Exploring Recursion with the Josephus Problem
(Or how to play "One Potato, Two Potato" for keeps)

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Summary

The Josephus problem is addressed in many discrete mathematics textbooks as an exercise in recursive modeling, with some books (e.g., [1] and [3]) even using it within the first few pages as an introductory problem to intrigue students. Since most students are familiar with the use of simple rhymes (like Eeny-meeny-miney-moe) for decision-making on the playground, they are comfortable with the physical process involved in this problem. For students who may wish to pursue this topic independently, [4] and [5] provide nice surveys and bibliographies, and the website [2] provides web-based tools for exploring the problem directly. The activities presented here are intended to be completed by students in a single class period early in the semester. We find that an opening student-centered problem can get the class involved and set a good tone for the semester. Moreover, we find that many issues arising from this particular problem can be built upon throughout the course. The next section provides some suggestions for connections to other parts of the course.

Notes for the instructor

The Josephus problem can be explored through role playing or through carefully constructed pencil and paper activities, depending on the amount of time one wishes to devote to it. We list below some of the things we discuss just before the activity as well as some of the contexts in which we have students revisit the problem later on.

- A good preliminary discussion on recursion can be initiated with the following problems.

    a. Pose the question, "What is $1 + 2 + 3 + \cdots + 19 + 20$?" This provides a good opportunity to share the creative idea of regrouping in order to sum 10 copies of 21 for a total of 210.
    b. Followup with the question, "What is $1 + 2 + 3 + \cdots + 20 + 21$?" Some students will try the regrouping trick, but at least one should point out that you can simply add 21 to the previous answer.
    c. This idea of using a "similar but simpler" problem that has been solved previously is the very essence of recursive thinking.

- The activities presented here have been written to be completed with paper and pencil, but with the investment of more time one can have students act out the roles. This is a good ice-breaking activity early in the semester, but it does take more time. Through role playing, students will discover for themselves issues like "We need to remember who was first," and "We need a system for describing who is the last one left."

- There will be several opportunities later in a discrete mathematics course when one can reprise the Josephus game as a source for exercises and motivational examples. Computer science courses often use this problem as an exercise in recursive programming or in maintaining circular linked lists. Hence, with some cooperation from a friendly computer science instructor, this problem can prove useful in more than one context.

a. (Mathematical induction) In the Josephus problem with skip number 2, prove that for all integers $n \geq 0$, if the game starts with 2^n players, then the person in position 1 will be the last person left. (This uses induction with the induction step involving the one pass all the way around the circle for the first time in which the even numbered people are eliminated.)

b. (Follow up) In the Josephus problem with skip number 2, if $0 \leq k < 2^n$ and the game starts with $2^n + k$ players, then the person in position $2k + 1$ will be the last person left. This is a non-inductive argument consisting of removing the first k (even numbered) people and then applying (a) to the remaining circle of size 2^n.

c. (Binary representation of numbers) Define the cyclic left shift of a binary numeral b as the number obtained from shifting the leading (i.e., leftmost) 1 bit to the rightmost end of the numeral. For example, the cyclic left shift of the binary numeral 1001101 is the numeral 0011011, which is the same as 11011. Show that if $0 \leq k < 2^n$, then the cyclic left shift of the binary representation of $2^n + k$ is the binary representation of $2k + 1$. Hence, the cyclic left shift of a number m gives the last person left in the m person Josephus game with skip number 2. This gives an "application flavor" to the study of binary numbers that may make them more intriguing.

d. (Modular arithmetic) When introducing modular arithmetic, an analogy can be made to the Josephus problem in which the original circle of people are numbered 0 through $n - 1$. In particular, the patterns within the tables of "last person left" all have the relationship "add k" but with the provision that the addition "wraps around the circle" to refer to the actual people.

Bibliography

[1] Ensley, D. E. and J. W. Crawley. *Discrete Mathematics: Mathematical Reasoning and Proof with Puzzles, Patterns and Games*, New York: John Wiley, 2006.

[2] Ensley, D. E. and J. W. Crawley. Companion website for *Discrete Mathematics*, at `http://webspace.ship.edu/~deensley/DiscreteMath/`

[3] Graham, R., D. Knuth and O. Patashnik. *Concrete Mathematics*, Reading, MA: Addison-Wesley, 1994.

[4] Herstein, I. N. and I. Kaplansky. *Matters Mathematical*, New York: Chelsea Publishing Company, 1974.

[5] Schumer, P. D. "The Josephus problem: Once more around," *Mathematics Magazine* 75 (2002) 12–17.

Worksheet on Exploring Recursion with the Josephus Problem (Or how to play "One Potato, Two Potato" for keeps)

Introduction

Ancient mathematics problems that still hold their own are always fun to play with. A particularly good one, which happens to be named for a first century historian, has its origins in the Jewish - Roman war. The historian Flavius Josephus was apparently trapped by the Romans in a cave with 40 fellow Jewish rebels. As good soldiers they decided on suicide rather than capture, so they formed a circle and agreed that every third person would be killed until no one was left.

Josephus and a friend were more keen on being captured than their colleagues, so they quickly found the spots to stand to ensure they were the two remaining at the end of the grisly proceedings. Hence, the mathematically inept suffered an untimely demise while Josephus and his friend lived to tell the tale.

This morbid story doesn't seem like much of a game or puzzle, but it has the same basic structure (with terminal consequences) as the age old way of choosing someone from a group: the "one potato, two potato" algorithm. We will spend some time in class today playing this type of game and analyzing our results.

Analyzing the Josephus Problem

In general, when we play the "Josephus game," there will be a certain number of people standing in a circle, and a "skip number" that tells us how many people to count before removing someone from the circle. In the classical example described above, the number of people is 41 and the skip number is 3.

Let's look at a simpler example. This time, there will be only six people in the circle, but we will keep the skip number at 3. We'll continue to play until there is only one person remaining. Let's say the people, named Ann, Beth, Chris, Dave, Emma, and Fred, are arranged as shown in Figure 1.

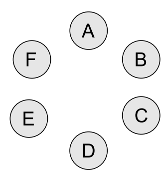

Figure 1. Six people in a circle.

In this case, we decide to start counting with Ann. We count Ann and Beth, and when we get to the third person, Chris, he is removed from the circle. With Chris gone, we continue counting with Dave. We count Dave and Emma, and when we get to Fred, he is eliminated from the circle. Now there are four people left in the circle: Ann, Beth, Dave, and Emma, and the counting continues with Ann. We count Ann and Beth, and then Dave is eliminated. The current situation is displayed in Figure 2.

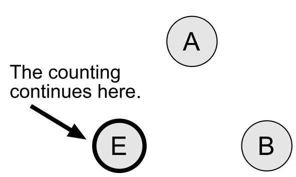

Figure 2. Three people remain.

We next count Emma and Ann, and remove Beth, and the counting once again continues with Emma. We count Emma, Ann, and then Emma is removed, so Ann is the person who is left standing at the end.

An important thing to notice about this process is that we need to know which person to start the counting with at each step, including the first step. If we remove a couple of people and then go on a coffee break, we might come back and forget who to resume the counting with.

For discussion: Can you think of a way that we could remember which person we need to start the counting with at each step?

One solution is for us to put a funny hat on the person we need to start the counting with at each step. In our diagrams, we will put a thick circle around the "starting person."

Let's try the game again, this time with seven people (named A, B, C, D, E, F, and G) and removing every fifth person. Recall that we say that the "skip number" is equal to 5. Figure 3 shows diagrams illustrating how such a game progresses. Note that the players are removed in the order E, C, B, D, G, A, and person F is the last one standing.

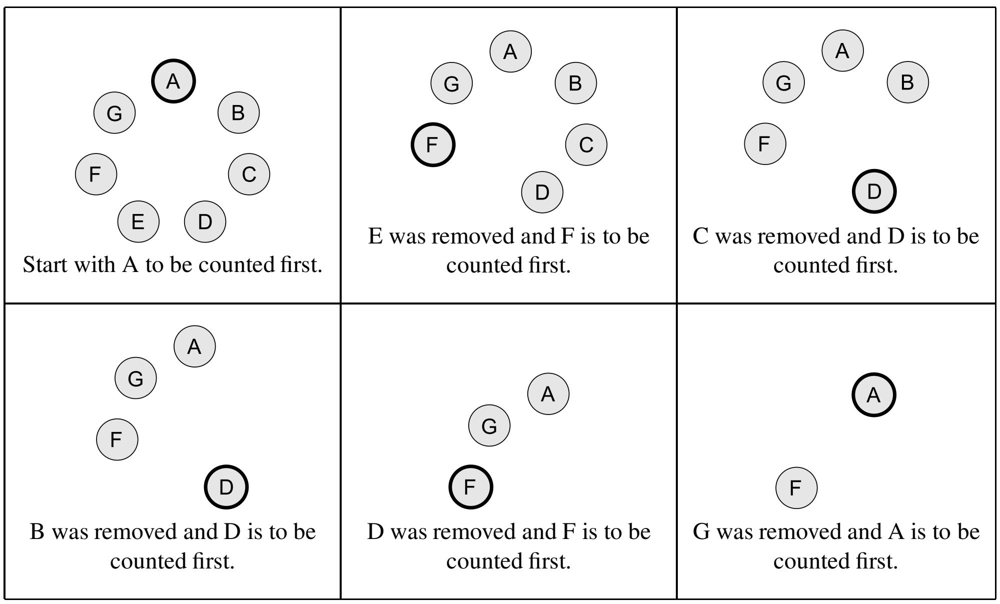

Figure 3. The game with seven people.

Exercise 1. On your own, play the Josephus game with n players and a skip number of k for each of the following values. Determine who is the last person standing.

a. $n = 6, k = 2$ b. $n = 10, k = 3$ c. $n = 11, k = 3$

Changing the Starting Player

What happens if we decide to keep the values of n and k the same, but change the person we start the game with? How does this affect the outcome? Let's go back to the example with seven people and a skip number of 5. Let's say the people are named Terry, Ursula, Vivian, Walter, Xander, Yolanda, and Zack, and we want to start the game with Walter as Figure 4 shows.

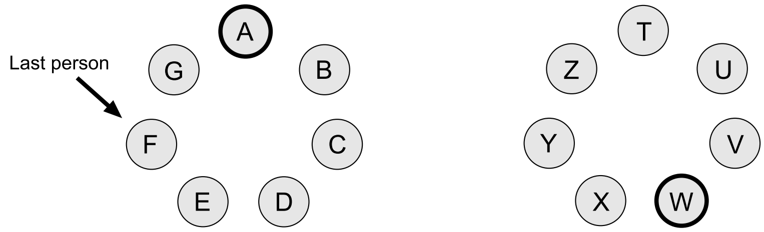

Figure 4. What's in a name?

For discussion: In the game shown on the left in Figure 4, F is the last person standing. Who will be the last person standing in the game shown on the right? Can you figure it out without playing the entire game again?

If you said that Ursula would be the last person standing, you are correct! When we have seven people and a skip number of 5, the last person standing is the sixth one around the circle from the starting player. (Here we count the starting person as the first player around the circle.) In mathematical notation, we will write this as $J(7, 5) = 6$.

The $J(n, k)$ notation is very handy for describing the last person left in the Josephus game that starts with n people in the circle and eliminates every k^{th} one. For example, the result of our first example can be described by simply writing $J(6, 3) = 1$.

Exercise 2. Go back to the three games you played in Exercise 1. Using the mathematical notation we have defined, find the value of $J(n, k)$ in each of the following cases.

a. $J(6, 2)$ b. $J(10, 3)$ c. $J(11, 3)$

Recursion: Using What Came Before

This idea of changing the starting player can be very helpful for finding patterns in the Josephus problem. Consider the game with eight people and a skip number of 5, as shown in Figure 5. After the first step of this game, E is eliminated

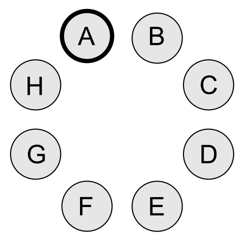

Figure 5. Beginning the game with eight people.

from the circle, and we have the situation in Figure 6.

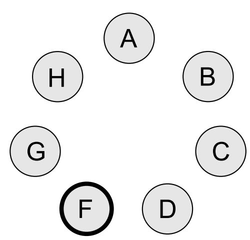

Figure 6. After person E is eliminated.

Now what? Well, we continue to play the game as before, or we might notice that we have seen this situation before. This is a game with **seven** players and a skip number of 5. We already determined that the last person standing in this game is the sixth person around the circle from the starting player. In this case, that means that C is the last person standing.

For discussion: Finish playing the game to verify that C is the last one standing.

Here is another example of this idea. In Exercise 2(c), you determined that $J(11, 3) = 7$. That is, in a game with eleven people and a skip number of 3, the seventh person around the circle from the starting person will be the last one standing. How does this help us determine the value of $J(12, 3)$? Consider the first step of the game with twelve people and a skip number of 3. The first person eliminated is person 3, and person 4 becomes the new starting player.

Now there are eleven people remaining, and we know that the last one standing will be the seventh person around the circle starting with person 4. This is person 10. You can verify on your own that $J(12, 3) = 10$.

Finding the Pattern

There is a pattern to how the position of the last survivor changes as we change the number of people initially standing in the circle. To see this pattern, we need to experiment and compute the answer for many different examples. In the table in Exercise 3, the top row shows the number of people in the circle, and the bottom row shows the position of the last person standing when the skip number is 3. The values we have already determined are filled in for you.

Exercise 3. Fill in the rest of the table, either by playing each game or by appealing to the "using-what-came-before" strategy.

n	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**
$J(n,3)$				1					7	10		

a. What pattern do you notice in the table?

b. Can you explain in terms of the "using-what-came-before" strategy why this pattern holds?

c. On your own, make a similar table but change the skip number to 4. Can you predict what pattern you will see?

An easier variation

A game that's a little better suited for detailed analysis is the variation where every second person is eliminated — that is, the skip number is 2. The game will officially be played with people named $1, 2, \ldots, n$ in a circle (with the numbers going clockwise). We go around the circle clockwise getting rid of every second person (Person 2 is the first to go) until no one is left. For example, if we start with four people, then the people are eliminated in the order $2, 4, 3, 1$, so person 1 is the last survivor.

We will let $J(n)$ denote the last survivor in the game which starts with n people and has a skip number of 2. (That is, we use $J(n)$ instead of $J(n, 2)$.)

Exercise 4. Fill in the rest of the table, either by playing each game or by appealing to the "using-what-came-before" strategy.

n	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**
$J(n)$				1												

n	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**
$J(n)$														

a. How is the value of $J(n)$ related to the value of $J(n-1)$?

b. What will be the next value of n for which $J(n) = 1$?

c. How would you describe a formula for $J(n)$ that would allow someone to quickly figure out the last place in line given any n?

Josephus and his buddy

In the original story, Josephus actually escapes with a friend, so in reality he had to know the positions of the last two survivors of this macabre game. To keep it simple, let's still use the game with skip number 2, but now we will use $F(n)$ to denote the required position of the friend in the Josephus game starting with n people.

Exercise 5. Play the Josephus game (with every second person eliminated, as above) for various n and record the numbers $J(n)$ and $F(n)$ of the last person alive and of the next-to-the-last person alive, respectively. Find more values than in the table below if you think it is helpful to do so. Remember to try to use things you already know as you tackle larger and larger values of n.

n	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**
$J(n)$													
$F(n)$													

a. How is the value of $F(n)$ related to the value of $F(n-1)$?

b. What will be the next value of n for which $F(n) = 1$?

c. Is there a direct relationship between $J(n)$ and $F(n)$?

Further questions for exploration

The following problems, as well as the ones above, can be explored with the applet found under Section 1.1 on the website

`http://webspace.ship.edu/~deensley/DiscreteMath/flash/`

Exercise 6. Fill in the following table using the "One potato, two potato" game on n people, starting the first "one potato" on person 1. For those not familiar with this method of choosing a person on the playground, this is simply the Josephus problem with every **eighth** person eliminated. That is, in the table below we use $P(n)$ to mean the same thing as $J(n, 8)$ from the previous discussion.

n	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**
$P(n)$															

a. If the ________ students in this class stand in a circle in alphabetical order and do "one potato, two potato", who will be the last person left?

b. Suppose in the game with 6 people, Josephus is person 1 but before the game starts, the Roman leader says, "Hey Joey, *you* pick the skip number." What should he say so that he is the last person left?

c. Is it possible for Josephus to always come up with a response to the previous question no matter how many people are originally in the circle?

Solutions

Exercise 1. We will use the conventions of labeling the people A, B, C, etc. clockwise around the circle and starting our count with person A.

a. For $n = 6$ and $k = 2$, the last person left is E.

b. For $n = 10$ and $k = 3$, the last person left is D.

c. For $n = 11$ and $k = 3$, the last person left is G.

Exercise 2.

a. $J(6, 2) = 5$.

b. $J(10, 3) = 4$.

c. $J(11, 3) = 7$.

Exercise 3. Here is the completed table:

n	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**
$J(n, 3)$	2	1	4	1	4	7	1	4	7	10	13	2

a. For all $n \geq 2$, person $J(n, 3)$ is three more around (clockwise) the original circle from person $J(n-1, 3)$.

b. If the k^{th} person around the circle of $n-1$ people is the last one remaining, then in the game that starts with n people, after one person is eliminated the first person in the remaining circle of $n-1$ is person 4. The k^{th} person in *this* circle, is the $(k+3)^{th}$ person in the original circle.

c. For all $n \geq 2$, person $J(n, 4)$ is four more around (clockwise) the original circle from person $J(n-1, 4)$.

Exercise 4. Here is the completed table:

n	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**
$J(n)$	1	1	3	1	3	5	7	1	3	5	7	9	11	13	15	1

n	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**
$J(n)$	3	5	7	9	11	13	15	17	19	21	23	25	27	29

a. For all $n \geq 2$, person $J(n)$ is two more around (clockwise) the original circle from person $J(n-1)$.

b. The next value of n for which $J(n) = 1$ will be $n = 32$. It appears that $J(n) = 1$ if and only if n is a power of 2.

c. Given n people originally, let m be the smallest power of 2 less than or equal to n. Eliminate people $2, 4, \ldots, 2(n-m)$. This leaves the game with m people, the first of whom is person $2(n-m)+1$. According to the observation in part (b) of this exercise, this person will be the last person left at the end of the entire process.

Exercise 5. Here is the completed table:

n	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**
$J(n)$	9	11	13	15	1	3	5	7	9	11	13	15	17
$F(n)$	1	3	5	7	9	11	13	15	17	19	21	23	1

a. For all $n \geq 2$, person $F(n)$ is two more around (clockwise) the original circle from person $F(n-1)$.

b. The next value of n for which $F(n) = 1$ will be $n = 48$. It appears that $F(n) = 1$ if and only if $n = 3 \cdot 2^k$ for some value of $k \geq 0$.

c. $J(n) - F(n) = 2^k$ when the integer k can be chosen so that $3 \cdot 2^{k-1} \leq n < 2^{k+1}$, and $F(n) - J(n) = 2^k$ when the integer k can be chosen so that $2^{k+1} \leq n < 3 \cdot 2^k$.

Exercise 6. Here is the completed table:

n	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**
$P(n)$	1	9	5	13	7	15	7	15	5	13	1	9	17	2	10

a. This answer will depend on the number of people in your class. Suppose there are 32 people in your class. Using the pattern of "adding 8" relative to the number in the circle, we find that $J(32, 8) = 17$.

b. Using a skip number of 60 will work for sure (see the next answer), but the smallest number that will work is $k = 3$.

c. For the game with n people, using k that is the least common multiple of the numbers in $\{1, 2, 3, \ldots, n\}$ is guaranteed to work, but there are typically much smaller values.