is quickly computable, a result that is originally due, in various guises, to Herzog [10], Greenberg [9], and Davison [7].

As much as there seems to be a well-defined border between the cases $d = 2$ and $d = 3$, there also seems to be such a border between the cases $d = 3$ and $d = 4$: Bresinsky [6] proved that for $d \geq 4$, there is no absolute bound for the number of terms in p, in sharp contrast to Denham's theorem.

On the other hand, Barvinok and Woods [1] proved recently that for fixed d, the rational generating function F can be written as a "short" sum of rational functions; in particular, F can be efficiently computed when d is fixed. A corollary of this fact is that the Frobenius number can be efficiently computed when d is fixed; this theorem is due to Kannan [11]. On the other hand, Ramírez-Alfonsín [13] proved that trying to efficiently compute the Frobenius number is hopeless if d is left as a variable. While these results settle the theoretical complexity of the computation of the Frobenius number, practical algorithms are a completely different matter. Both Kannan's and Barvinok-Woods' ideas seem complex enough that nobody has yet tried to implement them. The fastest known algorithm is due to Beihoffer, Nijenhuis, Hendry and Wagon [5]; it is currently being improved by Einstein, Lichtblau, and Wagon.

We conclude with a few projects. These differ distinctively from the questions of the worksheet in that they constitute open research problems. I list them in what I find decreasing order of difficulty (an estimate that is naturally subjective); the later projects are most suitable for undergraduate research and computational experiments that should bring new insights.

Project 1. Come up with a new approach or a new algorithm for the Frobenius problem in the $d \geq 3$ cases.

Project 2. There is a very good lower [7] and several upper bounds [15, Chapter 3] for the Frobenius number. Come up with improved upper bounds.

Project 3. Study vector generalizations of the Frobenius problem [16, 17], which seem for the most part unexplored.

Project 4. There are several special cases of $A = \{a_1, a_2, \ldots, a_d\}$ for which the Frobenius problem is solved, for example, arithmetic sequences [15, Chapter 3]. Extend these special cases and come up with new ones.

Project 5. Study the generalized Frobenius number g_j (defined in Question 11): Derive formulas for special cases, e.g., arithmetic sequences.

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Worksheet on how to change coins, M&M's, or chicken nuggets: The linear Diophantine problem of Frobenius

1 The Euclidean algorithm and its consequences

We approach the Frobenius problem through the following important consequence of the Euclidean algorithm.

Theorem 1. *Suppose a and b are relatively prime positive integers. Then there exist $m, n \in \mathbb{Z}$ such that $1 = ma + nb$.*

What we really need is the fact that one can find such an *integral linear combination* of a and b for any integer:

Corollary 2. *Suppose a and b are relatively prime positive integers. Given an integer k, there exist $m, n \in \mathbb{Z}$ such that $k = ma + nb$.*

Students who just learned about the Euclidean algorithm might find the Frobenius problem amusing, since this last corollary *almost* solves the Frobenius problem: in the latter, we're "only" asking that $m, n \in \mathbb{Z}$ are *nonnegative*. It is this tiny additional condition that makes the Frobenius problem so hard (and interesting!). Let's put the Euclidean algorithm to good use.

Question 1. Suppose a and b are relatively prime positive integers. Show that a given integer k can be *uniquely* written as

$$k = ma + nb\ ,$$

where $m, n \in \mathbb{Z}$ and $0 \leq m < b - 1$.

This gives a simple but useful criterion for k to be representable — recall that this means that k can be written as a nonnegative integral linear combination of a and b.

Question 2. Suppose a and b are relatively prime positive integers, and write $k \in \mathbb{Z}$ as $k = ma + nb$ where $m, n \in \mathbb{Z}$ with $0 \leq m \leq b - 1$. Show that k is representable (in terms of a and b) if and only if $n \geq 0$.

This observation allows us to conclude, among other things, that the Frobenius problem is well defined:

Question 3. Suppose a and b are relatively prime positive integers. Show that every sufficiently large integer is representable (in terms of a and b).

Question 4. Prove that the general Frobenius problem is well defined. That is, show that, given relatively prime $a_1, a_2, \ldots, a_d$, every sufficiently large integer is representable (in terms of $a_1, a_2, \ldots, a_d$).

Question 2 can be taken a step further to solve the Frobenius problem for $d = 2$:

Question 5. Prove that $g(a, b) = ab - a - b$.

Hint: Try to maximize possible non-representable integers, using Question 2.

Question 2 can also be used to prove Sylvester's Theorem. We start with the following:

Question 6. Suppose a and b are relatively prime positive integers and $0 < k < ab$ is not divisible by a or b. Prove that k is representable (in terms of a and b) if and only if $ab - k$ is not representable.

Hint: Use Question 2 for a representable integer k. Think about how you can strengthen the conditions of Question 2 using the divisibility properties.

Question 6 allows us to prove Sylvester's Theorem:

Question 7. Prove that there are $\frac{1}{2}(a - 1)(b - 1)$ non-representable integers.

2 A counting function

Now we study the counting sequence

$$r_k = \#\left\{(m,n) \in \mathbb{Z}^2 : \ m,n \geq 0, \ ma + nb = k\right\}$$

where a and b are fixed relatively prime positive numbers. In words, r_k counts the representations of $k \in \mathbb{Z}_{\geq 0}$ as nonnegative linear combinations of a and b. Question 3 states that this sequence has only finitely many r_k's that are 0, and the Frobenius problem asks for the largest among the r_k's that is 0.

Question 2 gives us the following almost-periodicity identity for r_k.

Question 8. Suppose a and b are relatively prime positive integers, and let r_k be given as above. Then

$$r_{k+ab} = r_k + 1 \ .$$

Remark: There is no analogous formula in the general case of d parameters $a_1, a_2, \ldots, a_d$. This is one reason why the Frobenius problem seems to be intractable for $d \geq 3$.

Let's take a moment to look at a geometric interpretation of r_k. As usual, fix two relatively prime positive integers a and b. Consider the line segment $L_k = \{(x,y) \in \mathbb{R}^2 : \ x,y \geq 0, \ ax + by = k\}$. The parameter k acts like a dilation factor of the line segment L_1 given by

$$L_1 = \left\{(x,y) \in \mathbb{R}^2 : \ x,y \geq 0, \ ax + by = 1\right\} .$$

Our counting sequence r_k enumerates integer points in $\mathbb{Z}^2$ that lie on the line segment L_k. As k increases, the line segment gets dilated. It is not too far fetched[1] to expect that the likelihood for an integer point to lie on the line segment L_k increases with k. In fact, one might even guess that this "probability" increases linearly with k, as the line segments are one-dimensional objects. Below we will give a formula (Theorem 3) which shows that this is indeed the case. Figure 1 shows the geometry behind the counting function r_k for the first few values of k in the case $a = 4, b = 7$. Note that

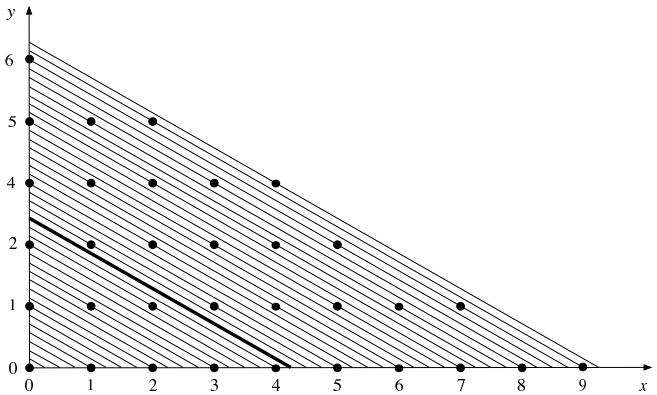

Figure 1. $4x + 7y = k \ , \qquad k = 1, 2, \ldots$

the thick line segment for the Frobenius number $k = 17 = 4 \cdot 7 - 4 - 7$ is the last one that does not contain any integer point.

Similar geometric pictures can be associated to the general Frobenius counting functions

$$\#\left\{(m_1, m_2, \ldots, m_d) \in \mathbb{Z}^d : \ \text{all } m_j \geq 0, \ m_1 a_1 + \cdots + m_d a_d = k\right\} .$$

Now the line segments get replaced by triangles ($d = 3$), tetrahedra ($d = 4$), and higher-dimensional simplices, but the general picture, namely that these counting functions enumerate integer points in $\mathbb{Z}^d$ in dilates of nice geometric

[1] However, one should be careful with such a statement—we invite the reader to prove that if a and b are not relatively prime, there are infinitely many line segments L_k that do not contain any integer point.

objects, stays the same. This geometric interpretation gives a glimpse into a subfield of discrete geometry called *Ehrhart theory*. It concerns the study of integer-point enumeration in *polytopes*, of which line segments, triangles, tetrahedra, etc., are special cases. The reader interested in these topics may consult [3]. There one can find a proof of the following beautiful formula for r_k due to Tiberiu Popoviciu, which we will use to re-derive some results on the Frobenius problem. First we need to define the *greatest-integer function* $\lfloor x \rfloor$, which denotes the greatest integer less than or equal to x. A close sibling to this function is the *fractional-part function* $\{x\} = x - \lfloor x \rfloor$.

Theorem 3 (Popoviciu)**.** *If a and b are relatively prime, the counting function*

$$r_k = \#\left\{(m,n) \in \mathbb{Z}^2 : \ m,n \geq 0, \ ma + nb = k\right\}$$

is explicitly given by

$$r_k = \frac{k}{ab} - \left\{\frac{b^{-1}k}{a}\right\} - \left\{\frac{a^{-1}k}{b}\right\} + 1 \, ,$$

where $b^{-1}b \equiv 1 \bmod a$ and $a^{-1}a \equiv 1 \bmod b$.

Remark: There are analogous formulas for the general Frobenius counting functions

$$\#\left\{(m_1, m_2, \ldots, m_d) \in \mathbb{Z}^d : \ \text{all } m_j \geq 0, \, m_1 a_1 + \cdots + m_d a_d = k\right\}$$

but they are not as simple as in Popoviciu's Theorem, even if $d = 3$. These "higher-dimensional" counting functions, nevertheless, give rise to generalized *Dedekind sums*, finite arithmetic sums that appear in various other mathematical contexts [2].

Question 9. Using Popoviciu's Theorem 3, give an alternative proof of formula $g(a,b) = ab - a - b$ for the Frobenius number by proving that $r_{ab-a-b} = 0$ and that $r_k > 0$ for every $k > ab - a - b$.

Hint: Use the periodicity of $\{x\}$ and the inequality $\left\{\frac{m}{a}\right\} \leq 1 - \frac{1}{a}$ for integers m, a.

Question 10. Using Popoviciu's Theorem 3, give an alternative proof that $r_k + r_{ab-k} = 1$ for any integer $1 \leq k \leq ab - 1$ that is not divisible by a or b (cf. Question 6), and use this to give another proof of Sylvester's Theorem.

Recall that Question 6 allowed us to prove Sylvester's Theorem, so Question 10 gives an alternate proof of Sylvester's Theorem.

Question 11. Given two relatively prime positive integers a and b, we say the integer k is *j-representable* if there are exactly j solutions $(m,n) \in \mathbb{Z}^2_{\geq 0}$ to $ma + nb = k$. We define g_j as the largest j-representable integer. (So g_0 is the Frobenius number.) Prove:

(a) g_j is well defined.

(b) $g_j = (j+1)ab - a - b$.

(c) Given $j \geq 2$, the smallest j-representable integer is $ab(j-1)$.

(d) There are exactly $ab - 1$ integers that are uniquely representable.

(e) Given $j \geq 2$, there are exactly ab j-representable integers.

Solutions

1. We mentioned already in Corollary 2 that any integer k can be written as

$$k = ma + nb$$

for some $m, n \in \mathbb{Z}$. From this representation we get others, for example,

$$k = (m + b)a + (n - a)b \qquad \text{or} \qquad k = (m - b)a + (n + a)b \; .$$

In fact, because a and b are relatively prime, *all* possible representations of k as integral linear combinations of a and b are given precisely by the expressions

$$k = (m + jb)a + (n - jb)b \; , \qquad j \in \mathbb{Z} \; .$$

By choosing j accordingly, we can force the coefficient of a to be in the interval $[0, b - 1]$.

2. If $n \geq 0$, then k is representable by definition, since both coefficients m and n in $k = ma + nb$ are nonnegative. Conversely, suppose k is representable, say $k = ja + lb$ for some nonnegative integers j and l. If $0 \leq j \leq b-1$, we are done; otherwise, we subtract enough multiples of b from j such that $0 \leq m = j - qb \leq b - 1$. Then the coefficient l has to be adjusted to $n = l + qa$, which is positive.

3. Question 2 implies that every integer $k \geq ab$ is representable, since when writing $k = ma + nb$ with $0 \leq m \leq b - 1$, n has to be positive.

4. Given an integer k, the Euclidean algorithm asserts the existence of integers $m_1, m_2, \ldots, m_d$ such that k can be represented as $k = m_1a_1 + m_2a_2 + \cdots + m_da_d$. With the same argument as in the solution to Question 1, we can demand that in this representation $0 \leq m_2, m_3, \ldots, m_d < a_1$, and by extension of Question 2, k is representable if and only if $m_1 \geq 0$. Hence certainly all integers beyond $a_1\,(a_2 + a_3 + \cdots + a_d)$ are representable in terms of $a_1, a_2, \ldots, a_d$.

5. By Question 2, we have to maximize the integral coefficients m and n in

$$k = ma + nb \; ,$$

subject to $0 \leq m \leq b - 1$ and $n < 0$ (so that k is not representable). The maximal choice is apparently

$$k = (b - 1)a + (-1)b = ab - a - b \; .$$

6. Suppose k is representable, so by Question 2 we can write

$$k = ma + nb$$

for some nonnegative integers m and n with $0 \leq m \leq b - 1$. Since k is not divisible by a or b, we have $m \neq 0$ and n is not divisible by a; in particular, n is positive. But then

$$ab - k = ab - ma - nb = (b - m)a - nb \; ,$$

and we note that $0 < b - m < b$ and $n > 0$. This means that $ab - k$ can be written in the form $ab - k = ja + lb$ with $0 \leq j \leq b - 1$ and $l < 0$, and by Question 2, $ab - k$ is not representable.

7. Question 6 implies that, for k between 1 and $ab - 1$ and not divisible by a or b, exactly one of k and $ab - k$ is representable. There are

$$ab - a - b + 1 = (a - 1)(b - 1)$$

integers between 1 and $ab - 1$ that are not divisible by a or b. Finally, if k is divisible by a or b then it is representable, simply by writing k as a multiple of a or b. Hence the number of nonrepresentable integers is $\frac{1}{2}(a - 1)(b - 1)$.

8. Question 2 implies that if k is representable then it can be written as

$$k = ma + nb$$

for some nonnegative integers m and n with $0 \le m \le b-1$. If $n \ge a$ then we get another representation, namely,

$$k = (m+b)a + (n-a)b\ .$$

We can continue the process of adding b to the coefficient of a and subtracting a from the coefficient of b, until the latter becomes negative, and those will be precisely the different representations of k. Suppose j is the largest integer such that $n - ja \ge 0$. That is, k has the $j+1$ representations

$$k = ma + nb = (m+b)a + (n-a)b = (m+2b)a + (n-2a)b = \cdots = (m+jb)a + (n-ja)b\ .$$

Then $k + ab$ has the $j+2$ representations

$$k + ab = ma + (n+a)b = (m+b)a + nb = (m+2b)a + (n-a)b = \cdots = (m+(j+1)b)a + (n-ja)b\ ,$$

precisely one representation more than k has.

9. We have to show that $r_{ab-a-b} = 0$ and that $r_{ab-a-b+n} > 0$ for any positive n. To prove the first assertion, we compute with Popoviciu's Theorem 3,

$$\begin{aligned} r_{ab-a-b} &= \frac{ab-a-b}{ab} - \left\{\frac{b^{-1}(ab-a-b)}{a}\right\} - \left\{\frac{a^{-1}(ab-a-b)}{b}\right\} + 1 \\ &= 2 - \frac{1}{a} - \frac{1}{b} - \left\{\frac{-b^{-1}b}{a}\right\} - \left\{\frac{-a^{-1}a}{b}\right\}\ . \end{aligned}$$

Since $b^{-1}b = 1 + ja$ for some integer j, $\left\{\frac{-b^{-1}b}{a}\right\} = \left\{\frac{-1}{a}\right\} = 1 - \frac{1}{a}$. With essentially the same argument, we conclude that $\left\{\frac{-a^{-1}a}{b}\right\} = 1 - \frac{1}{b}$, which implies that $r_{ab-a-b} = 0$.

To prove that $r_{ab-a-b+n} > 0$ for $n > 0$, we note that for any integer m, $\left\{\frac{m}{a}\right\} \le 1 - \frac{1}{a}$. Hence Popoviciu's Theorem 3 gives for any positive integer n,

$$r_{ab-a-b+n} \ge \frac{ab-a-b+n}{ab} - \left(1 - \frac{1}{a}\right) - \left(1 - \frac{1}{b}\right) + 1 = \frac{n}{ab} > 0\ .$$

10. By Popoviciu's Theorem 3,

$$\begin{aligned} r_{ab-k} &= \frac{ab-k}{ab} - \left\{\frac{b^{-1}(ab-k)}{a}\right\} - \left\{\frac{a^{-1}(ab-k)}{b}\right\} + 1 \\ &= 2 - \frac{k}{ab} - \left\{\frac{-b^{-1}k}{a}\right\} - \left\{\frac{-a^{-1}k}{b}\right\} \\ &\overset{(\star)}{=} -\frac{k}{ab} + \left\{\frac{b^{-1}k}{a}\right\} + \left\{\frac{a^{-1}k}{b}\right\} \\ &= 1 - r_k\ . \end{aligned}$$

Here, $(\star)$ follows from the fact that $\{-x\} = 1 - \{x\}$ if $x \notin \mathbb{Z}$.

11. (a) Since every integer beyond $ab-a-b$ has at least one representation, every integer beyond $(j+1)ab-a-b$ has at least $j+1$ representations, by Question 8.

(b) As we just showed, every integer beyond $(j+1)ab-a-b$ has at least $j+1$ representations. Furthermore, by the formula for $g(a,b)$ and Question 8, $(j+1)ab - a - b$ has exactly j representations, and so $g_j = (j+1)ab - a - b$.

(c) Let n be a nonnegative integer. Then

$$\begin{aligned} r_{ab(j-1)-n} &= \frac{ab(j-1)-n}{ab} - \left\{ \frac{b^{-1}(ab(j-1)-n)}{a} \right\} - \left\{ \frac{a^{-1}(ab(j-1)-n)}{b} \right\} + 1 \\ &= j - \frac{n}{ab} - \left\{ \frac{-b^{-1}n}{a} \right\} - \left\{ \frac{-a^{-1}n}{b} \right\} . \end{aligned}$$

If $n = 0$, this equals j. If n is positive, we use the fact that $\{x\} \geq 0$ to see that

$$r_{ab(k-1)-n} \leq j - \frac{n}{ab} < j \ .$$

(d) In the interval $[1, ab]$, there are, by Sylvester's Theorem and the fact that ab is the smallest 2-representable integer,

$$ab - \frac{(a-1)(b-1)}{2} - 1$$

1-representable integers. With Question 8 and again Sylvester's Theorem, we see that there are

$$\frac{(a-1)(b-1)}{2}$$

1-representable integers above ab. Hence there is a total of $ab - 1$ uniquely representable integers.

(e) It suffices to prove this result for $j = 2$; then the general statement follows by induction with Question 8. By the previous proof and Question 8, there are $ab - \frac{(a-1)(b-1)}{2} - 1$ integers with two representations in the interval $[ab + 1, 2ab]$, and $\frac{(a-1)(b-1)}{2}$ such integers beyond $2ab$. Hence, together with the 2-representable integer ab, there are precisely ab integers with two representations.