Exploring Polyhedra and Discovering Euler's Formula[1]

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Summary

The activities and exercises collected here provide an introduction to Euler's formula, an introduction to interesting related topics, and sources for further exploration. While Euler's formula applies to any planar graph, a natural and accessible context for the study of Euler's formula is the study of polyhedra.

The article includes an introduction to Euler's formula, four student activities, and two appendices containing useful information for the instructor, such as an inductive proof of Euler's theorem and several other interesting results that may be proved using Euler's theorem. Each activity includes a discussion of connections to discrete mathematics and notes to the instructor. Three of the activities have worksheets at the end of this article, followed by solutions and a template for a toroidal polyhedron.

A brief introduction to Euler's formula

Theorem (Euler's formula, polyhedral version)**.** *Let P be any polyhedron topologically equivalent to a sphere. Let V be the number of vertices, E the number of edges and F the number of faces of P. Then $V - E + F = 2$.*

A graph is said to be a *simple graph* if it is an undirected graph containing neither loops nor multiple edges. A graph is a *plane graph* if it is embedded in the plane without crossing edges. A graph is said to be *planar* if it admits such an embedding.

Theorem (Euler's formula, graph version)**.** *Let G be any simple plane graph. Let V be the number of vertices, E the number of edges and F the number of faces (alternately, the number of connected sets in the complement) of G. Then $V - E + F = 2$.*

Note that the number of faces F includes what is sometimes called the outside face, the connected component containing the unbounded region in the complement of the graph. The supposition that G is a simple graph is not necessary for the proof (the result holds for graphs with loops and multiple edges), but only simple graphs arise in the polyhedral context, and adding this assumption simplifies the inductive proof. Note that the adjective "simple" means different things for graphs and polyhedra; simple polyhedra will be defined later.

Throughout this treatment we are defining polyhedra as embedded compact two-dimensional polygonal surfaces with planar, nonself-intersecting faces. While some of these criteria are dropped in other contexts (the Kepler-Poinsot polyhedra are either immersed surfaces or have self-intersecting faces), omitting them introduces complexities we do not feel are appropriate to the activities collected here.

[1] Thanks to Doris Schattschneider and Gary Gordon who provided ideas and templates for early versions of some of these activities.

1 Activity: Polyhedra Exploration

The *Polyhedra Exploration* worksheet is the centerpiece of an activity designed to lead to the discovery of, and build confidence in, Euler's formula. This activity is a discovery-based activity in which students are asked to build models of various polyhedra, given a description of some feature of the polyhedron, and then asked to collect data on the number of vertices, edges, and faces of their polyhedron. After all students have contributed data, the class is asked to make conjectures about relationships between the numbers of vertices, edges, and faces. The obvious conjecture is Euler's formula. The worksheet is included at the end of this article.

Connection to discrete mathematics

One of the most important aspects of writing a proof is determining the appropriate proposition to prove. Thus, practice making conjectures is a useful part of a discrete mathematics course. The activity also leads into the presentation of a proof of Euler's formula; one of our favorite proofs of this formula is by induction on the number of edges in a graph. This is an especially nice proof to use in a discrete mathematics course, because it is an example of a nontrivial proof using induction in which induction is done on something other than an integer.

Notes for the instructor

We have had the most success with this activity, and it takes the least amount of time, if students have access to large numbers of *Polydrons* or *GeoFix* shapes. When they are available, this activity can take as little as 20 minutes. Alternatively, tape and polygons made from card stock may be used, though assembly in this case will usually take more time. If you are using homemade polygons it is important that the length of every side is the same for all polygons. At a minimum you will need 76 triangles, 29 squares, 14 pentagons, 20 hexagons and 6 octagons. We build a tetrahedron (polyhedron 0 in the worksheet list) as a demonstration.

At the beginning of the activity students are assigned (either individually or in groups) one of the items on the list to investigate. Students who finish early may either be assigned a second item from the list or additional assembly problems, given below. We usually handle problem assignments in both cases by printing polyhedron descriptions on half sheets of paper or index cards passed out randomly or by randomly assigning numbers to groups. The completed table is provided in the Solutions section, including the common names for the various polyhedra.

The additional assembly problems below are particularly important for the activities described in the next two Activities. If you plan on using the additional assembly problems, you will need a large selection of the above shapes, including at least 30 more triangles, 18 more squares and 34 more pentagons.

Additional assembly problems

The reader may be surprised at how many different examples students will generate if they are challenged to come up with unique examples subject to minimal constraints. Here are some supplementary exercises for the *Polyhedra Exploration* worksheet.

- Using only triangles, make two distinct polyhedra.
- Using only squares, make two distinct polyhedra.
- Using only pentagons, make two distinct polyhedra.
- Using triangles and squares, make a polyhedron where four triangles and a square come together at each corner. Can you make two different polyhedra?[2]
- Pick any two polygon shapes and make a polyhedron.
- Make a polyhedron using as many different shapes as you can.

Note that these additional assembly problems are, with the exception of the snub cube, open-ended. If students make spherical polyhedra, then as with the other worksheet exercises, they will find that $V + F$ is two more than E. However, be on the lookout for the clever students who decide to make toroidal polyhedra!

[2] The snub cube, which comes in two mirror image forms.

2 Activity: "Some of these things are not like the others"

This activity asks students to determine features that various polyhedra have in common, and it encourages them to carefully articulate those features.

Depending on the number of terms selected and the temperament of the students, this activity can take as little as 20 minutes, or as long as a full class. Depending on time availability, it may either be used directly as a follow on to the *Polyhedra Exploration*, or the models generated during the *Polyhedra Exploration* may be saved for a later class period for use in this activity.

Connection to discrete mathematics

Precision in vocabulary is an important feature in learning how to communicate mathematics effectively; not only are students exposed to vocabulary about polyhedra, but they should begin to recognize why having the vocabulary is useful. Students also make and test conjectures in the process of articulating features that various groups of polyhedra have.

Notes for the instructor

Once a large collection of polyhedral models has been assembled, it is possible to explore a number of questions about geometric and combinatorial properties of polyhedra. Our preference is to do this before students have been exposed to very much vocabulary about polyhedra.

We gather the students around a table where all of the interesting looking distinct models have been collected. Then we sort the models into two different piles. All of the polyhedra in one pile will have a chosen property, and all of the polyhedra in the remaining pile will not have this property. Part of the goal here is to provide students a context in which they can be challenged with several important mathematical problems: pattern recognition, geometric reasoning, hypothesis testing and development of appropriate nomenclature. Students often struggle with each of these, and only after students are able to describe in words correctly what distinguishes the chosen collection of objects are they told the formal term they have just defined. A selection of such terms and related definitions is included below.

It is important when doing this activity to wait a long time for students to make suggestions, to steer as little as possible, and to avoid giving any hints, although you should feel free to make more precise definitions that students have created. The level of depth and precision for these definitions should be fitted to the other instructional needs of the course but may also provide nice opportunities to emphasize the importance of making sure that a definition says what its author thinks it does (a common problem in geometry). Be careful not to reject examples of polyhedra or define them in ways that would artificially limit your ability later to talk about any of the definitions given. The goal is to have the students thinking as creatively and broadly as possible.

Regular Polyhedron/Platonic Solid This is a term that can be defined several ways. For convex polyhedra, it suffices to define a regular polyhedron as one in which all of the faces are regular polygons of exactly one type and each vertex has the same number of polygons around it. (It would be more precise to say that they are the polyhedra that are isogonal, isotoxal and isohedral; however there exists a combinatorial (nonconvex) polyhedron that satisfies all three of these conditions without meeting the modern definition of regularity.[3]) In most modern contexts, to be regular means that the polyhedron is "flag transitive." A flag is a vertex, edge and face of the polyhedron, all mutually incident, and to be flag transitive is to require that any two such flags may be mapped to each other by a symmetry of the polyhedron.

Convex Polyhedron Loosely, this means that the polyhedron lacks dents. Precisely, a polyhedron is convex if every pair of points on or in the polyhedron determines a segment contained entirely on or inside the polyhedron.

Symmetric Polyhedron Any polyhedron with a nontrivial symmetry mapping the polyhedron to itself.

[3] This is the Petrie dual (or Petrial, or Petrie polyhedron) of the cuboctahedron. For more information about Petrie duals we recommend Cromwell's discussion of Petrie polyhedra in [3]. More technical treatments are available in Coxeter's [2], Grünbaum's [4], and McMullen and Schulte's [6].

Isogonal Polyhedron Any vertex of the polyhedron may be mapped to any other vertex under a symmetry of the polyhedron. The triangular prism is isogonal, while the square pyramid is not isogonal, since the vertex at the apex of the pyramid can't be mapped to any vertex at the base.

Isohedral Polyhedron Any face of the polyhedron may be mapped to any other face under a symmetry of the polyhedron. Nonregular examples include the triangular bipyramid (or any of the other Catalan solids); Archimedean solids, such as the cuboctahedron, are not isohedral.

Isotoxal Polyhedron Any edge of the polyhedron may be mapped to any other edge under a symmetry of the polyhedron. The cuboctahedron is a good example of this phenomenon, while the triangular bipyramid does not have this property.

Uniform or Semiregular Polyhedron An isogonal polyhedron in which all of the faces are regular polygons, not all congruent.

Vertex Star The collection of polygons incident to a vertex. The combinatorial type of a vertex star is a listing between parentheses, separated by periods, in cyclic order, of the number of sides of the polygons incident to the vertex. A single such sequence is sufficient to characterize a uniform polyhedron. For example, a uniform polyhedron with vertex stars of type (5.6.6) must necessarily be the truncated icosahedron (soccer ball) in which every vertex is surrounded by a pentagon and two hexagons.

Archimedean Polyhedron A polyhedron in which all of the vertex stars are congruent, and the faces are all regular polygons.[4]

Prism A polyhedron with two parallel congruent faces connected by parallelograms. If the congruent faces may be obtained one from the other by translating in a direction perpendicular to the plane in which each lies, and so the remaining faces are rectangles, it is a *right* prism. If the rectangles are in fact squares, and the parallel polygons are regular, then the prism is also uniform.

Antiprism A convex uniform polyhedron with two parallel congruent regular n-gons and $2n$ equilateral triangles. The $2n$ triangles form a strip connecting the two parallel congruent n-gons. The vertex stars are all of combinatorial type $(3.3.3.n)$.

Spherical polyhedron A polyhedron that is topologically equivalent to a sphere.

Toroidal polyhedron A polyhedron that is topologically equivalent to a torus.

Simple polyhedron A polyhedron in which the number of edges (*degree*) at each vertex is three. Like other occurrences of the word "simple" in mathematics, some authors may mean something different when they use this term to describe a polyhedron, such as what we refer to as spherical.

Simplicial polyhedron A polyhedron made up entirely of triangles. Note that the dual of a simplicial polyhedron is simple and vice versa.

3 Activity: Symmetric Polyhedra and Angle Deficiency

A central theme in discrete mathematics is counting things, especially when counting things known to be equal by two different methods yields a useful equality. The activity on *Symmetric Polyhedra and Angle Deficiency* has two parts. The first part, which is discovery-based, asks students to make a conjecture about the total angle deficiency of a convex polyhedron, and the second part asks them to prove their conjecture using Euler's formula. The worksheet is included at the end of this article.

[4] Many sources use Archimedean and uniform as synonyms. However, there exists an example of an Archimedean polyhedron that is *not* uniform, the pseudorhomicuboctahedron (Johnson solid J_{37}).

Connection to discrete mathematics

An important class of proofs is "proofs by counting" — i.e., combinatorial proofs. This activity leads students through a nontrivial proof that proceeds by counting things two ways, as well as providing an application of Euler's formula.

Notes for the instructor

Exercise 5 on the *Symmetric Polyhedra and Angle Deficiency* worksheet (included at the end of the article) provides a good example of proving results by counting things two ways.

Here are two additional exercises you may want to use with this activity. Exercise 0 is a natural question to ask as a precursor for the worksheet, and Exercise 6 is a natural extension of the questions presented in the worksheet. Note that Exercise 6 may be answered positively, with prisms and antiprisms providing the remaining possible examples.

Exercise 0. Prove, using induction, that the sum of the angles in a convex n-gon is $180(n-2)$. Using this fact, determine the measure of a single angle in a regular n-gon.

Exercise 6. Using your answer from Exercise 0, determine whether there are any polyhedra other than those listed in the table of Exercise 3 of the *Symmetric Polyhedra and Angle Deficiency* worksheet that: (i) have faces that are regular polygons where every corner "looks" the same and (ii) has at least two kinds of faces.

Solutions for these exercises and the questions on the worksheet are provided in the solutions section that follows the worksheets.

4 Activity: Poincare's formula and higher genus polyhedra

Euler's formula only applies to polyhedra topologically equivalent to a sphere. This activity encourages students to construct nonspherical polyhedra, notice that Euler's formula does not hold, and make conjectures about what an appropriate generalization might be. It then asks students to think about a complicated polyhedron and, after making a guess, determine the genus using Poincaré's generalization of Euler's formula.

Connection to discrete mathematics

Paying attention to hypotheses is important; Euler's formula does not hold for nonspherical polyhedra! What is an appropriate generalization?

Notes for the instructor

Instruct students to construct "donuts" using the template provided on the last page of this article. If they then count up the numbers of vertices, edges and faces, they will find that the results do not obey the basic formulation of Euler's formula. Instead, the formula needs to take into account the genus of the object. Define $\chi(g) = 2 - 2g$ where g is the genus of the surface in question; Euler's formula generalizes to Poincaré's formula $V - E + F = \chi(g)$. The worksheet for this activity consists of two exercises that formalize this process, and solutions are provided in the solutions section.

Appendix A: Inductive proof of Euler's theorem using graphs

Every spherical polyhedron corresponds to a *simple* (no loops or multiple edges) *plane* (no edges cross) *connected* (you can get to any vertex from any vertex) graph. One way to see this is to imagine that the polyhedron is drawn on the surface of an extremely flexible balloon, with the opening of the balloon (where you blow) in the interior of a face. Pull the opening of the balloon wide and flatten the balloon to a disk in the plane; then the vertices, edges and faces of the polyhedron will form a graph in the plane. As long as we regard the outside of the graph as a face, there is a one-to-one correspondence between the vertices, edges and faces of the original polyhedron (on the blown-up balloon) and the vertices, edges, and faces (or regions) of the flattened-out balloon. Alternately, we may view the polyhedron as a transparent model, position an eye close enough to a face and trace the edges and vertices we see in that face (forming a *Schlegel diagram*). We can then transcribe the resulting figure into the plane and now the outside face corresponds to the face we traced the other faces into during our tracing process.

Therefore, if we can prove the stronger result that Euler's formula holds for *any* connected simple plane graph, we will have shown that Euler's formula must hold for the subclass of simple plane graphs which correspond to polyhedra.

Note that for students who have had exposure to proofs by induction, the following method of proving Euler's theorem, if approached casually, is a nice discovery-based learning topic, which can be used to introduce graph theory and also serves as an interesting, non-trivial example of a proof by induction, especially in classes where most of the proofs by induction deal only with properties of sequences of integers.

Theorem 1. *Suppose G is a connected simple plane graph with V vertices, E edges, and F faces (or regions). Then*

$$V - E + F = 2.$$

Proof. The proof is by (strong) induction on the number of edges, E. We assume that the "outside" of the graph is a face.[5] Let G be any connected simple plane graph with V vertices, E edges, and F faces (or regions).

Base Case $E = 0$. If $E = 0$ and G is a graph, then G is the unique connected graph with a single vertex and a single face, so $V = 1$, $E = 0$, $F = 1$ and $V - E + F = 2$.

Induction Hypothesis Suppose $E \geq 0$, and suppose for any k with $0 \leq k \leq E$ that if Q is any connected graph with V_Q vertices, $E_Q = k$ edges and F_Q faces, then $V_Q - E_Q + F_Q = 2$.

Let G be an arbitrary connected simple plane graph with $n+1$ edges, and let e be an arbitrary edge of G. Removing e yields a graph H—not necessarily connected—with n edges. There are two cases to consider (see Figure 1).

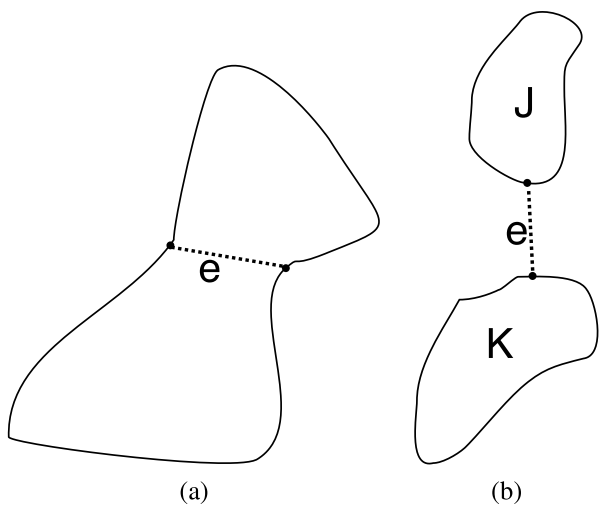

Figure 1. A graph G with a chosen edge e indicated as a dashed line segment; removal of e forms a graph H. (a) Case 1: H is connected; (b) Case 2: H consists of two connected components.

Case 1: H is connected, so e connects two vertices in H (Figure 1(a)) Note that since H is connected and has n edges, by the induction hypothesis, $V_H - E_H + F_H = 2$. In this case, because G can't have edges that cross, e must connect two vertices that share a face in H. When e is added back in to H to form G, the face in H becomes two faces in G. Therefore, $V = V_H$, $E = E_H + 1$, and $F = F_H + 1$, so

$$\begin{aligned} V - E + F &= V_H - (E_H + 1) + (F_H + 1) \\ &= (V_H - E_H + F_H) - 1 + 1 \\ &= 2. \end{aligned}$$

[5] The Jordan Curve Theorem allows us to use the graph embedding to divide the plane into an "inside" and an "outside."

Case 2: H **consists of two connected components (Figure 1(b))** Suppose that removing e from G disconnects the graph, and H consists of two connected components, J and K, with E_J and E_K edges respectively. (Note that one of the connected components could consist of a single vertex, so that one of E_J or E_K could equal zero!) Since $E_J + E_K = n$, it follows that $0 \leq E_J \leq n$ and $0 \leq E_K \leq n$, so we can apply the induction hypothesis to each component individually to conclude that $V_J - E_J + F_J = 2$ and $V_K - E_K + F_K = 2$. Note that $V = V_J + V_K$ and $E = E_J + E_K + 1$. However, F_J counts the outside face, and F_K also counts the outside face, so $F = F_J + F_K - 1$. Therefore,

$$\begin{aligned} V - E + F &= (V_J + V_K) - (E_J + E_K + 1) + (F_J + F_K - 1) \\ &= (V_J - E_J + F_J) + (V_K - E_K + F_K) - 2 \\ &= 2 + 2 - 2 \\ &= 2. \end{aligned}$$

Therefore, by induction on the number of edges, every connected simple plane graph with V vertices, E edges and F faces satisfies $V - E + F = 2$. □

Appendix B: Other interesting applications of Euler's formula

Euler's formula provides a number of nice opportunities to discuss a variety of methods suitable to a discrete mathematics course. Its proof provides a wonderful nontrivial example of the use of induction as a proof method. In application, Euler's formula is a rich source of examples of the classic combinatorial argument involving counting things two different ways. We have collected here some of our favorite examples; they make excellent demonstrations or homework exercises depending on the level and preparation of the students. Throughout this section we assume all polyhedra are spherical (so that Euler's theorem applies).

Our first example is a classical result known since antiquity. Note that every convex polyhedron is spherical, so that Euler's theorem applies to proofs involving convex polyhedra.

Theorem 2. *There are no more than five regular convex polyhedra.*

Proof. We start by observing that there must be at least three polygons at a vertex and that each polygon must have at least three vertices and three edges. Let P be a regular convex polyhedron, let d be the number of edges emanating from each vertex (the degree) and let k be the number of sides to a face. (These must all be the same since P is regular.) Let F be the number of faces, let E be the number of edges, and let V be the number of vertices. If we count all the edges around each face, we will have counted each edge twice (once for each edge it appears on) so $Fk = 2E$ or $F = 2E/k$. Likewise, if we count the number of edges coming out of each vertex, we will have again counted each edge twice (once for each vertex at its ends) so $Vd = 2E$ or $V = 2E/d$. Since P is a polyhedron, Euler's formula applies, so $V - E + F = 2$. Substituting for V and F in Euler's formula we see that $\frac{2E}{d} - E + \frac{2E}{k} = 2$, so if we divide both sides of the equation by $2E$ we obtain $\frac{1}{d} - \frac{1}{2} + \frac{1}{k} = \frac{1}{E}$. Note that $\frac{1}{E} > 0$ since $E > 1$, so $\frac{1}{d} + \frac{1}{k} > \frac{1}{2}$. We therefore need only check to see for which values of d and k the inequality holds, as in Table 1, using the fact that both numbers have to be at least three. Observe that only the first 5 rows are admissible cases (each corresponding to a known regular polyhedron), while the last three exhaust all other possibilities, completing the proof. □

It helps to have models of these polyhedra around when discussing this argument, particularly to demonstrate that five Platonic solids do exist. The tetrahedron, cube (hexahedron), octahedron, dodecahedron and icosahedron form the complete set of regular convex polyhedra.

Euler's theorem also has applications to the study of polyhedra that lack the high degree of symmetry seen in the Platonic solids. One interesting class consists of polyhedra whose faces all have five or more sides. Surprisingly, Euler's theorem constrains the number of pentagons in such polyhedra.

Theorem 3. *Any polyhedron made up entirely of polygons with five or more sides must have at least 12 pentagons.*

d	k	$\frac{1}{d}+\frac{1}{k}$	Polyhedron
3	3	2/3	Tetrahedron
3	4	7/12	Cube
4	3	7/12	Octahedron
3	5	8/15	Dodecahedron
5	3	8/15	Icosahedron
3	6	1/2	Hexagonal tessellation
6	2	1/2	Tessellation by equilateral triangles
4	4	1/2	Square tessellation
$d>4$	$k>4$	$<1/2$	

Table 1. Table of admissible and inadmissible values for d and k.

Proof. Let f_n be the number of n-gons in our polyhedron P. Note that f_3 and f_4 are both zero by assumption. The number of faces F in the polyhedron is then $F=\sum_{n=5}^{\infty} f_n$. If we count all the edges around each face and add them up we will have counted each edge twice, so $\sum_{n=5}^{\infty} n\ f_n = 2E$, where E is the number of edges in P. The degree at each vertex of the polyhedron must be greater than or equal to three (as in Theorem 2), so if we count up the number of edges around each vertex, again we would have counted each edge twice, so $3V \le 2E$ or $V \le 2E/3$. The numbers of vertices, edges and faces of P must obey Euler's formula, so $V-E+F=2$. Substituting for the number of vertices and faces from our calculations above we see that

$$
\begin{aligned}
2 &\le \frac{2}{3}E - E + F\\
2 &\le -\frac{1}{3}E + F\\
2 &\le -\frac{1}{3}\cdot\left(\frac{1}{2}\sum_{n=5}^{\infty} n\ f_n\right) + \sum_{n=5}^{\infty} f_n\\
12 &\le -\sum_{n=5}^{\infty} n\ f_n + \sum_{n=5}^{\infty} 6f_n\\
12 &\le -5f_5 - 6f_6 - \sum_{n=7}^{\infty} n\ f_n + 6f_5 + 6f_6 + \sum_{n=7}^{\infty} 6f_n\\
12 &\le f_5 - \sum_{n=7}^{\infty}(n-6)f_n.
\end{aligned}
$$

Since $(n-6)f_n \ge 0$ for all $n \ge 7$, the only way the inequality can hold is if $f_5 \ge 12$. □

This problem can be made simpler for students by limiting the types of polygons to pentagons and hexagons.

Theorem 4. *Any polyhedron made up entirely of regular pentagons and hexagons must have exactly 12 pentagons.*

Proof. The proof follows from use of the notion of deficiency at a vertex, namely that at each vertex, the sum of the angles around that vertex must be less than 360°. Therefore, the degree at each vertex cannot be more than three since the angle contribution of each polygon at a vertex is either 108° or 120°. Let P be a polyhedron made up entirely of regular pentagons and hexagons; then the number of faces F is the sum of the number of pentagons p and the number of hexagons h. Note that since P is a polyhedron, the degree at each vertex d must be at least 3 and due to the deficiency constraint it can be no more than 3, so $d=3$. If we count up all the edges around all the vertices we will have counted each edge twice, so $3V=2E$ or $V=2E/3$. As in the proof of Theorem 3, $5p+6h=2E$. Since P is a polyhedron, the number of vertices V, edges E and faces F of the polyhedron must satisfy $V-E+F=2$.

Substituting for V and F we obtain

$$\begin{aligned} 2 &= \frac{2E}{3} - E + F \\ 2 &= -\frac{1}{3}E + F = -\frac{1}{3} \cdot \frac{1}{2}(5p + 6h) + (p + h) \\ 12 &= -5p - 6h + 6p + 6h = p \end{aligned}$$

completing the proof. □

Theorem 4 also holds if we allow nonregular pentagons and hexagons, but require the polyhedron to be simple. On the other hand, there exist nonsimple polyhedra with nonregular pentagonal and hexagonal faces that have more than 12 pentagonal faces.

A simple polyhedron made up entirely of (not necessarily regular) pentagons and hexagons is called a *fullerene*; certain fullerenes have important applications in chemistry. A famous example is buckminsterfullerene, C_{60}, also known as buckyballs, which was first synthesized in 1985 by Robert F. Curl [1]. Every fullerene contains exactly 12 pentagons, but Euler's formula places no constraints on the number of hexagons. Long before the first buckyballs were studied in the lab, H. S. M. Coxeter raised the question of what values were possible for f_6 in a simple polyhedron made up of only pentagons and hexagons. This question was answered in 1963 by Branko Grünbaum and Theodore Motzkin, who were able to demonstrate that while it was impossible to build a fullerene with exactly one hexagon, it is possible to build one with any other number of hexagons [5]. Readers interested in related topics should consider either [1] or [5].

An argument very similar to that used in Theorems 3 and 4 may be used to demonstrate that every simple polyhedron satisfies the equality

$$\sum_{n=3}^{\infty}(6-n) \cdot f_n = 12.$$

This sum, in turn, is very useful for determining constraints on the number of faces of polyhedra subject to constraints about what kinds of faces are available. For example, any simple polyhedron must have at least some faces with five or fewer sides, any simple polyhedron made up of quadrangles must be combinatorially equivalent to a cube, and in any figure made up of pentagons, hexagons and heptagons, the number of pentagons and heptagons will differ by 12.

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Worksheet: Polyhedra Exploration

Given the polygon pieces, follow the directions below assigned to you by the instructor for assembling one or more polyhedra. After you make the construction(s), follow the directions on the next page. We will then inventory the results for the whole class.

0. (Demonstration) Given: 4 equilateral triangles. Make a polyhedron.

1. Given: 14 equilateral triangles. Make two polyhedra, following the directions in the next two parts.
   (a) At every corner, make four triangles come together.
   (b) Make a polyhedron that has at least two different numbers of faces meeting at a corner using the remaining pieces.

2. Given: 8 regular hexagons and 4 equilateral triangles.
   (a) Make a polyhedron using four hexagons and four triangles; every corner is the same.
   (b) Can you make a polyhedron using only regular hexagons? Give reasons for your answer.

3. Given: 7 squares and 4 equilateral triangles. Make two polyhedra:
   (a) Use exactly six squares.
   (b) Use the remaining square and four triangles.

4. Given: 6 squares and 8 equilateral triangles. Make a polyhedron with every corner the same: triangle, square, triangle, square, in that order.

5. Given: 12 regular pentagons. Make a polyhedron with these.

6. Given: 2 regular pentagons, 2 equilateral triangles, 8 squares. Make two polyhedra:
   (a) Use the triangles and exactly three squares; make every corner the same.
   (b) Use the pentagons and the remaining squares; make every corner the same.

7. Given: 20 equilateral triangles. Make a polyhedron with every corner the same: five triangles come together.

8. Given: 2 squares, 8 equilateral triangles. Make a polyhedron with these. Make every corner the same: three triangles and one square come together.

9. Given: 8 equilateral triangles and 6 octagons. Make every corner the same.

10. Given: 6 squares and 8 hexagons. Make every corner the same.

After you have constructed the polyhedra, follow these directions:

For each polyhedron you have constructed, count the number of its corners or *vertices* (V), the number of its edges (E), and the number of its faces (F). Enter these numbers in the chart as well as the sum $V + F$. If you know the name of the polyhedron, fill it in.

Polyhedron	Name	V (# vertices)	F (# faces)	E (# edges)	$V + F$
0	tetrahedron	4	4	6	8
1 (a)					
1 (b)					
2 (a)					
2 (b)					
3 (a)					
3 (b)					
4					
5					
6 (a)					
6 (b)					
7					
8					
9					
10					

What do you notice?

Worksheet: Symmetric Polyhedra and Angle Deficiency

Working in teams of 2 or 3, carry out the steps indicated below, and record your observations.

Part I, Some simple figures

Exercise 1. Take the pieces necessary to make up one corner of a cube. Split them along one of the edges, and lay them flat. What is the angle measure of the gap formed along the separated edge?

What you have just measured is the *angle deficiency* of a vertex of the cube. In general, the angle deficiency of a vertex of a polyhedron is the sum of the angles formed by the polygons that meet at that vertex, measured in radians, subtracted from 2π.

When describing the vertices of a symmetric polyhedron made up of regular polygons, a convenient way of cataloging them is to list the numbers of edges that belong to each face at a vertex. For example, in the cube, three squares meet at each vertex. We would denote such a figure by the symbol (4.4.4), because each square has four edges, and call the faces that make up that corner a *vertex star*. In general, a vertex star is a collection of faces that meet at a vertex.

The *total angle deficiency* of a polyhedron is the sum of the angle deficiencies of the vertices of the polyhedron. In other words, if a figure has 8 vertices (like the cube), you would add up the deficiencies at each of the vertices. In the case of the cube this would be 4π.

For the following exercises it may be helpful to recall that the interior angle of a regular n-gon is $\pi(n-2)/n$ radians.

Exercise 2. Build vertex stars for at least two of the polyhedra described below, collect the appropriate data and complete the table.

name	vertex symbol	# vertices	angle deficiency at a vertex	total angle deficiency
tetrahedron	(3.3.3)	4		
octahedron	(3.3.3.3)	6		
icosahedron	(3.3.3.3.3)	12		
cube	(4.4.4)	8		4π
dodecahedron	(5.5.5)	20		

Worksheet: Symmetric Polyhedra and Angle Deficiency

Part II, Some slightly more complicated figures

An important class of polyhedra have faces that are regular polygons, with vertices such that every vertex "looks the same," and at least two kinds of faces. These polyhedra have very high degrees of symmetry, and traditionally they are called *Archimedean solids*.

Exercise 3. Figure out the missing entries in the table below.

name	vertex symbol	# vertices	angle deficiency at a vertex	total angle deficiency
truncated tetrahedron	(3.6.6)	12		
truncated cube	(3.8.8)	24		
truncated octahedron	(4.6.6)	24		
truncated dodecahedron	(3.10.10)	60		
truncated icosahedron	(5.6.6)	60		
cuboctahedron	(3.4.3.4)	12		
icosidodecahedron	(3.5.3.5)	30		
snub dodecahedron	(3.3.3.3.5)	60		
rhombicuboctahedron	(3.4.4.4)	24		
great rhombicosidodecahedron	(4.6.10)	120		
rhombicosidodecahedron	(3.4.5.4)	60		
great rhombicuboctahedron	(6.4.8)	48		
snub cube	(3.3.3.3.4)	24		

Exercise 4. Based on the results from Exercise 3, what do you conjecture?

Exercise 5. Recall that the angle deficiency at a vertex of a polyhedron $\mathcal{P}$ is the sum of the angles at the corners of the faces that meet at that angle subtracted from 2π. Suppose the polyhedron $\mathcal{P}$ has V vertices, E edges and F faces, and suppose that

$$f_1, f_2, \ldots, f_F \text{ are the faces of } \mathcal{P},$$
$$v_1, v_2, \ldots, v_V \text{ are the vertices of } \mathcal{P}.$$

(a) Explain why $\sum_{i=1}^{V}(\text{sum of angles around } v_i) = \sum_{j=1}^{F}(\text{sum of angles in } f_i)$.

(b) Suppose that n_{f_j} is the number of sides of face f_j. What is the sum of the angles in face f_j, in radians?

(c) What is $\sum_{j=1}^{F} n_{f_j}$ in terms of the number of edges, E, of the polyhedron? Why?

(d) Note that by definition,

$$\text{total angle deficiency} = \sum_{i=1}^{V}\left(2\pi - (\text{sum of the angles around } v_i)\right).$$

Use the facts you showed from (a), (b) and (c) and Euler's theorem to determine the total angle deficiency of $\mathcal{P}$. Does the answer you get agree with your conjecture from Exercise 4?

Worksheet: Poincare's Formula and Higher Genus Polyhedra

Exercise 1. Using the template provided, construct a polyhedral "donut" (formally, a torus, plural tori).

(a) Count the number of vertices, edges and faces. What is $V - E + F$?

(b) Take two of the polyhedral tori, and glue them along an outer rectangle. You have now created a surface with two holes, called a two-holed torus. Count the number of vertices, edges and faces, ignoring the glued face. What is $V - E + F$?

(c) Take three of the polyhedral tori, and glue them along two of the outer rectangles. You have now created a surface with three holes, called a three-holed torus. Count the number of vertices, edges and faces, ignoring the glued faces. What is $V - E + F$?

(d) If you were to create a four-holed torus, what do you predict for the value of $V - E + F$?

(e) What about an n-holed torus?

Exercise 2. Consider a solid cube with holes punched out of it in the following way:

- Subdivide each face of a solid cube into nine equal squares whose sides are one third the length of the side of the cube.
- For each pair of opposing middle squares, cut out a square hole going all the way through the cube.

Without calculating anything, try to determine how many holes there are on the surface formed. Note that the three tubular holes all meet in a smaller cube ($1/27^{th}$ the size) at the center of the cube. Once you think you have a reasonable guess, use Poincaré's formula to check your intuition. Note that when you are counting faces of the punched-out cube, a face cannot itself have a hole in it!

Solutions to Worksheets and Additional Exercises

Worksheet Solutions: Polyhedra Exploration

Polyhedron	Name	V (# vertices)	F (# faces)	E (# edges)	V+F
0	tetrahedron	4	4	6	8
1 (a)	octahedron	6	8	12	14
1 (b)	triangular bipyramid	5	6	9	11
2 (a)	truncated tetrahedron	12	8	18	20
2 (b)	(impossible)				
3 (a)	cube	8	6	12	14
3 (b)	square pyramid	5	5	8	10
4	cuboctahedron	12	14	24	26
5	dodecahedron	20	12	30	32
6 (a)	triangular prism	6	5	9	11
6 (b)	pentagonal prism	10	7	15	17
7	icosahedron	12	20	30	32
8	square antiprism	8	10	16	18
9	truncated cube	24	14	36	38
10	truncated octahedron	24	14	36	38

We notice that in all these cases, $V + F = E + 2$.

Exercise and Worksheet Solutions: Symmetric Polyhedra and Angle Deficiency

Exercise 0, from Notes to the instructor. Prove that the sum of the angles in a convex n-gon is $180(n-2)$. Determine the measure of a single angle in a regular n-gon.

Base case: For a single triangle ($n = 3$), the sum of the angles is 180°.

Induction hypothesis: Suppose, for any convex k-gon with $k < n$, that the sum of the angles in the convex k-gon is $180(k-2)$.

We need to show that given any $n > 3$, the sum of the angles in the convex n-gon is $180(n-2)$. Label the vertices of the n-gon cyclically (say, counterclockwise) as $v_1, v_2, \ldots, v_n$, and construct the diagonal $v_1 v_3$; note that since the n-gon is convex, this diagonal must lie inside the polygon. The diagonal divides the n-gon into two smaller polygons, one triangle and one $(n-1)$-gon. Applying the induction hypothesis to each piece, we conclude that the sum of the angles of the triangle is 180, and the sum of the angles of the $(n-1)$-gon is $180(n-3)$; adding the sums together, we see that the total angle sum of the n-gon is $180(n-2)$, as desired.

With this, we see that a single angle of a regular n-gon is $180(n-2)/n$ degrees.

Exercise 1, from Worksheet Part I. The angle measure of the gap is $90° = \frac{\pi}{2}$ radians.

Exercise 2, from Worksheet Part I.

name	vertex symbol	# vertices	angle deficiency at a vertex	total angle deficiency
tetrahedron	(3.3.3)	4	π	4π
octahedron	(3.3.3.3)	6	$\frac{2\pi}{3}$	4π
icosahedron	(3.3.3.3.3)	12	$\frac{\pi}{3}$	4π
cube	(4.4.4)	8	$\frac{\pi}{2}$	4π
dodecahedron	(5.5.5)	20	$\frac{\pi}{5}$	4π

Exercise 3, from Worksheet Part II.

name	vertex symbol	# vertices	angle deficiency at a vertex	total angle deficiency
truncated tetrahedron	(3.6.6)	12	$\frac{\pi}{3}$	4π
truncated cube	(3.8.8)	24	$\frac{\pi}{6}$	4π
truncated octahedron	(4.6.6)	24	$\frac{\pi}{6}$	4π
truncated dodecahedron	(3.10.10)	60	$\frac{\pi}{15}$	4π
truncated icosahedron	(5.6.6)	60	$\frac{\pi}{15}$	4π
cuboctahedron	(3.4.3.4)	12	$\frac{\pi}{3}$	4π
icosidodecahedron	(3.5.3.5)	30	$\frac{2\pi}{15}$	4π
snub dodecahedron	(3.3.3.3.5)	60	$\frac{\pi}{15}$	4π
(small) rhombicuboctahedron	(3.4.4.4)	24	$\frac{\pi}{6}$	4π
great rhombicosidodecahedron	(4.6.10)	120	$\frac{\pi}{30}$	4π
(small) rhombicosidodecahedron	(3.4.5.4)	60	$\frac{\pi}{15}$	4π
great rhombicuboctahedron	(6.4.8)	48	$\frac{\pi}{12}$	4π
snub cube	(3.3.3.3.4)	24	$\frac{\pi}{6}$	4π

Exercise 4, from Worksheet Part II. We conjecture that the total angle deficiency is 4π.

Exercise 5, from Worksheet Part II.

(a) Both sums $\sum_{i=1}^{V}$(sum of angles around v_i) and $\sum_{j=1}^{F}$(sum of angles in f_i) give the total measure of all the angles around all vertices. The left-hand sum does it by summing the angle measure around each vertex, while the right-hand sum does it by summing the angle measure around each face.

(b) Note that a if polygon face f_j has n_{f_j} edges, it also has n_{f_j} vertices. Therefore, the sum of the angles in face f_j is $\pi(n_{f_j} - 2)$.

(c) The sum $\sum_{j=1}^{F} n_{f_j}$ equals twice the number of edges. To see this, imagine each time a vertex is counted, a little bit of each edge that is incident with that vertex is highlighted. Since the sum counts all the vertices exactly once, and each edge is incident with two vertices, if we count the "highlighted" parts of the edges as we count the vertices, each edge will be counted twice. That is, $\sum_{j=1}^{F} n_{f_j} = 2E$.

(d) Note that

$$\begin{aligned}
\text{total angle deficiency} &= \sum_{i=1}^{V} (2\pi - (\text{sum of the angles around } v_i)) \\
&= 2\pi V - \sum_{i=1}^{V} (\text{sum of the angles around } v_i) \\
&= 2\pi V - \sum_{i=1}^{F} (\text{sum of the angles in } f_i) && \text{(by (a))} \\
&= 2\pi V - \left(\sum_{i=1}^{F} \pi (n_{f_j - 2}) \right) && \text{(by (b))} \\
&= 2\pi V - \left(\sum_{i=1}^{F} \pi n_{f_j} - \sum_{i=1}^{F} 2\pi \right) \\
&= 2\pi V - 2\pi E + 2\pi F && \text{(by simplifying and (c))} \\
&= 2\,(V - E + F)\,\pi \\
&= 2(2)\pi && \text{(by Euler's theorem).}
\end{aligned}$$

That is, the total angle deficiency equals 4π, which is what we had conjectured.

Exercise 6, from Notes for the instructor. Yes. The prisms and antiprisms each have two kinds of faces; an n-prism has each corner consisting of a regular n-gon and two squares, while an n-antiprism has each corner consisting of a regular n-gon and three equilateral triangles.

5 Worksheet: Poincare's Formula and Higher Genus Polyhedra

Exercise 1.

(a) There are 9 vertices, 9 faces, and 18 edges, so $V - E + F = 0$.

(b) Each torus has 9 vertices, 9 faces, and 18 edges. We need to remove entirely one face of each torus to glue them together, so the two-holed torus has $(9 - 1) + (9 - 1) = 16$ faces. Also, we need to remove four edges and four vertices from one of the tori, so the two-holed torus has $(9 - 4) + 9 = 14$ vertices and $(18 - 4) + 18 = 32$ edges. Therefore, the two-holed torus has $V - E + F = -2$.

(c) Reasoning as before, the three-holed torus has $(16 - 1) + (9 - 1) = 23$ faces, $(14 - 4) + 9 = 19$ vertices, and $(32 - 4) + 18 = 46$ edges, so $V - E + F = -4$.

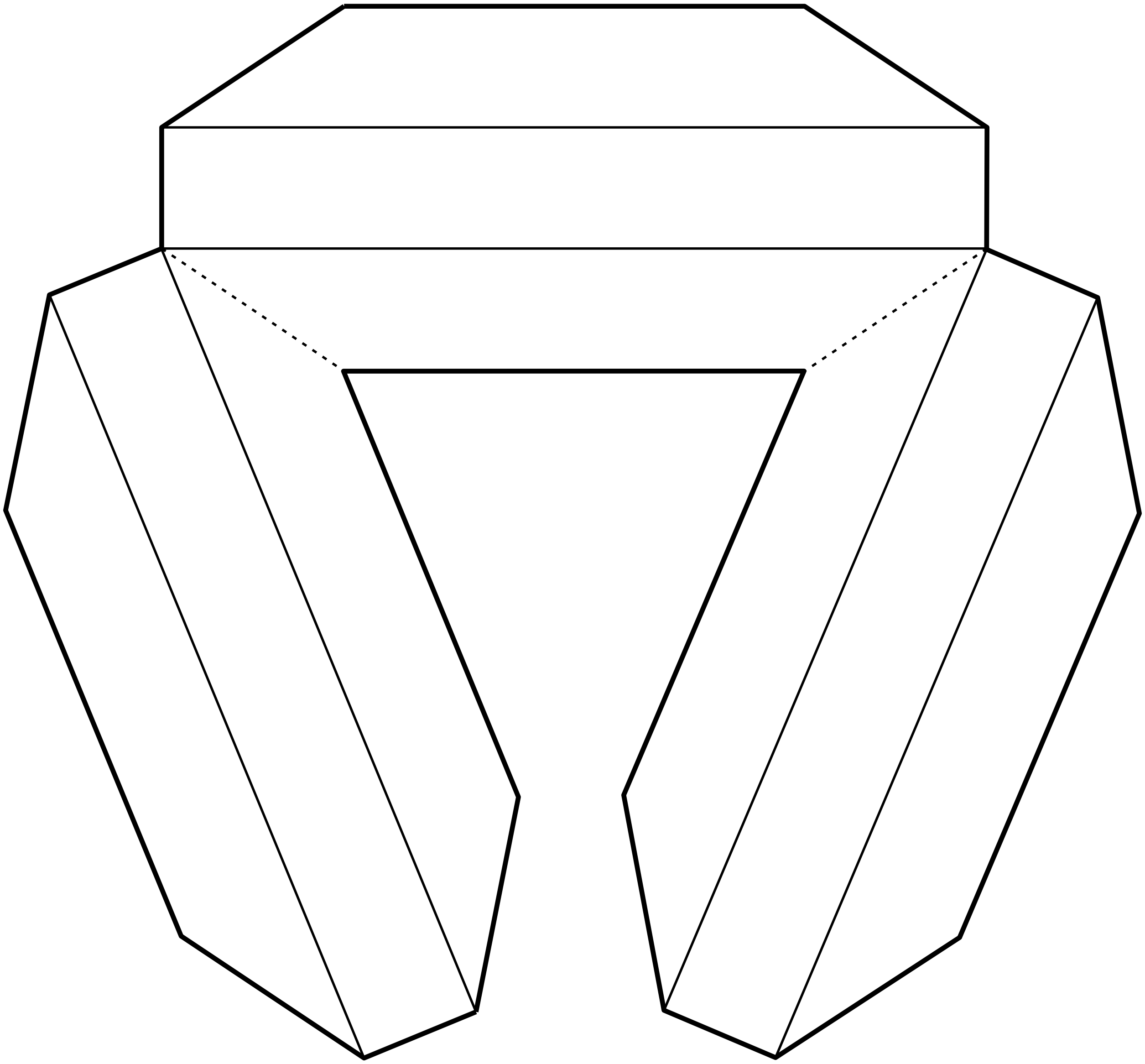
(d) For a four-holed torus, we predict $V - E + F = -6$.

(e) For an n-holed torus, we predict $V - E + F = 2 - 2n$.

Exercise 2. The correct answer to this question is, somewhat surprisingly, five holes. One way to understand this is that the first square hole punches one hole out of the cube, but the remaining two punches each cut two more holes (since they pass through the space punched out by the first punch). If we break up the exterior so that each face that has been punched through now has eight squares on it, using Poincaré's formula we note that there are 64 vertices, 144 edges and 72 faces, so we have $64 - 144 + 72 = -8 = 2 - 2(5)$, or genus 5. Alternately, one may cut the exterior faces that had the holes punched through them along their diagonals into two nonconvex c-shaped pieces, in which case we have 40 vertices, 84 edges and 36 faces, yielding the same conclusion.

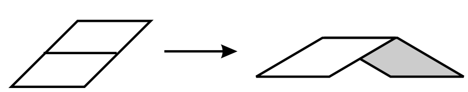
It is worth noting that this construction is the first iteration in the construction of the Menger Sponge.

Template for a paper torus



cut

mountain fold (fold away from you)



valley fold (fold towards you)