Generalizing Pascal: The Euler Triangles

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Summary

This project investigates generalizing Pascal's Triangle to generate the coefficients appearing in the expansion of powers of multinomials of the form $1 + x + x^2 + \cdots + x^k$. Students will initially use a *counting paths* method and then relate the procedure to a generalized triangle, known as an Euler Triangle.

Notes for the instructor

This activity is appropriate for mature Algebra and Precalculus students and can be accomplished in two or three one-hour class periods, depending on the backgrounds of the students. It fits nicely after a unit on binomial expansion. A peg board, with some type of peg or stud, can be used to string paths that can then be counted to generate the coefficients.

Bibliography

[1] A.W.F. Edwards. *Pascal's Arithmetical Triangle*, Johns Hopkins University Press, 2002.

[2] Ronald Graham, Donald Knuth, and Oren Patashnik. *Concrete Mathematics*, Addison Wesley, 1994.

[3] Hilton, Holton, & Pedersen. *Mathematical Reflections*, Springer Verlag, 1997.

[4] Hilton, Holton, & Pedersen. *Mathematical Vistas*, Springer Verlag, 2002.

[5] `http://arxiv.org/pdf/math.HO/0505425`

[6] `http://binomial.csuhayward.edu/Euler.html`

[7] `mathforum.org/workshops/usi/pascal/pascal_intro.html`

[8] `mathworld.wolfram.com/PascalsTriangle.html`

[9] `www.shodor.org/interactivate/activities/pascal1/index.html`

1 Generalizing Pascal: The Euler Triangles

Most high school students are familiar with Pascal's Triangle (Fig. 1) as a way to find the coefficients of the binomial $(1+x)^n$. By adding pairs of coefficients in a given row of the triangle, one can generate the binomial coefficients that appear in the row immediately following.

Recall:

$$
\begin{array}{l ccccccccccc}
(1+x)^0 & & & & & & \mathbf{1} & & & & & \\
(1+x)^1 & & & & & \mathbf{1} & & \mathbf{1} & & & & \\
(1+x)^2 & & & & \mathbf{1} & & \mathbf{2} & & \mathbf{1} & & & \\
(1+x)^3 & & & \mathbf{1} & & \mathbf{3} & & \mathbf{3} & & \mathbf{1} & & \\
(1+x)^4 & & \mathbf{1} & & \mathbf{4} & & \mathbf{6} & & \mathbf{4} & & \mathbf{1} & \\
(1+x)^5 & \mathbf{1} & & \mathbf{5} & & \mathbf{10} & & \mathbf{10} & & \mathbf{5} & & \mathbf{1}
\end{array}
$$

Figure 1. Pascal's Triangle showing the binomial coefficients up to the 5th power.

The coefficients can then be used to write out the expansion. For example, using the triangle above:

$$(1+x)^4 = \mathbf{1} + \mathbf{4}x + \mathbf{6}x^2 + \mathbf{4}x^3 + \mathbf{1}x^4.$$

The coefficients can also be determined by counting the numbers of paths from the apex of the triangle to the entry in question. For example, in the above expansion of $(1+x)^4$ the coefficient of x^3 is 4. This enumerates the following four paths, each of which must descend at each step to one of the two entries nearest to and below (see Figure 2):

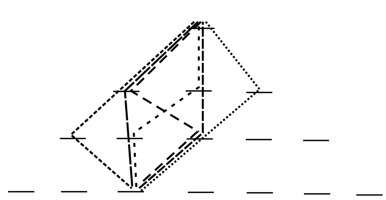
$$
\begin{array}{ccccccccc}
 & & & & \mathbf{1} & & & & \\
 & & & 1 & & \mathbf{1} & & & \\
 & & 1 & & 2 & & \mathbf{1} & & \\
 & 1 & & 3 & & 3 & & \mathbf{1} & \\
1 & & 4 & & 6 & & \mathbf{4} & & 1
\end{array}
\qquad
\begin{array}{ccccccccc}
 & & & & \mathbf{1} & & & & \\
 & & & 1 & & \mathbf{1} & & & \\
 & & 1 & & 2 & & \mathbf{1} & & \\
 & 1 & & 3 & & \mathbf{3} & & 1 & \\
1 & & 4 & & 6 & & \mathbf{4} & & 1
\end{array}
$$

$$
\begin{array}{ccccccccc}
 & & & & \mathbf{1} & & & & \\
 & & & 1 & & \mathbf{1} & & & \\
 & & 1 & & \mathbf{2} & & 1 & & \\
 & 1 & & 3 & & \mathbf{3} & & 1 & \\
1 & & 4 & & 6 & & \mathbf{4} & & 1
\end{array}
\qquad
\begin{array}{ccccccccc}
 & & & & \mathbf{1} & & & & \\
 & & & \mathbf{1} & & 1 & & & \\
 & & 1 & & \mathbf{2} & & 1 & & \\
 & 1 & & 3 & & \mathbf{3} & & 1 & \\
1 & & 4 & & 6 & & \mathbf{4} & & 1
\end{array}
$$

Figure 2. The four paths are indicated by boldface entries.

Is there a similar way to generate the coefficients for $(1+x+x^2)^n$? For example, if $n=3$, can we use this method to generate the coefficients for $(1+x+x^2)^3$?

Students can count the number of paths to a particular entry by drawing out the individual paths on a template[1] of the Euler triangle to be used for trinomials (Figure 3), or by creating the paths with strings (or rubber bands) on a pegboard version of the triangle. (See Photos 1 and 2). In this case, paths must descend at each step to one of the three spaces nearest to and below. For example, there are six paths that can be taken to reach the desired location indicated in Figure 3.



$$
\begin{array}{ccccccc}
 & & & \mathbf{1} & & & \\
 & & \mathbf{1} & \mathbf{1} & \mathbf{1} & & \\
 & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{2} & \mathbf{1} & \\
\mathbf{1} & \mathbf{3} & \boxed{\mathbf{6}} & \mathbf{7} & \mathbf{6} & \mathbf{3} & \mathbf{1}
\end{array}
$$

Figure 3. Template for trinomial Euler triangle and completed version.

[1] To create the template for the trinomial triangle, we recognize that the number of terms in the expansion of $(1+x+x^2)^n$ is precisely $2n+1$. The template, then, will have rows of $1, 3, 5, \ldots$ entries.

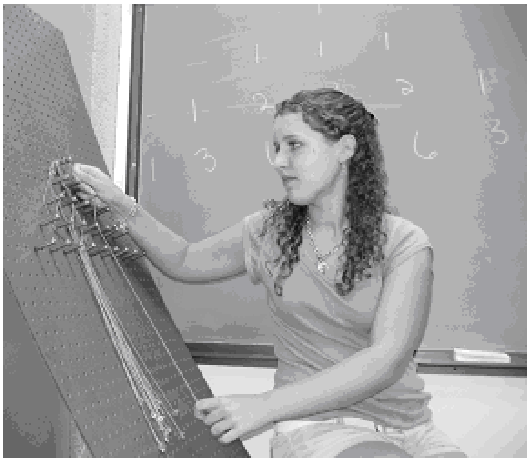


Photo 1. Stringing paths on an Euler Triangle.

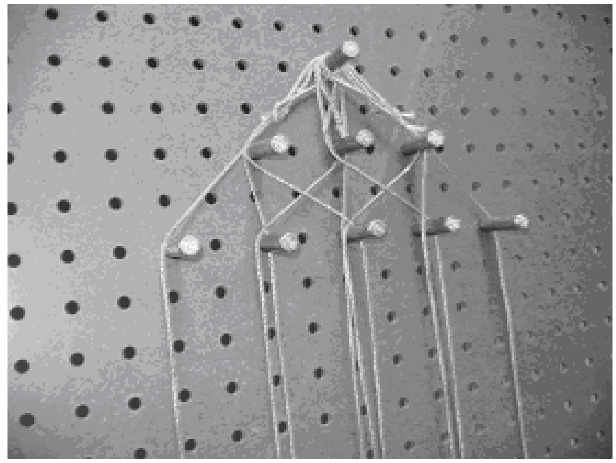


Photo 2. The 1-2-3-2-1 pattern for paths representing the coefficients of $(1 + x + x^2)^2$.

Exercise 1.

Complete the table for $(1 + x + x^2)^n$ for $n = 1, 2$, and 3 by stringing and counting paths.

$$
\begin{array}{ccccccc}
 & & & \underline{1} & & & \\
 & & \text{—} & \text{—} & \text{—} & & \\
 & \text{—} & \text{—} & \text{—} & \text{—} & \text{—} & \\
\text{—} & \text{—} & \text{—} & \text{—} & \text{—} & \text{—} & \text{—}
\end{array}
$$

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2 A recursive method for calculating coefficients

A different — and more convenient — method of obtaining the coefficients for the expansion can be obtained as follows: for each entry other than the initial 1, sum those values that are both adjacent to and above that entry. For this **tri**nomial (Figure 4), you add the **three** entries above (if possible), just as in the expansion of a **bi**nomial, you add the **two** entries above in Pascal's Triangle. In the case of Pascal's triangle, these entries are just the *binomial coefficients*, $\binom{n}{k}$, which represent the coefficients of x^k in the expansion of $(1 + x)^n$. We recall that these binomial coefficients, sometimes read *n choose k*, can be computed using the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. That one can obtain an entry by adding the two previous entries above is a consequence of the recursive identity $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

Euler introduced the notation $\binom{n}{k}_3$ to represent the trinomial coefficients. For example, back in Figure 3, we found the entry represented by $\binom{3}{2}_3$, the coefficient of the x^2 term in $(1+x+x^2)^3$. More generally, the values $\binom{n}{k}_m$ represent the m-nomial coefficients. These can, like the binomial coefficients, be defined recursively, as follows:

$$
\binom{n}{k}_3 = \begin{cases} 0, & n < 0 \text{ or } k < 0 \text{ or } k > 2n, \\[2ex] 1, & n = k = 0, \\[2ex] \binom{n-1}{k-2}_3 + \binom{n-1}{k-1}_3 + \binom{n-1}{k}_3, & n \text{ and } k \text{ not both} = 0. \end{cases}
$$

Can you tease out the reason for this rather complicated looking definition? If we examine, the $\binom{4}{3}_3 = 16$ entry from the Euler trinomial triangle in Figure 4, we see that it is equal to $\binom{3}{1}_3 + \binom{3}{2}_3 + \binom{3}{3}_3 = 3 + 6 + 7$. However, if we look at, say, $\binom{3}{5}_3 = 3$, we see that this equals $\binom{2}{3}_3 + \binom{2}{4}_3 + \binom{2}{5}_3 = 1 + 1 + 0$. Even though the triangle itself doesn't have an entry for $\binom{2}{5}_3$, our recursive definition tells us this must be 0, since $k > 2n$ in this instance.

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & 1 & 1 & & & \\
 & & 1 & 2 & 3 & \mathbf{2} & \mathbf{1} & & \\
 & & & & & & \searrow\downarrow\swarrow & & \\
 & 1 & \mathbf{3} & \mathbf{6} & \mathbf{7} & 6 & \mathbf{3} & 1 & \\
 & & & \searrow\downarrow\swarrow & & & & & \\
1 & 4 & 10 & \mathbf{16} & 19 & 16 & 10 & 4 & 1
\end{array}
$$

Figure 4. Method for calculating values for entries in the trinomial Euler Triangle.

Exercise 2

- Find the next two rows for the triangle in Figure 4. ______________________________

  ____

- Using your triangle, expand the following:

$(1 + x + x^2)^3 =$ ____

$(1 + x + x^2)^6 =$ ____

There are a number of interesting observations about the trinomial Euler triangle. Some of these are:

1. The triangle is symmetric, each row being palindromic. Row n follows the pattern 1 $n \cdots n$ 1. (For example, row 3 in Figure 4 above is: 1 3 6 7 6 3 1.)
2. Each new row has two more entries than the previous one.
3. In Pascal's Triangle (for **bi**nomial expansions) the row sum is $\mathbf{2^n}$. For the **tri**nomial Euler Triangle, what do you note about the sum of the entries in row n?
4. One also notes that the triangular numbers appear along the third diagonal of this Euler triangle. (See Figure 5.)

						1	.	.	.	.	.	.	.	.	.	.	1
					1	1	1	.	.	.	.	.	.	.	.	.	3
				1	2	3	2	1	.	.	.	.	.	.	.	.	9
			1	3	6	7	6	3	1	.	.	.	.	.	.	.	27
		1	4	10	16	19	16	10	4	1	.	.	.	.	.	.	81
	1	5	15	30	45	51	45	30	15	5	1	.	.	.	.	.	243
1	6	21	50	90	126	141	126	90	50	21	6	1	.	.	.	.	729

Figure 5. Sum of the coefficients in the nth row equals 3^n.

3 Generalizations

Will the processes that we have discussed so far generalize to higher degree polynomial powers? The following exercises will help us discover an answer to this.

Exercise 3. Complete the first three rows of the Euler Triangle for the **quadri**nominal $(1+x+x^2+x^3)^n$ by counting paths. Then check your answer by adding the **four** entries above each entry. Verify this algebraically by expanding $(1+x+x^2+x^3)^4$.

(Use *DERIVE* or some other Computer Algebra System, if possible.)

Exercise 4. a) What is the sum of the entries in rows 0, 1, 2, 3, and 4 of this triangle? In row n?

b) Using the triangle, write the complete expansion of $(1+x+x^2+x^3)^4$.

Of course, we can continue to look at powers of higher and higher degree polynomials — quintinomials, 6-nomials, 7-nomials, etc. Figure 6 gives the first few rows of the quintinomial Euler triangle. An observation we can make that can help determine the shape of the Euler triangle is based on the number of entries in each successive row. For the binomial case (Pascal), each new row had one additional entry. For the trinomial Euler triangle, each new row had two additional entries and for the quadrinomial case, three. We would expect for the quintinomial that the number of entries in each row would increase by four and be determined by adding the nearest five entries from the row above (allowing for blank entries if necessary). For instance, in the center of the triangle in Figure 6, we have $19 = 3+4+5+4+3$.

								1								
						1	1	1	1	1						
				1	2	**3**	**4**	**5**	**4**	**3**	2	1				
		1	3	6	10	15	18	**19**	18	15	10	6	3	1		
1	4	10	20	35	52	68	80	85	80	68	52	35	20	10	4	1

Figure 6. The first few rows of the Euler Triangle for the quintinomial $(1+x+x^2+x^3+x^4)^n$.

Exercise 5.

1. Find the next row for the triangle in Figure 6.
2. What is the sum of the coefficients in each row of the quintinomial Euler triangle (Figure 6)?
3. Create the first few rows of the 6-nomial Euler triangle.
4. What is the sum of the coefficients in each row of the 6-nomial Euler triangle?
5. Generalize how to find the coefficients of any multinomial power $(1+x+\cdots+x^{m-1})^n$.
6. Show that for an m-nomial Euler triangle, the sum of the entries in row n is just m^n.

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4 Research Questions

1. For the binomial case, we know that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. Find a closed form for the trinomial coefficients $\binom{n}{k}_3$.

2. Do the same for the general m-nomial coefficient $\binom{n}{k}_m$.

3. Explain why the $\binom{n}{3}_m$ entry is the nth triangular number ($n \geq 1$).

Solutions

Exercise 1.

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & 1 & 1 & & \\ & 1 & 2 & 3 & 2 & 1 & \\ 1 & 3 & 6 & 7 & 6 & 3 & 1 \end{array}$$

Exercise 2.

1 5 15 30 45 51 45 30 15 5 1

1 6 21 50 90 126 141 126 90 50 21 6 1

$1 + 3x + 6x^2 + 7x^3 + 6x^4 + 3x^5 + x^6$

$1 + 6x + 21x^2 + 50x^3 + 90x^4 + 126x^5 + 141x^6 + 126x^7 + 90x^8 + 50x^9 + 21x^{10} + 6x^{11} + x^{12}$

Exercise 3.

$$\begin{array}{ccccccc} & & & 1 & & & \\ & 1 & 1 & & 1 & 1 & \\ 1 & 2 & 3 & 4 & 3 & 2 & 1 \end{array}$$

Exercise 4.

a) The sum of the entries for rows 0, 1, 2, 3, 4, and n is respectively 1, 4, 16, 64, 256, and 4^n.

b) $(1+x+x^2+x^3)^4 = 1+4x+10x^2+20x^3+31x^4+40x^5+44x^6+40x^7+31x^8+20x^9+10x^{10}+4x^{11}+1x^{12}$.

Exercise 5.

1) 1 5 15 35 70 121 185 255 320 365 381 365 320 255 185 121 70 35 15 5 1

2) The sum of the coefficients on row n is 5^n.

3)

$$\begin{array}{cccccccccccccccc} & & & & & & & 1 & & & & & & & & \\ & & & & & 1 & 1 & 1 & 1 & 1 & 1 & & & & & \\ & & & 1 & 2 & 3 & 4 & 5 & 6 & 5 & 4 & 3 & 2 & 1 & & \\ 1 & 3 & 6 & 10 & 15 & 21 & 25 & 27 & 27 & 25 & 21 & 15 & 10 & 6 & 3 & 1 \end{array}$$

4) The sum of the coefficients in row n is 6^n.

5) For the Euler Triangle associated with powers of the m-nomial $1 + x + \cdots + x^{m-1}$, one only needs to add the m entries nearest to and above the entry to be filled. We have more generally

$$\binom{n}{k}_m = \binom{n-1}{k-m+1}_m + \cdots + \binom{n-1}{k-1}_m + \binom{n-1}{k}_m = \sum_{j=0}^{m-1} \binom{n-1}{k-j}_m.$$

6) The sum of the coefficients in the expansion of $(1 + x + \cdots + x^{m-1})^n$ can be obtained by setting $x = 1$ in the expanded form. But this is the same as replacing x by 1 in the unexpanded expression, which gives

$$(\underbrace{1 + 1 + \cdots + 1}_{m \text{ times}})^n = m^n.$$