Fun and Games with Squares and Planes

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Summary

This project is intended to introduce students to the concepts of mutually orthogonal Latin squares and their relationship to finite affine planes. These topics are introduced in the first section. After describing how tic-tac-toe is played on an affine plane, the second section explores player strategies. Playing the game will help students understand the combinatorial and geometric notions described, and build geometric intuition for these objects.

Notes for the instructor

Students must play the game in order to understand the strategy arguments. You can have them play against each other in class or turn in their game sheets as an exercise. While the last exercise may be an exercise in frustration, it is important for students to repeatedly play the game.

For a class project, have the members of your class play a "Tic-tac-toe on the affine plane of order 4" tournament. Our Mathematics Club holds an annual tic-tac-toe tournament at the University of Scranton with prizes for the top finishers. We have best-of-three matches to decide the winner of each random pairing, with a toss of a coin deciding who makes the first move. While perfect play will result in a win for the first player, you can rely on your students to make mistakes!

As an additional project, ask your students to create a new game to play on an affine plane. For example, how about a game where the player who claims the last unclaimed point on a line loses? As another example, what happens on π_n if a player needs only $n-1$ of their marks on any line to win? Some of these games may be easy to analyze strategically while others could be extremely difficult.

There are other examples of generalizations of tic-tac-toe. Several examples of these are given in the reading list below. For those who wish to learn about the weight function techniques we used to prove our game theoretic results, see our paper listed below.

Bibliography

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1 Worksheet on Latin squares

In 1782, the great Leonhard Euler posed the following question. Can 36 officers, with six different ranks from six different regiments, be arranged in a 6 by 6 square such that each row and column has each rank and regiment exactly once? With this deceptively easy question Euler started one of the most widely studied and productive branches of discrete mathematics.

As often happens in mathematics, the ease with which a question is posed can be adversely proportional to the difficulty in finding a solution. A solution to Euler's question remained unpublished until 1901 when a French colonel named Tarry computed (by hand) all possible arrangements. It took Tarry one year to do what would now take a computer under an hour, eventually finding that there was no such arrangement. This is exactly what Euler had predicted, but could not prove.

Euler approached the problem in the following way. He used Latin letters (A,B,C,D, ...) to represent the ranks and Greek letters ($\alpha, \beta, \gamma, \ldots$, ...) to represent the regiments. So each officer now had a unique representation consisting of one Latin letter and one Greek letter, an ordered pair. Euler then split the problem into arranging the regiments and arranging the ranks. To arrange either you simply had to make a square in which each letter appeared exactly once in each row and each column. This object has come to be known as a Latin square regardless of the alphabet employed. For example here are two 4 by 4 Latin squares.

$$\begin{pmatrix} A & B & C & D \\ B & A & D & C \\ C & D & A & B \\ D & C & B & A \end{pmatrix} \qquad \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ \delta & \gamma & \beta & \alpha \\ \beta & \alpha & \delta & \gamma \\ \gamma & \delta & \alpha & \beta \end{pmatrix} \tag{1}$$

If Euler had posed a 16 officer problem, a solution could be constructed by overlapping the two Latin squares above so that each ordered pair of Latin and Greek letters appears exactly once. We say that the two Latin squares are *orthogonal* if this occurs. Overlapping the two squares above produces the Graeco-Latin square:

$$\begin{pmatrix} A\alpha & B\beta & C\gamma & D\delta \\ B\delta & A\gamma & D\beta & C\alpha \\ C\beta & D\alpha & A\delta & B\gamma \\ D\gamma & C\delta & B\alpha & A\beta \end{pmatrix} \tag{2}$$

Exercise 1.1. Verify that each letter appears exactly once in each row and column of both squares of (1), and that each ordered pair appears exactly once in (2).

This Graeco-Latin square proves that it is possible to arrange 16 officers in a 4 by 4 square such that each of the ranks and regiments appears exactly once in each row and column. The ease of this construction makes the impossibility of the 36 officer problem even stranger. To further add to the mysterious nature of this problem, it *is* known that n^2 officers can be arranged in such an n by n square for all n except $n = 2$ and $n = 6$.

Interestingly, we can expand the structure in (2) by using the the natural numbers, $0, 1, 2, 3, \ldots,$ as our alphabet. This is the alphabet that is typically used for all Latin squares since it allows for the use of the algebra of the natural numbers in constructions. Consider the following Latin square:

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \\ 1 & 0 & 3 & 2 \end{pmatrix} \tag{3}$$

We can overlap this 4 by 4 square with each of the others in (1) as shown below.

$$\begin{pmatrix} 0A & 1B & 2C & 3D \\ 2B & 3A & 0D & 1C \\ 3C & 2D & 1A & 0B \\ 1D & 0C & 3B & 2A \end{pmatrix} \qquad \begin{pmatrix} 0\alpha & 1\beta & 2\gamma & 3\delta \\ 2\delta & 3\gamma & 0\beta & 1\alpha \\ 3\beta & 2\alpha & 1\delta & 0\gamma \\ 1\gamma & 0\delta & 3\alpha & 2\beta \end{pmatrix} \tag{4}$$

Exercise 1.2. Verify that each ordered pair appears exactly once in both squares of (4).

In (2) and (4) we have shown that the Latin squares given in (1) and (3) are Mutually Orthogonal Latin Squares (MOLS) of order 4. This means that there are three 4 by 4 squares, each pairing of which is orthogonal. Here is a way to represent these three MOLS using only the first four natural numbers as our alphabet.

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 2 & 3 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \\ 1 & 0 & 3 & 2 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \end{pmatrix} \tag{5}$$

Exercise 1.3. Construct another Latin square of order 4, different from those in (5) but using the same alphabet.

For the following exercise, consider the following Latin squares of order 4.

$$P = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 0 & 1 & 2 \\ 2 & 3 & 0 & 1 \\ 1 & 2 & 3 & 0 \end{pmatrix} \qquad Q = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 2 & 3 & 0 & 1 \\ 1 & 2 & 3 & 0 \\ 3 & 0 & 1 & 2 \end{pmatrix} \tag{6}$$

Exercise 1.4.
(a) Are P and Q orthogonal to each other?
(b) Is P orthogonal to the Latin squares in (5)? Is Q?
(c) Using the same alphabet, can you find another Latin square, R, for which P, Q and R are MOLS?

At this point you may have conjectured that there are at most three MOLS of order 4. We shall now show that the largest number of MOLS of order n is $n-1$. Suppose we have a set of MOLS of order n. We can assume that the first row of each is arranged in ascending order, $0, 1, 2, 3, 4, \ldots, n-1$, since if it were not we could simply rename the elements so that it fits this arrangement. Consider the elements in the first column of the second row in these squares. These elements must be distinct and different from 0. Why? They must be distinct since the ordered pairs $(0,0), (1,1), \ldots, (n-1, n-1)$ are produced in the first row when overlapping any two of these squares. They must be different from 0 since there can only be one 0 in the first column. Therefore we only have $n-1$ possibilities, and we can have at most $n-1$ MOLS of order n.

If we have a set of $n-1$ MOLS of order n then we say that it is a complete set of MOLS of order n. Here is the principal question which has generated over 200 years of research and remains largely unsolved.

Fundamental Question: For which n are there complete sets of MOLS?

This is another example of a question which is easily stated, can be explained to almost anyone, and has withstood the attempts of a great number of mathematicians for hundreds of years. It is precisely because of this that Gary Mullen, a well-known mathematician in this field, recently suggested it as the next Fermat problem. As we will see, this question is equivalent to one of the most important questions of finite geometry, namely when do projective and affine planes exist.

There are some simple constructions for a complete set of MOLS of prime order which we shall now describe. First we recall some basic definitions of modular arithmetic. We say that $a \equiv b \pmod{n}$ if $a - b$ is a multiple of n. We can take as the representatives of all natural numbers modulo n the set $\{0, 1, 2, \ldots, n-1\}$, and use this set as our alphabet. When we add or multiply a and b we take the representative from this set that is equivalent to the sum or product modulo n. For example, $3 + 5 \equiv 1 \pmod{7}$ and $3(4) \equiv 5 \pmod{7}$. Using these operations, we can construct tables not unlike elementary school multiplication tables, but utilizing modular arithmetic. As an example, addition and multiplication tables for modulo 3 arithmetic are shown below. Notice that each cell in the grid is the result of its row heading and column heading under the specified operation modulo 3.

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

*	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Exercise 1.5. Construct the addition and multiplication tables for modulo 5 arithmetic.

We shall now attempt to construct $n-1$ MOLS, $L^1, L^2, \ldots, L^{n-1}$, of order n using the set $\{0, 1, 2, \ldots, n-1\}$ as our alphabet. We let L^k_{ij} denote the entry in the i-th row and j-th column of the Latin square, or matrix, L^k.

Orthogonal Construction: Define $L^k_{ij} = ki + j \pmod{n}$, where i, j and k are members of our alphabet set.

For example, when $n = 3$ we have

$$L^1 = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad L^2 = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix}. \tag{7}$$

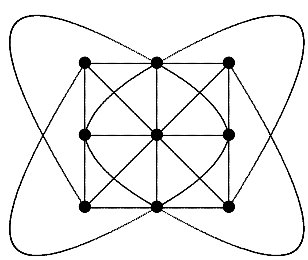
Exercise 1.6. Verify that L^1 and L^2 are orthogonal.

Notice that this construction technique has produced the largest possible set of MOLS of order 3 since we know there can be at most two. Will this construction technique work for all possible values of n?

Exercise 1.7. Construct the L^k's when $n = 4, 5$ and 6:
(a) Use the Orthogonal Construction technique to form L^1, L^2, and L^3 when $n = 4$.
(b) Use the construction to form L^1, L^2, L^3 and L^4 when $n = 5$.
(c) Use the construction to form L^1, L^2, L^3, L^4 and L^5 when $n = 6$.
(d) Did the construction yield $n-1$ MOLS of order n in these three cases? If not, do you have a conjecture as to when this construction will yield $n-1$ MOLS?

If you conjectured that this construction technique only works for prime n then you are correct. (You will be asked to supply a proof for this in the last project.) A similar construction technique is used to generate the three MOLS of order 4 but the addition and multiplication operations are defined in a different way. Specifically, it is not modulo 4 arithmetic but the arithmetic of the finite field of order 4.

The relationship between MOLS and finite geometry was hinted at in reference to the Fundamental Question. Let's look at an example to explore the nature of this connection. Arrange nine points in a square and connect each set of three horizontal points and each set of three vertical points, using a different colored marking device to connect each set. For the next part, think of your grid of points as corresponding to L^1 in (7), and connect each set of three points that have a common symbol, using a different color for each set. Do the same with L^2. The result should be similar to the following diagram, but much more colorful.



Each connection of three points is called a line. How many lines are there? Your answer should be the same as the number of colors used, namely twelve. Notice that any two lines either meet once or are parallel (no points in common). This object is known as an affine plane of order 3. In the following four exercises you will construct MOLS for higher order affine planes. These constructions are needed to proceed to the games in the second part of the project.

Exercise 1.8. Construct the affine plane of order 4: Put 16 points in a square and draw a line for each set of four vertical and horizontal points. Next, using the three MOLS of order 4 given in (5), connect each set of four points with the corresponding common symbols. Use a different color for each line to make the connections more visible.

Exercise 1.9. Use the 4 MOLS of order 5 from Exercise 1.7 to draw the corresponding affine plane of order 5 using the method outlined in Exercise 1.8.

Exercise 1.10. Use the Orthogonal Construction technique to construct 6 MOLS of order 7 and then use them to draw the corresponding affine plane of order 7 as in Exercise 1.8.

Exercise 1.11. Prove the following result. The Orthogonal Construction technique produces $p-1$ MOLS of order p if and only if p is a prime. Here is an outline of one way to proceed: First show that L^k is a Latin square, that is, show that each element appears once in each row and column. Then prove that if p is a prime then L^k and L^j are orthogonal by showing that each ordered pair is represented exactly once. The following fact will be useful. If p is prime and $ax \equiv b \pmod{p}$ then there is a unique x in $\{0, 1, \ldots, p-1\}$ satisfying the equation. Next, show it will not work when p is not a prime. An easy way to do this is to show that the construction will not produce Latin squares in all cases.

Epilogue

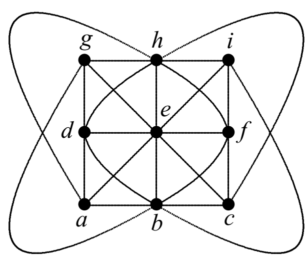
Now that you have developed an understanding for the Fundamental Question, you may be wondering whether there has been any progress towards a solution. Well, it is known that a complete set of MOLS exists for all orders p^k where p is prime and k is a positive integer. It is not known if there exists a complete set of MOLS for any other value. The problem has proven to be unbelievably difficult. For example, it has been shown that there is no complete set of MOLS of order 10. It took numerous theoretical papers and one year of computation on a supercomputer to settle this particular case. It did not determine the largest number of MOLS of order 10, only that there are not 9 of them. The difficulty encountered with the $n = 10$ case is what is known as combinatorial explosion. While it is easy to settle the 6 by 6 case with the aid of a computer, it is impossible for a computer to completely settle the 10 by 10 case since the number of Latin squares increases too quickly.

2 Worksheet on Games

The great power of describing mathematics in geometric terms is that it allows us to apply the intuition we have developed since we were children to very complicated problems. For example, understanding the possible simultaneous solutions to equations of the form $(x-h)^2 + (y-k)^2 = r^2$ and $ax + by + c = 0$ is greatly simplified upon realizing that these equations may be described geometrically as a circle and a line, respectively. Likewise, while it is possible to study complete sets of MOLS simply by their definition, our understanding will be greatly enhanced by viewing them as a geometry. Specifically, they can be seen as an affine plane which satisfies axioms similar to those we know well from the Cartesian plane. Even though we will tinker slightly with the definitions of point, line and plane in this new context, we can still apply a good deal of the intuition that we have developed for the Cartesian plane. In this section, we shall describe a very effective way to understand the geometry described in the previous section by using an unlikely learning tool, tic-tac-toe. By playing this game, we will develop geometric intuition which will aid in the understanding of MOLS and affine planes.

Tic-tac-toe and variations on its theme are abundant, appearing in children's books, journals, game theory textbooks and on the Internet. The rules for tic-tac-toe are simple enough for a young child to learn. A player must mark all of the positions on a line in order to win the game. While the standard version of tic-tac-toe is played on a 3 by 3 grid, it can easily be expanded to any n by n grid, with the usual horizontal, vertical and diagonal lines as the winning lines. We could take this generalization a bit further by playing on an $n \times n \times n$ cube or a hypercube, that is, a cube in n dimensions. You can challenge the computer to a game of tic-tac-toe played on a torus or a Klein bottle on the Internet at `www.geometrygames.org`. We will use a different game board, namely the affine plane that was introduced in the previous section. That is, we shall describe the game played on a complete set of MOLS.

The diagrams created in the previous section will be used to develop intuition for affine planes. As a start, let's look at the representation of the affine plane of order 3 constructed in the previous section. It may be helpful to use your colored diagram as a reference. Labels have been inserted so that we may refer to individual points.



This plane has 9 points and 12 lines, each of these lines consists of three points and, in your representation, each has a different color. As you may have speculated, this is called the affine plane *of order 3* since each line has exactly three points, and it is denoted as π_3. Referring to each line as a set of three points, the lines are:

$$\{a,b,c\},\{d,e,f\},\{g,h,i\},$$
$$\{a,d,g\},\{b,e,h\},\{c,f,i\},$$
$$\{a,e,i\},\{c,e,g\},\{a,h,f\},$$
$$\{g,b,f\},\{i,b,d\},\{c,h,d\}.$$

These lines appear to be quite different from those found in the Cartesian plane but we shall see that they satisfy similar axioms. Lines in the Cartesian plane are continuous, contain infinitely many points, and their representation is described as straight. Lines on this affine plane are discrete, contain a finite number of points, and their representation cannot necessarily be drawn with a straightedge. It is an example of a finite plane. Though we have "connected the dots" in our representation, there are no points in between which belong to the plane. When we connect the dots we are merely representing the simple fact that the connected points form a line. Therefore, our representation of any line in a finite affine plane does not need to be straight.

The concept of parallel lines remains unchanged in this context. They are simply lines having no points in common, for example $\{g, b, f\}$ and $\{c, h, d\}$. Additionally, we say that a line is parallel to itself in the same way we do on the Cartesian plane. The lines $\{g, b, f\}$ and $\{i, b, d\}$ are not parallel since they meet at the point b, in other words b is a point on both lines. Notice that an intersection of lines in the representative diagram does not necessarily indicate a point common to both lines. For example, the lines $\{i, b, d\}$ and $\{c, h, d\}$ appear to meet in the diagram, but upon further inspection you will notice that they do not have a point in common.

Exercise 2.1. Questions about the affine plane of order 3, π_3:
(i) List the lines which are parallel to $\{a, b, c\}$.
(ii) List the lines which are parallel to $\{a, d, g\}$.
(iii) List the lines which are parallel to $\{g, b, f\}$.
(iv) List the lines which are parallel to $\{c, e, g\}$.

The answers to parts (i), (ii), (iii) and (iv) are called parallel classes. The affine plane of order 3 has 4 parallel classes of 3 lines each since it has 12 lines in total.

Exercise 2.2. Questions about the affine plane of order 4, π_4, constructed in Exercise 1.8 in the previous section: First, label the 16 points in the same manner as in π_3, starting with the letter a in the lower left corner and proceeding accordingly.
(i) List the lines which are parallel to $\{a, b, c, d\}$.
(ii) List the lines which are parallel to $\{a, e, i, m\}$.
(iii) List the lines which are parallel to $\{d, g, j, m\}$.
(iv) List the lines which are parallel to $\{a, g, l, n\}$.
(v) List the lines which are parallel to $\{a, h, j, o\}$.

Exercise 2.3. One of the primary techniques of mathematics is generalizing to all cases from a few known cases. In this exercise, use your answers from the previous two exercises to finish filling in the first two columns. Using similar techniques, consult your solutions to Exercises 1.9 and 1.10 to fill in the third and fourth columns. Try to recognize the patterns emerging in order to formulate some conjectures about the general case, π_n.

Number of $\Downarrow$ in $\Rightarrow$	π_3	π_4	π_5	π_7	π_n
points	9	16			
lines	12				
points on each line	3	4			
lines through any point					
points where two non-parallel lines meet					
lines parallel to any given line [Note: Every line is parallel to itself.]	3				
parallel classes	4				

An example of the method for constructing these finite affine planes from their MOLS, specifically for π_3, was given after Exercise 1.7, but the reason for this method was not. It is clear from these exercises that the underlying

structure has a great deal of symmetry. We cannot take any finite set of points, connect them as we wish, and call it an affine plane. As is always the case in mathematics, there are rules to be followed. Specifically, an affine plane is a nonempty set of points, P, and a nonempty collection of subsets of points (called lines) which satisfy the following three axioms:
(1) through any two distinct points there exists a unique line;
(2) if p is a point, ℓ is a line, and p is not on line ℓ, then there exists a unique line, m, that passes through p and is parallel to ℓ;
(3) there are at least two points on each line, and there are at least two lines.

Let's verify one instance of the second axiom on the affine plane of order 4. The point f is not on the line $\{a, g, l, n\}$. There are five lines through f, namely $\{e, f, g, h\}$, $\{b, f, j, n\}$, $\{a, f, k, p\}$, $\{d, f, i, o\}$ and $\{c, f, l, n\}$, of which only $\{d, f, i, o\}$ is parallel to $\{a, g, l, n\}$.

Exercise 2.4. Verify that your representation of π_3 satisfies these axioms.

When these rules are followed we can create a finite or an infinite affine plane. The Cartesian plane, with points and lines defined as usual, is the example we typically envision when reading this definition. It is an example of an infinite affine plane. Since we intend to play tic-tac-toe on finite planes, we will focus on these. Every line on a finite affine plane has the same number of points, and as you have surely noticed, this number is called the order of the plane. In an affine plane of order n there are n^2 points. If you are thinking ahead towards playing the game, you may notice that n^2 is the perfect number of points needed for an $n \times n$ grid.

Since we may not always have a representative diagram as a reference, how can we easily identify the lines through any given point without the use of one? Well, the horizontal and vertical lines should be clearly visible. As for the others, you may recall that in the previous section we constructed the representative diagrams by using the MOLS. So, even if we did not have a diagram we could still easily find the other lines by consulting the MOLS. In each Latin square, the points corresponding to identical symbols form a line. So, we can find one line per Latin square through any given point. On the affine plane of order 4, for example, through any given point we have one horizontal line, one vertical line, and one line from each of the three MOLS of order 4, for a total of five lines. For example, consulting both the MOLS in (5) and your labeled representation of π_4 from Exercise 2.2 we can easily see the five lines through the point labeled “a” on π_4:

$\{a, b, c, d\}$	horizontal line
$\{a, e, i, m\}$	vertical line
$\{a, f, k, p\}$	line determined by the symbol 3 in first Latin square
$\{a, g, l, n\}$	line determined by the symbol 1 in second Latin square
$\{a, h, j, o\}$	line determined by the symbol 2 in third Latin square

Exercise 2.5.
(a) Using the labeled representation of π_3, identify the two lines determined by the symbol 1 in the associated MOLS L^1 and L^2 which were given in (7) in the previous section.
(b) Using your labeled representation of π_4 from Exercise 2.2, identify the three lines determined by the symbol 2 in the associated MOLS given in (5).
(c) Label the points of your representation of π_5 from Exercise 1.9 by starting with the letter a in the lower left corner and proceeding as we have on the other planes. Identify the four lines determined by the symbol 3 in the associated MOLS found in Exercise 1.7(b).

Now that you have an understanding for our new game board, let's play tic-tac-toe on these planes. When we play on what looks to be a standard 3×3 board, we will actually be playing the game on the affine plane of order 3. If we were playing standard 3×3 tic-tac-toe, there would be 8 ways to win, namely the 3 horizontal, 3 vertical, and 2 diagonal lines. Since we are playing on the affine plane of order 3 there are 12 ways to win, namely the 12 lines of the plane. The old ways to win are still there, but we've added four new winning lines which are given below.

	X	
		X
X		

		X
X		
	X	

	X	
X		
		X

X		
		X
	X	

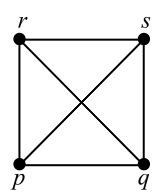
We will assume that X always makes the first move and we will use subscripts to denote the order of play. So, in the example given below, X_2 is the second move of player X, and O_1 is the first move of player O. Here is a sample game where player X has won. Can you identify the winning line?

X_1	X_3	O_3
	O_1	X_2
X_4	O_2	

Exercise 2.6. Play several games of tic-tac-toe on the affine plane of order 3. Be sure to play some games as X and others as O if you have an actual opponent.

Did you notice one of the ways this game differs from standard 3×3 tic-tac-toe is the placement of the first move? In standard 3×3 tic-tac-toe the center point sits on 4 lines, the corner points sit on 3 lines, and all other points sit on 2 lines. This difference places greater importance on the first move of the game. On any affine plane, the first player's move could just as well be made with his eyes closed since every point sits on the same number of lines. In other words, there is no advantage to be gained by claiming a particular first point on the plane.

In the standard 3×3 version of tic-tac-toe, we usually learn as young children that the second player can always force a draw as long as they make the correct choices. Determining the outcome of play assuming that players make the best possible choices is a basic question in game theory. Let's start small and see what conclusions we can make about playing on the smallest affine plane, that of order 2. The diagram for this plane of 4 points and 6 lines is given below. (There is only one Latin square associated with this plane. What is it?)



Notice that once X_1 and O_1 are chosen, player X will win on his second move since there exists a line between any two points on an affine plane. In this situation we say that X has a winning strategy. This was too easy. Let's consider the affine plane of order 3.

Try to play tic-tac-toe on π_3 until you can determine the outcome of play on this plane. You will find that player X always wins. By the reasoning given above, X_1 and O_1 may be assumed to be chosen arbitrarily. Choose X_2 as any point not on the line containing X_1 and O_1. Since there exists a line between any two points on an affine plane, O_2 must be placed on the line containing X_1 and X_2 (otherwise X wins on the next move). Likewise, X_3 must block the line containing O_1 and O_2. Player O must now block either the line containing X_1 and X_3 or the line containing X_2 and X_3, (it is a simple matter to see that O does not already have these lines blocked). Now, X_4 completes the line that O_3 did not block, and X wins the game. We have shown that X has a winning strategy by providing the method by which X can win any game on the affine plane of order 3. Now that we've provided a winning strategy on this plane, let's move on to the plane of order 4.

Exercise 2.7. Play tic-tac-toe on π_4 (use either the diagram created in Exercise 1.8 or the MOLS given in (5)), and try to determine the outcome of play on this plane.

As soon as we venture beyond the first two affine planes we find that the complexity of the game increases dramatically. This jump in difficulty when moving from one case to the next is not uncommon in mathematics. Indeed, we saw this very phenomenon in the attempt to find a complete set of MOLS of order 10 in the previous section. As for tic-tac-toe on the affine plane of order 4, the additional points and lines generate a far greater number of possible moves for each player. If you found that you cannot provide an easy move-by-move analysis as we did for the previous two planes then you are not alone. Perhaps we do not possess the computational fortitude that Tarry showed when he solved the 36 officer problem by hand, or perhaps we are merely availing ourselves of our computational technology, but in either case we relied on a computer to tell us that player X has a winning strategy by exhausting all possible outcomes. We do not, however, know what this winning strategy is in general. This means that we cannot describe an easily applied winning algorithm (for a person to follow) which prescribes each move. We can, however, teach a computer to play so that it never loses, but a vast number of tree searches are required at each move, placing this algorithm far beyond the capabilities of a human player.

Epilogue

What about the affine planes of order 5 and order 7 that you constructed in the previous section? We determined that player O can always force a draw on these planes just as the second player can in the standard 3×3 tic-tac-toe you played as a child. In fact, player O can force a draw on all affine planes of order greater than 4. If you presume that since a move-by-move analysis is not possible for the affine plane of order 4 then it surely must not be possible for higher ordered planes, then you are correct. We did not, however, have to rely on a computer to determine the outcome of play on these higher ordered planes. We were able to prove this result by using a technique that goes beyond the scope of this project, namely weight functions.

Solutions

Solution 3 Numerous examples can be given, for example: $\begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 0 & 1 & 2 \\ 2 & 3 & 0 & 1 \\ 1 & 2 & 3 & 0 \end{pmatrix}$

Solution 4 (a) No, for example the pair $(0,3)$ appears twice but $(0,1)$ does not appear. (b) Neither P nor Q are orthogonal. (c) No.

Solution 5

$$\left(\begin{array}{c|ccccc} + & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 0 & 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 & 4 & 0 \\ 2 & 2 & 3 & 4 & 0 & 1 \\ 3 & 3 & 4 & 0 & 1 & 2 \\ 4 & 4 & 0 & 1 & 2 & 3 \end{array}\right) \qquad \left(\begin{array}{c|ccccc} * & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 2 & 3 & 4 \\ 2 & 0 & 2 & 4 & 1 & 3 \\ 3 & 0 & 3 & 1 & 4 & 2 \\ 4 & 0 & 4 & 3 & 2 & 1 \end{array}\right)$$

Solution 1.7

(a)

$$L^1 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{pmatrix} \quad L^2 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 2 & 3 & 0 & 1 \\ 0 & 1 & 2 & 3 \\ 2 & 3 & 0 & 1 \end{pmatrix} \quad L^3 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 0 & 1 & 2 \\ 2 & 3 & 0 & 1 \\ 1 & 2 & 3 & 0 \end{pmatrix}$$

(b)

$$L^1 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 0 \\ 2 & 3 & 4 & 0 & 1 \\ 3 & 4 & 0 & 1 & 2 \\ 4 & 0 & 1 & 2 & 3 \end{pmatrix} \quad L^2 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 0 & 1 \\ 4 & 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 & 0 \\ 3 & 4 & 0 & 1 & 2 \end{pmatrix} \quad L^3 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 3 & 4 & 0 & 1 & 2 \\ 1 & 2 & 3 & 4 & 0 \\ 4 & 0 & 1 & 2 & 3 \\ 2 & 3 & 4 & 0 & 1 \end{pmatrix} \quad L^4 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 4 & 0 & 1 & 2 & 3 \\ 3 & 4 & 0 & 1 & 2 \\ 2 & 3 & 4 & 0 & 1 \\ 1 & 2 & 3 & 4 & 0 \end{pmatrix}$$

(c)

$$L^1 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 & 0 \\ 2 & 3 & 4 & 5 & 0 & 1 \\ 3 & 4 & 5 & 0 & 1 & 2 \\ 4 & 5 & 0 & 1 & 2 & 3 \\ 5 & 0 & 1 & 2 & 3 & 4 \end{pmatrix} \quad L^2 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 0 & 1 \\ 4 & 5 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 0 & 1 \\ 4 & 5 & 0 & 1 & 2 & 3 \end{pmatrix} \quad L^3 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 0 & 1 & 2 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 0 & 1 & 2 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 0 & 1 & 2 \end{pmatrix}$$

$$L^4 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 0 & 1 & 2 & 3 \\ 2 & 3 & 4 & 5 & 0 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 0 & 1 & 2 & 3 \\ 2 & 3 & 4 & 5 & 0 & 1 \end{pmatrix} \quad L^5 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 5 & 0 & 1 & 2 & 3 & 4 \\ 4 & 5 & 0 & 1 & 2 & 3 \\ 3 & 4 & 5 & 0 & 1 & 2 \\ 2 & 3 & 4 & 5 & 0 & 1 \\ 1 & 2 & 3 & 4 & 5 & 0 \end{pmatrix}$$

(d) For $n = 4$ and $n = 6$ the construction did not even produce Latin squares for L^i when i was not relatively prime to n. For $n = 5$ the construction did work. The correct conjecture is that it will work if and only if n is prime.

Solution 1.8 Simply connect the points corresponding to each number in the given Latin squares.

Solution 1.9 Simply connect the points corresponding to each number in the given Latin squares.

Solution 1.10

$$L^1 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 & 0 \\ 2 & 3 & 4 & 5 & 6 & 0 & 1 \\ 3 & 4 & 5 & 6 & 0 & 1 & 2 \\ 4 & 5 & 6 & 0 & 1 & 2 & 3 \\ 5 & 6 & 0 & 1 & 2 & 3 & 4 \\ 6 & 0 & 1 & 2 & 3 & 4 & 5 \end{pmatrix} \quad L^2 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 0 & 1 \\ 4 & 5 & 6 & 0 & 1 & 2 & 3 \\ 6 & 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 & 0 \\ 3 & 4 & 5 & 6 & 0 & 1 & 2 \\ 5 & 6 & 0 & 1 & 2 & 3 & 4 \end{pmatrix} \quad L^3 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 0 & 1 & 2 \\ 6 & 0 & 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 & 0 & 1 \\ 5 & 6 & 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 5 & 6 & 0 \\ 4 & 5 & 6 & 0 & 1 & 2 & 3 \end{pmatrix}$$

$$L^4 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 & 5 & 6 & 0 \\ 5 & 6 & 0 & 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 & 6 & 0 & 1 \\ 6 & 0 & 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 & 0 & 1 & 2 \end{pmatrix} \quad L^5 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 0 & 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 & 0 & 1 & 2 \\ 1 & 2 & 3 & 4 & 5 & 6 & 0 \\ 6 & 0 & 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 6 & 0 & 1 & 2 & 3 \\ 2 & 3 & 4 & 5 & 6 & 0 & 1 \end{pmatrix} \quad L^6 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 0 & 1 & 2 & 3 & 4 & 5 \\ 5 & 6 & 0 & 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 0 & 1 & 2 & 3 \\ 3 & 4 & 5 & 6 & 0 & 1 & 2 \\ 2 & 3 & 4 & 5 & 6 & 0 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 0 \end{pmatrix}$$

Solution 1.11 The Orthogonal Construction technique produces $p-1$ MOLS of order p if and only if p is a prime.
Proof. Assume p is a prime. If $L^k_{i,j} = L^k_{i,j'}$ then $ki + j = ki + j'$ which implies $j = j'$. Thus no element appears more than once in a row. If $L^k_{i,j} = L^k_{i',j}$ then $ki + j = ki' + j$ implies $ki = ki'$ and then $k(i - i') = 0$. There is a unique solution to $kx = 0 \pmod p$, namely 0 and so $i = i'$ and no element appears more than once in a column and L^k is a Latin square.

Next we shall show that L^k and $L^{k'}$ are orthogonal. Assume $(L^k_{i,j}, L^{k'}_{i,j}) = (L^k_{i',j'}, L^{k'}_{i',j'})$ then $ki + j = ki' + j'$ and $k'i + j = k'i' + j'$. This gives that $k(i - i') = j' - j$ and $k'(i - i') = j' - j$. However, there is a unique solution to $(i - i')x = (j' - j)$ and so $k = k'$. Hence if $k \neq k'$ the same ordered pair cannot appear more than once.

If p is not a prime then $p = ab$ for $1 < a, b < p$. Consider L^a: in the row corresponding to b we have $L^a_{b,j} = ab + j = j$ and in the 0th row we have $L^a_{a0+j} = j$ so that L^a is not a Latin square. □

Solution 2.1

(i) $\{a,b,c\}, \{d,e,f\}, \{g,h,i\}$
(ii) $\{a,d,g\}, \{g,e,h\}, \{c,f,i\}$
(iii) $\{g,b,f\}, \{h,d,c\}, \{i,e,a\}$
(iv) $\{c,e,g\}, \{a,f,h\}, \{b,d,i\}$

Solution 2.2

(i) $\{a,b,c,d\}, \{e,f,g,h\}, \{i,j,k,l\}, \{m,n,o,p\}$
(ii) $\{a,e,i,m\}, \{b,f,j,n\}, \{c,g,k,o\}, \{d,h,l,p\}$
(iii) $\{d,g,j,m\}, \{c,h,i,n\}, \{b,e,l,o\}, \{a,f,k,p\}$
(iv) $\{a,g,l,n\}, \{b,h,k,m\}, \{d,f,i,o\}, \{c,e,j,p\}$
(v) $\{a,h,j,o\}, \{c,f,l,m\}, \{d,e,k,n\}, \{b,g,i,p\}$

Solution 2.3

Number of $\Downarrow$ in $\Rightarrow$	π_3	π_4	π_5	π_7	π_n
points	9	16	25	49	n^2
lines	12	20	30	58	n^2+n
points on each line	3	4	5	7	n
lines through any point	4	5	6	8	$n+1$
points where two non-parallel lines meet	1	1	1	1	1
lines parallel to any given line [Note: Every line is parallel to itself.]	3	4	5	7	n
parallel classes	4	5	6	8	$n+1$

Solution 2.5

(a) $\{a, f, h\}, \{h, d, c\}$
(b) $\{o, l, e, b\}, \{o, i, f, d\}, \{0, j, h, a\}$
(c) $\{x, r, l, f, e\}, \{x, q, o, h, a\}, \{x, p, m, j, b\}, \{x, t, k, g, c\}$

Solution 2.6 You should find that X has a winning strategy.

Solution 2.7 Player X has a winning strategy but, as the next paragraph in the worksheet notes, this strategy is not easy to find. After playing the game extensively by hand, you may appreciate the online version at

`http://academic.scranton.edu/faculty/carrollm1/tictactoe/tictactoea4.html`