Coloring and Counting Rectangles on the Board

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Summary

We describe the Rectangles on the Board project, an adaptation of an activity for elementary and middle school students that appears in [1]. Students are challenged to determine the coloring of the instructor's 10×10 board, given the restrictions that (1) all 100 squares are colored in one of four colors and (2) the colors form four rectangular regions, one in each color. Our extensions of this project involve counting, symmetry, geometry, and logical reasoning. For example, given the color of some squares, students infer the color of other squares based on geometry. In turn, they use logic and their understanding of this geometry to count the minimal number of squares needed to be revealed so that they can determine all of the squares' colors. Similar reasoning leads them to a best next "guess," when playing the game. Students use combinatorics and symmetry to count the number of ways to color the board.

Notes for the instructor

This project is suitable for mathematics courses at all levels. We have implemented versions of the activity in elementary, middle, and high school classes, as well as in undergraduate mathematics and graduate mathematics education courses. The game-like aspect of the project is engaging and appealing to students. The project can be used to spur all-class discussions, or can be used to promote cooperative learning. Depending on how deeply your class desires to delve into the activity and its extensions, it can take from one to three 50-minute class periods to complete. The worksheet does well to foster communication; hence, we recommend that you have students work in groups and end the project with a whole-class discussion of the solutions. Because many acceptable answers exist for some of the problems, grading many worksheets (e.g., if a large class turns in individual worksheets) can be tedious.

The project extends an activity that appears in Math for Girls and Other Problem Solvers [1]. The book is intended for use in elementary and middle school classrooms to encourage equity and participation. We describe the original activity, called the Multi-Color game, and problems that extend from the activity. The problems related to combinatorics, decision making, proofs, and logic can be used in collegiate discrete mathematics courses. Students are asked not only to provide solutions, but also to explain their reasoning, thereby promoting mathematical communication. We have used versions of this project in courses for pre- and in-service teachers to encourage mathematical reasoning, communication, and cooperative learning. In those classes, we lead discussions about whether or not this activity constitutes mathematics, given many secondary school students' rigid belief that mathematics is about computing solutions and manipulating equations. Middle and elementary school teachers often see possibilities for using the activity to discuss geometry, percents, and fractions.

Before class, the instructor colors a 10×10 board on a piece of paper using four colors, each of which forms exactly one rectangle of integer dimensions (see problem 9 on the worksheet for sample boards). We follow [1] which uses a 10×10 board though any size rectangular board will suffice. Using the matrix coordinates, students "guess" by asking the instructor to reveal the color of one square at a time. The instructor colors the corresponding square on an overhead transparency of a blank 10×10 board. After each square is revealed, students are asked whether the color of that square determines the colors of other squares — called "freebies." After a few guesses, students see that if they use logic and what they know about rectangles, they can conserve guesses by asking for squares that reveal more information about the structure of the board. The object of the game is for the class to determine the coloring of the board in as few guesses as possible. After playing the game with the entire class, have students work on the worksheet in small groups.

For the worksheet, we represent four colors by suits of a deck of cards, ♡, ◇, ♣, and ♠. The board is viewed as a matrix in which individual squares are given in (row, column) position with X denoting 10. The first two questions relate to freebies and other strategy-related considerations that students may have encountered during the game. Questions 3–6 concern the different ways in which the board can be partitioned into four rectangles. We lead students to discover that there are six configuration classes if rotation, reflection, and size of rectangles on the board are disregarded. Students are exposed to this idea in two ways — first by identifying equivalent boards based on geometry and second by noticing that the decisions one makes in partitioning the board force the configuration class. In the last four questions of the worksheet, students are asked to use what they know about strategies and configuration classes to count the total number of ways in which the board can be colored, and to determine the minimal number of guesses necessary to complete a given board.

Bibliography

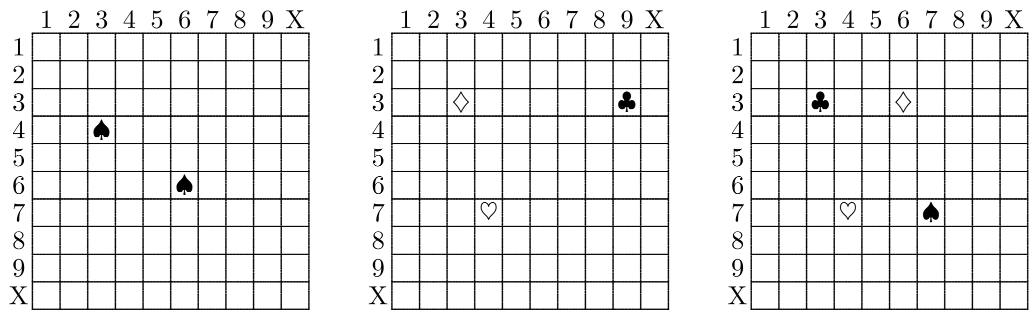
[1] D. Downie, T. Slesnick, and J. K. Stenmark, *Math for Girls and Other Problem Solvers.* EQUALS, Lawrence Hall of Science, Berkeley, CA, 1981.

Worksheet for Coloring and Counting Rectangles on the Board

1. Some partially filled boards are shown in Figure 1. For each board, answer the following questions:

   i) Based on what has already been revealed, how many "freebies" are there, if any? Identify them and justify your answers.

   ii) What is a good next guess? Why?



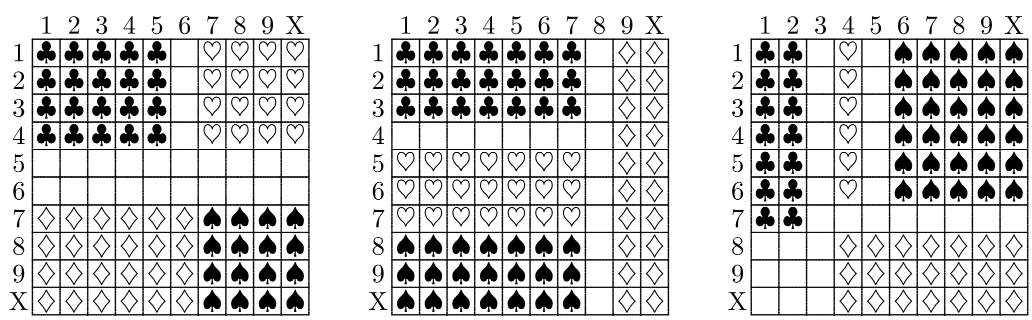
a) *b*) *c*)

Figure 1

2. The boards below in Figure 2 are partially filled in. Assuming that freebies are correctly determined by the guess, answer the following questions:

   i) Best case scenario: What is the minimum number of additional guesses needed to complete the board? Give examples of which squares you would guess, and what suits those squares would have to be in order for you to complete the board in this best case scenario.

   ii) Worst case scenario: What is the maximum number of additional guesses needed to complete the board? Give a sequence of guesses, along with the suits of the revealed squares, that would result in this worst case scenario.



a) *b*) *c*)

Figure 2

3. For each row of four boards in Figure 3, indicate which board is not like the others. Why?

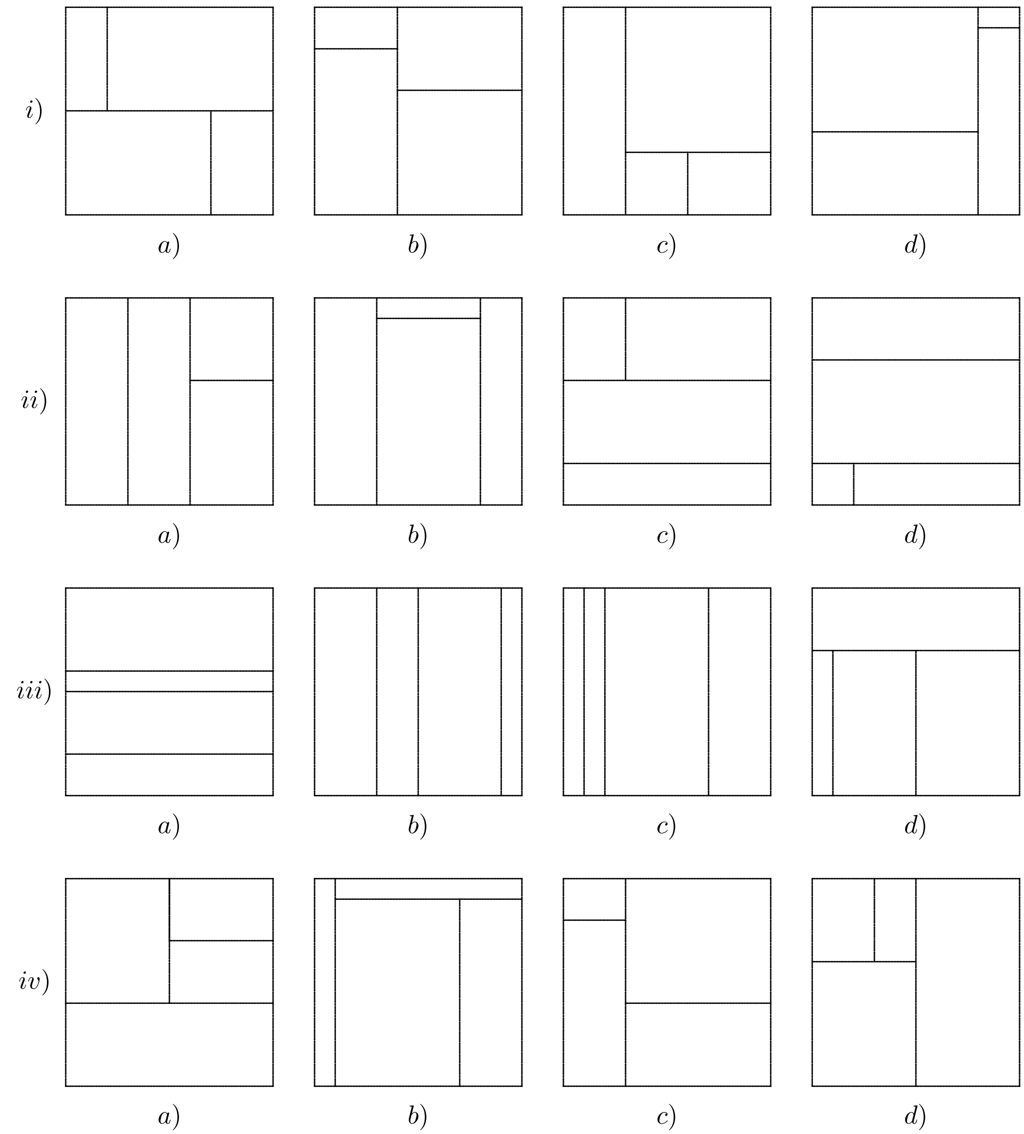


Figure 3

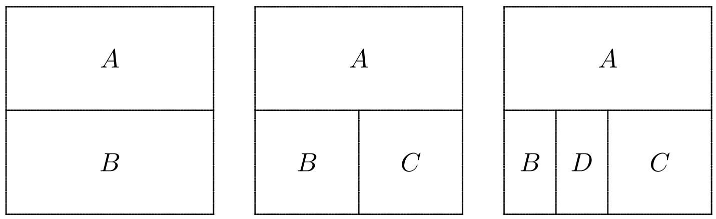
The partitions that belong in the same group for each question in Problem 3 are defined to be in the same *configuration class*. We say that two boards are in the same configuration class if they are equivalent under rotation and reflection, without regard to the size, location, and suit of each rectangle. For example, you should have found that $c)$ in Figure $3i)$ is not equivalent to the others. You cannot rotate, reflect, or "slide" any of the line segments from $a)$, $b)$, or $d)$ to produce $c)$. We will explore another way to consider configuration classes in the next two problems.

4. Notice that each of the partitions in Problem 3 has at least one vertical or horizontal line segment that forms a border between suits and extends from one edge of the board to the opposite edge. Prove that such a segment always exists for a partition of a square into four rectangles.

5. Partitioning the square into four rectangles can be viewed as an iterative process of three steps such that at each step exactly one rectangle is divided into two. The first step is to divide the 10×10 square into two rectangles with a vertical or horizontal line segment. For example, a horizontal segment has been used to divide a square

into rectangles A and B in Figure 4a). The next step is to divide rectangle A or B into two rectangles using either a horizontal or vertical line. For example, B is divided into two rectangles B and C using a vertical line in Figure 4b). To finish the process of dividing the board into four rectangles, one of the rectangles A, B, or C must be divided into two rectangles. In Figure 4c), we divide rectangle B into rectangles B and D. Note that if A is divided (instead of B), the result is different.

Make a decision tree to indicate all possible ways that a board can be divided into four rectangles. Realize that certain decisions are equivalent. A horizontal first line and a vertical first line are indistinguishable because the resulting boards are equivalent under rotation. Partitions are distinguished from one another by the relative positions of the rectangles to one another, but not by the size of the rectangles.

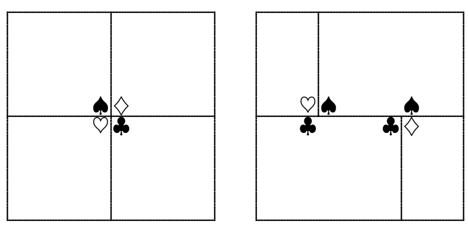


a) b) c)

Figure 4

6. The terminal nodes of the decision tree from problem 5 (here, the nodes resulting from three decisions) also represent configuration classes. How many different configuration classes were produced from your decision tree?

7. For each of the configuration classes you found in problem 6, find the minimal number of guesses required to complete the board. Keep in mind that within a single configuration class, different numbers of minimal guesses may be required depending on the positions of the borders. You can view this as moving the boundaries while staying within the class, but changing the minimal number of guesses. For example, Figure 5 shows boards from the same configuration class that require different minimal numbers of guesses.



a) 4 guesses b) 6 guesses

Figure 5

For each of the other configuration classes find all possible numbers of minimal guesses by drawing representative boards.

8. How many different colorings of the board are possible? Hint: Use the configuration classes to guide your work. Realize that two boards in the same configuration class that are equivalent under rotation produce different colorings. Further, positions of the borders and the distribution of the suits produce different colorings too.

9. For the completed boards in Figure 6, what is the minimum number of squares that can remain unknown so that more than one board is possible? Give examples of which squares would have to remain unknown in order to achieve this minimal number.

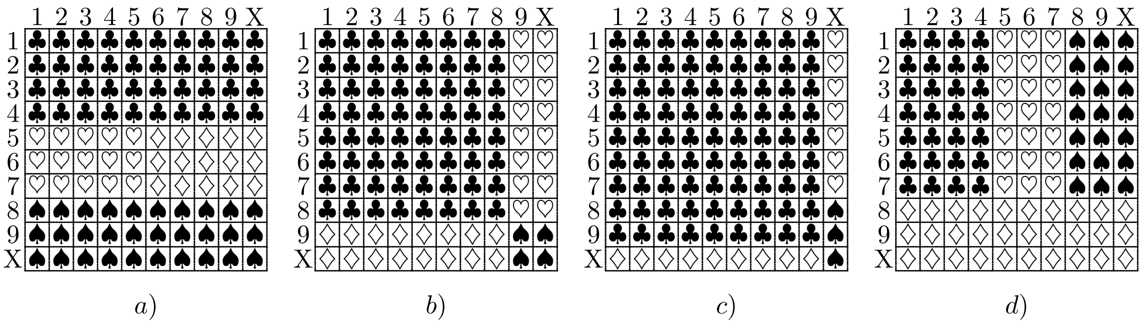


Figure 6

10. Determine the values of n such that knowing a specific n squares still does not complete the board, but knowing an $(n + 1)$st square allows you to complete the board. Use sample boards to explain.

Solutions

1. *i*) Solutions for Figures 1*a*), 1*b*), and 1*c*) appear below.

   1*a*) There are 10 freebies. They are in the rectangle with vertices at (6,3), (6,6), (4,6), and (4,3).

   1*b*) No freebies.

   1*c*) There are 28 freebies. There are 8 ♣ in the rectangle defined by vertices (3,1), (3,3), (1,3), and (1,1). There are 15 ♠ freebies in the rectangle defined by vertices (X,7), (X,X), (7,X), and (7,7). There are 3 ♡ freebies at (8,4), (9,4), and (X,4) and 2 ♢ freebies at (1,6) and (2,6).

   ii) Answers vary. A good guess has the potential to reveal many freebies. For example, in Figure 1*b*), a guess in the rectangle with vertices (X,5), (X,X), (4,X), and (4,5) could reveal a ♠ and lead to many freebies, as the board would have the same structure as Figure 1*c*). Also, students might look for borders between suits or guess squares that have more possibilities for their suits. For the latter, consider the board in Figure 1*b*): squares in the rectangle given by vertices (6,4), (6,8), (1,8), and (1,4) can be any of the four suits.

2. *i*) Answers vary. Examples of acceptable solutions are given for each figure.

   2*a*) 1 guess: If (6,6) is ♡, then we're done.

   2*b*) 1 guess: If (4,8) is ♣, then we're done.

   2*c*) 2 guesses: If (8,3) is ♣, and (7,5) is ♠, then we're done.

   ii) Answers vary. Examples of acceptable solutions are given for each figure.

   2*a*) 5 guesses; 1 guess to reveal the vertical border between ♡ and ♣ and a maximum of 2 each to determine the horizontal borders between ♣ and ♢ and between ♡ and ♠. For example, if (1,6) is ♣; (6,1) is ♢; (5,1) is ♣; (6,X) is ♠; and (5,X) is ♡, then the board is completed.

   2*b*) 2 guesses; 1 guess to reveal the vertical border between ♢ and the other suits and 1 guess to reveal the horizontal border between ♣ and ♡. For example, if (X,8) is ♢, and (4,1) is ♣, then the board is completed.

   2*c*) 4 guesses; 1 guess to reveal the horizontal border between ♢ and the 2 suits ♡ and ♠ above it, 1 guess to reveal the vertical border between ♡ and ♠, 1 guess to reveal the vertical border between ♣ and ♡, and 1 guess to reveal the border between ♣ and ♢. For example, if (7,X) is ♠; (1,5) is ♠; (1,3) is ♡; (X,1) is ♢, then the board is completed.

3. 3*i*) *c*; 3*ii*) *b*; 3*iii*) *d*; 3*iv*) *c*.

4. Consider the four corners of the board. If any adjacent pair is the same color, then one of the four rectangles would extend from one edge of the board to the opposite edge. This rectangle creates either a vertical or horizontal border that traverses the board.

   We find it useful to solve this problem by viewing the board as a unit square, partitioned into four rectangles. If the four corners of the board are colored distinctly, then each colored rectangular region has a single vertex in the interior of the square; the other vertices that define the rectangle are on the boundary of the square. Denote the four rectangular regions as A, B, C, and D such that these contain the vertices of the unit square $(0,1)$, $(1,1)$, $(1,0)$, and $(0,0)$, respectively, and their interior vertices are denoted a, b, c, and d, respectively. Let $a=(x_a,y_a)$, $b=(x_b,y_b)$, $c=(x_c,y_c)$, and $d=(x_d,y_d)$.

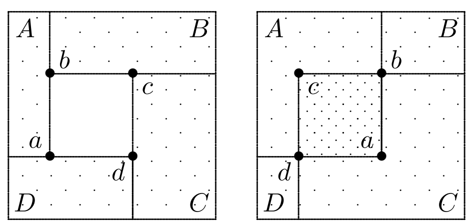
   Rectangles A and B must share a vertical border which implies that $x_a = x_b$. Similarly, C and D share a vertical border and $x_c = x_d$. Because A and D share a horizontal border, then $y_a = y_d$. Likewise, B and C share a horizontal border such that $y_b = y_c$. There are nine ways in which the x and y values of the vertices can be ordered:

$$
\begin{array}{llll}
1)\ y_a < y_b & & & \\
 & a)\ x_a < x_c & b)\ x_a = x_c & c)\ x_a > x_c \\
2)\ y_a = y_b & & & \\
 & a)\ x_a < x_c & b)\ x_a = x_c & c)\ x_a > x_c \\
3)\ y_a > y_b & & & \\
 & a)\ x_a < x_c & b)\ x_a = x_c & c)\ x_a > x_c
\end{array}
$$

Cases $1b$), $2b$) and $3b$) all have $x_a = x_c$. Thus, these colorings have a vertical border that traverses the board. Similarly, cases $2a$), $2b$) and $2c$) all have $y_a = y_b$; these colorings have a horizontal border that traverses the board.

In case $1a$), $y_a < y_b$ and $x_a < x_c$ as pictured in Figure $7a$). The vertices a, b, c and d form a rectangle that is not colored. Hence, this case cannot occur. Due to symmetry, case $3c$) cannot occur for the same reason.

In case $1c$), $y_a < y_b$ and $x_a > x_c$ as pictured in Figure $7b$). The vertices c, b, a and d form a rectangle that is colored twice (as part of rectangles A and C). So, this case cannot occur. Due to symmetry, case $3a$) cannot occur for the same reason.



a) b)

Figure 7

5. The decision tree appears in Figure 8. Because an initial horizontal or vertical division of the square into two rectangles is rotationally equivalent, we identify the first action by the notation $h \sim v$. From solution 4, there exists at least one line segment that extends from one edge of the board to the opposite edge. There can be 3, 2, or 1 such parallel line segments. For 3 parallel line segments, there is only one configuration, a). For 2 such segments, the two possibilities are given by b) and c). For 1, all configurations are accounted for by d), e), and f).

   Notice that the same sequential division of the tree into rectangles does not yield all ways that a square can be divided into five rectangles. As an example, the division of the 10×10 square into rectangles in Figure $7a$) cannot be generated by extending the decision tree approach. This follows because no border traverses the entire length of the 10×10 board while the first line drawn in the decision tree approach traverses the entire board.

6. There are 6 configuration classes. The 2 boards denoted by c) in Figure 8 are equivalent. The other 5 terminal nodes in the decision tree in Figure 8 belong to unique configuration classes.

7. Figure 9 consists of boards for each configuration class that yield all of the possible minimal numbers of guesses. We consider the boards from the terminal nodes of the decision tree in left-to-right order. Terminal node e) is described as part of the problem statement and is shown in Figure 5.

8. Using the terminal nodes of the tree in Figure 8, we count how many ways there are to determine the suits of each configuration class. For node a), of the 9 possible horizontal borders in the 10×10 board, 3 such borders

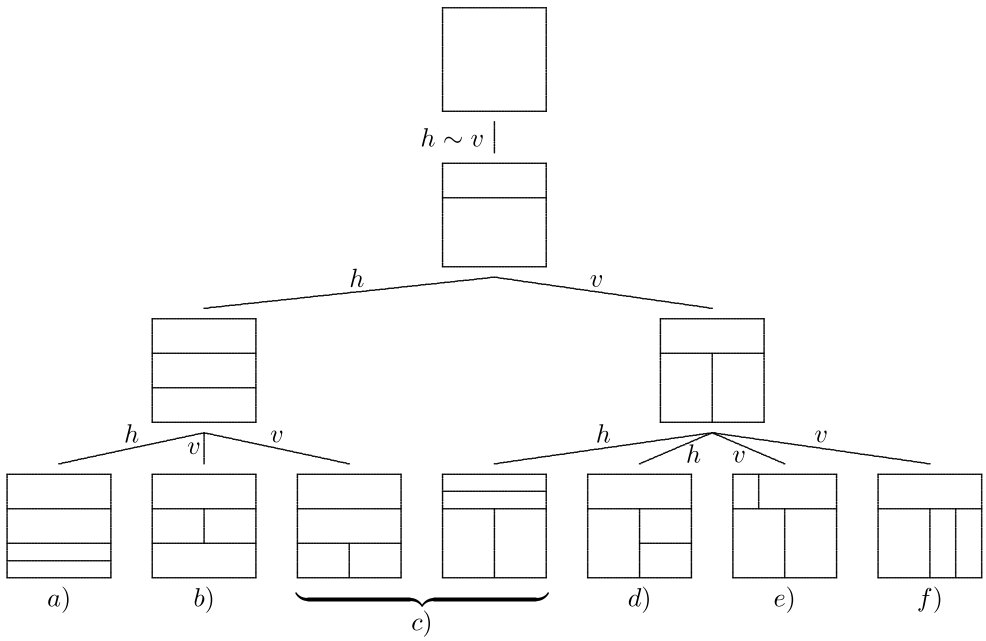


Figure 8

define the 4 regions. There are $\binom{9}{3}$ ways to choose these horizontal borders. Suits in each of these boards can be allocated in 4! ways. And, because horizontal and vertical borders are in the same configuration class under rotation, the total number of boards in this configuration class is $2 \cdot 4! \cdot \binom{9}{3}$.

For node b), there are $\binom{9}{2}$ ways to select the horizontal borders. For each of these pairs of horizontal borders, there are 9 ways to choose the vertical border making the cross-bar of the H. Suits for the 4 regions can be allocated in 4! ways. Once again, the initial borders could have been vertical, instead of horizontal. Hence, there are $2 \cdot 4! \cdot 9 \cdot \binom{9}{2}$. For nodes c), the only difference is that the vertical border is in the top or bottom region. Therefore, there are twice as many boards for configuration class c as for node b), or $2 \cdot \left[2 \cdot 4! \cdot 9 \cdot \binom{9}{2}\right]$ completed boards.

For node d), there are $\binom{9}{2}$ ways to select the horizontal borders. The vertical border determines how far across from the right side the lower horizontal border goes. There are 9 ways to select the vertical border that extends from the upper horizontal border to the bottom of the board. Because the lower horizontal border could extend from either the left or the right edge to the vertical border, choosing left or right doubles the number of boards. Also, there are 2 choices for which horizontal border traverses the board, either the upper or the lower, doubling again the number of boards. Because the initial borders could have been vertical and considering the 4! ways to allocate suits to the 4 regions, there are $2 \cdot 2 \cdot \left[2 \cdot 4! \cdot 9 \cdot \binom{9}{2}\right]$ completed boards.

For node e), it is possible for both the horizontal and vertical borders to traverse the board, making a '$+$,' as in Figure 5a). To prevent double counting, we first consider those that do not make a '$+$' sign. There are 9 choices for the horizontal border that traverses the board. The top region's vertical border can be selected in 9 ways, while the bottom region's vertical border can be selected in 8 ways, to prevent the '$+$'. Suits can be allocated to the four regions in 4! ways. With the cases in which the vertical border is the only one that traverses the board, there are $2 \cdot [4! \cdot 9 \cdot 9 \cdot 8]$ ways to complete the board in this configuration class without '$+$' cases. The '$+$' cases can be completed in $4! \cdot 9 \cdot 9$ ways because the horizontal border can be chosen in 9 ways, the vertical border can be chosen in 9 ways, and the four suits can be allocated to the four regions in 4! ways. Hence, there are $2 \cdot [4! \cdot 9 \cdot 9 \cdot 8] + 4! \cdot 9 \cdot 9$ distinct boards in this configuration class.

For node f), there are 9 ways to choose the horizontal border, $\binom{9}{2}$ ways to choose the two vertical borders, and the vertical borders could be above or below the horizontal border. The four suits can be allocated to the four

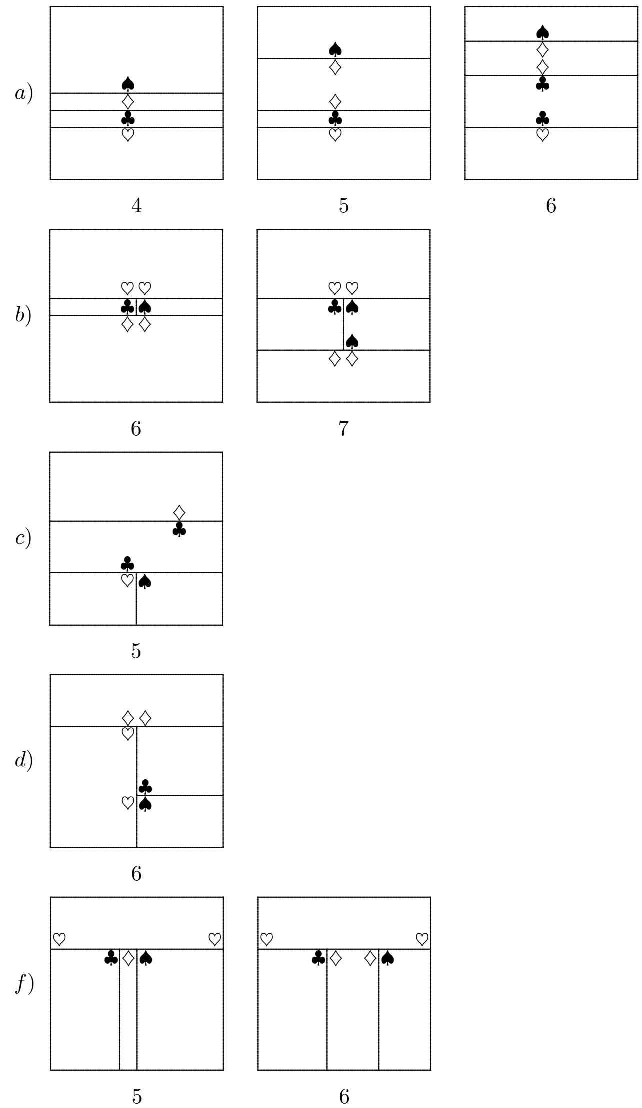


Figure 9

regions in 4! ways. Because the initial horizontal border could have been vertical, we multiply our result by 2. Hence, there are $2 \cdot 4! \cdot 2 \cdot \binom{9}{2} \cdot 9$ completed boards in this configuration class.

Adding the total number of boards for each configuration class there are

$$4! \left[17 \cdot 9^2 + 2 \cdot 9^2 \cdot \binom{9}{2} + 2 \cdot \binom{9}{3} \right]$$

or 177048 possible boards.

9. Answers vary. Examples of acceptable solutions are given.

   a) 3 squares: (7,6), (6,6), and (5,6).

   b) 2 squares: (8,9) and (8,X).

   c) 1 square: (8,X).

   d) 7 squares: (1,5), (2,5), (3,5), (4,5), (5,5), (6,5), and (7,5).

10. All values of n from 3 to 99. For $n = 3$, the board in Figure 5a) is an example where knowing three squares (♠, ♡, and ♢) isn't enough to complete the board, but knowing a specific 4th square (♣) does complete the board. For $n = 99$, knowing the suits of all squares except for (8,X) in Figure 6c) is not enough to complete the board because (8,X) could be ♡ or ♠. Other values can be constructed by generalizing the boards in Figure 6.