Some notes on generating functions for Math 22

The **generating function** for the sequence $a_0, a_1, a_2, ..$ is defined as
$F(x) = a_0 + a_1 x + a_2 x^2 + \ldots$

The generating function for the sequence 1,1,1,... is the standard geometric sequence $1 + x + x^2 + \ldots = \frac{1}{1-x}$. We ignore issues of convergence, because we will only use the algebraic properties of such functions, both in their role as "infinite polynomials," and as rational functions, to aid in solving counting problems.

Example

$$\frac{d}{dx}(1 + x + x^2 + \ldots) = 1 + 2x + 3x^2 + \ldots + kx^{k-1} + \ldots = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}.$$

Therefore $\frac{1}{(1-x)^2}$ is the generating function for the positive integers!

Generalization of the binomial theorem

$$(1-x)^{-n} = \left(\frac{1}{1-x}\right)^n = (1 + x + x^2 + \ldots)(1 + x + x^2 + \ldots) \ldots (1 + x + x^2 + \ldots) \qquad (n \text{ factors})$$

What is the coefficient of each x^k in the expression on the right? By the distributive property, we seek the number of ways to choose one term from each of the *n* factors such that the sum of the exponents of these terms is *k*. That is, if we choose term x^{k_i} from the *i*th factor, then we get a product of the form $x^{k_1} x^{k_2} x^{k_3} \ldots x^{k_n} = x^{k_1 + k_2 + \ldots k_n} = x^k$.

The total number of ways to make these choices is the number of ways of choosing *k* identical balls (the powers of *x*) from *n* distinct urns (the *n* factors). As we have seen before, this is $\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$. Therefore,

$$(1-x)^{-n} = \binom{n+k-1}{0} \cdot 1 + \binom{n+k-1}{1} x + \binom{n+k-1}{2} x^2 + \ldots + \binom{n+k-1}{k} x^k + \ldots$$

Example

Suppose 12 identical candy bars are to be distributed to three students, Dustin, Radhika, and Daxi. Dustin is to receive at least one, but no more than five, Radhika will receive at least two, but no more than six. Daxi will receive at least three, but no more than seven. How many ways are there to distribute the candy bars?

We seek the coefficient of x^{12} in the expression

$$(x^1 + x^2 + x^3 + x^4 + x^5)(x^2 + x^3 + x^4 + x^5 + x^6)(x^3 + x^4 + x^5 + x^6 + x^7)$$

since that coefficient is actually the number of ways of choosing an exponent from each of the three factors such that the sum of all three exponents is 12.
First we factor out the highest power of x from each of the factors above:

$$(x^1 + x^2 + x^3 + x^4 + x^5)(x^2 + x^3 + x^4 + x^5 + x^6)(x^3 + x^4 + x^5 + x^6 + x^7)$$
$$= x^6(1 + x^1 + x^2 + x^3 + x^4)^3$$

Thus we really seek the coefficent of $x^{12\text{-}6} = x^6$ in

$$(1 + x^1 + x^2 + x^3 + x^4)^3$$

Using the standard formula for the sum of a geometric progression,

$$(x^1 + x^2 + x^3 + \ldots + x^m) = \frac{1 - x^{m+1}}{1 - x},$$

we see that the expression above can be rewritten as

$$(\frac{1 - x^5}{1 - x})^3 = (1 - x^5)^3(1 - x)^{-3}$$

Now we expand the first factor using the standard form of the binomial theorem, and the second factor using the generalization to negative exponents:

$$(1 - x^5)^3(1 - x)^{-3} = (1 - 3x^5 + 3x^{10} - x^{15})(\binom{2}{0} + \binom{3}{1}x + \binom{4}{2}x^2 + \binom{5}{3}x^3 + \ldots + \binom{3+k-1}{k}x^k + \ldots)$$

The only terms with exponents six arise from the following terms:

$$1 \cdot \binom{8}{6}x^6 - 3x^5 \cdot \binom{3}{1}x$$

so the answer to our problem is $\binom{8}{6} - 3\binom{3}{1} = 28 - 9 = 19$.

(Alternatively, you might submit
$(x^1 + x^2 + x^3 + x^4 + x^5)(x^2 + x^3 + x^4 + x^5 + x^6)(x^3 + x^4 + x^5 + x^6 + x^7)$
to Wolfram Alpha, and just glance at the coefficient of x^{12}!)

Homework:

Use a generating function to solve the following problem. A computer company is donating 20 identical new computers to four schools: Evergreen, De Anza, Foothill, and Mission Community Colleges. Evergreen and De Anza will each receive at least two computers and no more than five, and Foothill and Mission will each receive at least three computers and no more than six. Use a generating function to show how many ways are there to distribute the computers.