Math 1D, Exam 2 Sample Test, Fall 08, Solutions

(1) To find the critical points, we solve $f_x = 0$ and $f_y = 0$ for x and y. Solving

$$f_x = 3x^2 - 3 = 0$$
$$f_y = 3y^2 - 12y = 0$$

shows that $x = -1$ or $x = 1$ and $y = 0$ or $y = 4$. There are four critical points: $(-1, 0)$, $(1, 0)$, $(-1, 4)$, and $(1, 4)$.

We have

$$D = (f_{xx})(f_{yy}) - (f_{xy})^2 = (6x)(6y - 12) - (0)^2 = (6x)(6y - 12).$$

At critical point $(-1, 0)$, we have $D > 0$ and $f_{xx} < 0$, so f has a local maximum at $(-1, 0)$.
At critical point $(1, 0)$, we have $D < 0$, so f has a saddle point at $(1, 0)$.
At critical point $(-1, 4)$, we have $D < 0$, so f has a saddle point at $(-1, 4)$.
At critical point $(1, 4)$, we have $D > 0$ and $f_{xx} > 0$, so f has a local minimum at $(1, 4)$.

(2)

(a) The revenue $R = p_1q_1 + p_2q_2$. Profit $= P = R - C = p_1q_1 + p_2q_2 - 2q_1^2 - 2q_2^2 - 10$.

$$\frac{\partial P}{\partial q_1} = p_1 - 4q_1 = 0 \quad \text{gives } q_1 = \frac{p_1}{4}$$
$$\frac{\partial P}{\partial q_2} = p_2 - 4q_2 = 0 \quad \text{gives } q_2 = \frac{p_2}{4}$$

Since $\frac{\partial^2 P}{\partial q_1^2} = -4$, $\frac{\partial^2 P}{\partial q_2^2} = -4$ and $\frac{\partial^2 P}{\partial q_1 \partial q_2} = 0$, at $(p_1/4, p_2/4)$ we have that the discriminant, $D = (-4)(-4) > 0$ and $\frac{\partial^2 P}{\partial q_1^2} < 0$, thus P has a local maximum value at $(q_1, q_2) = (p_1/4, p_2/4)$. Since P is quadratic in q_1 and q_2, this is a global maximum. So $P = \frac{p_1^2}{4} + \frac{p_2^2}{4} - 2\frac{p_1^2}{16} - 2\frac{p_2^2}{16} - 10 = \frac{p_1^2}{8} + \frac{p_2^2}{8} - 10$ is the maximum profit.

(b) The rate of change of the maximum profit as p_1 increases is

$$\frac{\partial}{\partial p_1}(\max\ P) = \frac{2p_1}{8} = \frac{p_1}{4}.$$

(3) (a) We solve

$$2x = 2xy^2\lambda$$
$$2y = 2x^2y\lambda$$
$$x^2y^2 = 4,$$

giving $x^2 = y^2$, so $(x^2)^2 = 4$, so (since we are working in the first quadrant) $x = y = \sqrt{2}$

(b) We solve

$$2xy^2 = \lambda$$
$$2x^2y = \lambda$$
$$x + y = 4.$$

Dividing the first two equations gives $x = y$, so (since we are working in the first quadrant) $x = y = 2$.

(To see these are mins and maxes, examine the contour curves for these functions!)

(4) This was a homework problem!

(a) We first find an over- and underestimate of the integral, using four subrectangles. On the first subrectangle ($0 \le x \le 3$, $0 \le y \le 4$), the function $f(x, y)$ appears to have a maximum of 100 and a minimum of 79. Continuing in this way, and using the fact that $\Delta x = 3$ and $\Delta y = 4$, we have

$$\text{Overestimate} = (100 + 90 + 85 + 79)(3)(4) = 4248,$$

and

$$\text{Underestimate} = (79 + 68 + 61 + 55)(3)(4) = 3156.$$

A better estimate of the integral is the average of the overestimate and the underestimate:

$$\text{Better estimate} = \frac{4248 + 3156}{2} = 3702.$$

(b) The average value of $f(x, y)$ on this region is the value of the integral divided by the area of the region. Since the area of R is $(6)(8) = 48$, we approximate

$$\text{Average value} = \frac{1}{\text{Area}} \int_R f(x, y) dA \approx \frac{1}{48} \cdot 3702 = 77.125.$$

We see in the table that the values of $f(x, y)$ on this region vary between 55 and 100, so an average value of 77.125 is reasonable.

(5) Ch. 16.2: # 22. (This was a homework problem which we went over in class.)

(6)

The required volume, V, is given by

$$\begin{aligned} V &= \int_0^{10} \int_0^{10-x} \int_{x+y}^{10} dz\, dy\, dx \\ &= \int_0^{10} \int_0^{10-x} (10 - (x + y))\, dy dx \\ &= \int_0^{10} \left[10y - xy - \frac{1}{2} y^2 \right]_{y=0}^{y=10-x} dx \\ &= \int_0^{10} \frac{1}{2} (10 - x)^2\, dx \\ &= \frac{500}{3} \end{aligned}$$

(7)

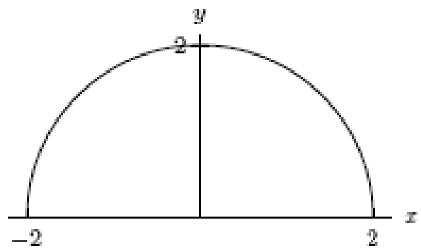

Figure 16.4.100

$$
\begin{aligned}
\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} e^{-(x^2+y^2)}\,dy\,dx &= \int_0^{\pi}\int_0^{2} e^{-r^2} r\,dr\,d\theta \\
&= \int_0^{\pi} -\frac{e^{-r^2}}{2}\bigg|_0^2\,d\theta \\
&= \int_0^{\pi} -\frac{e^{-4}}{2} - (-\frac{e^{-0}}{2})\,d\theta \\
&= \int_0^{\pi} (\frac{1}{2} - \frac{e^{-4}}{2})\,d\theta \\
&= (\frac{1}{2} - \frac{e^{-4}}{2})\int_0^{\pi} d\theta \\
&= \frac{(1-e^{-4})\pi}{2}
\end{aligned}
$$

(8) (a) Surface. This is the half-plane $y = 0$, $x \le 0$, which is vertical and perpendicular to the y-axis.

(b) Surface of the cylinder of radius 3 centered on the z-axis.

(c) Line, parallel to the z-axis, with $x = 0$, $y = 3$.

(d) Solid region. A solid cylinder of radius 4, centered on the z-axis from $z = -5$ to $z = 2$, with the central cylindrical core of radius 1 removed.

(e) Solid region. A solid ball of radius 4, centered at the origin, with the smaller ball of radius 1 removed from its interior.

(f) Point (–0.90, 0.13, –0.42), since $x = \cos 3 \sin 2 = -0.90$, $y = \sin 3 \sin 2 = 0.13$, $z = \cos 2 = -0.42$.

(9)

(a) The Jacobian is

$$\frac{\partial(x,y)}{\partial(s,t)} = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 1 & -2 \end{vmatrix} = (1)(-2) - (3)(1) = -5.$$

The absolute value of the Jacobian $= |-5| = 5$.

(b) Under the change of variables $x = s + 3t, y = s - 2t$, the line $2x + 3y = 1$ becomes $2(s+3t) + 3(s-2t) = 1$, or equivalently $5s = 1$. Similarly, the line $2x + 3y = 4$ becomes $5s = 4$, the line $x - y = -3$ becomes $5t = -3$ and the line $x - y = 2$ becomes $5t = 2$. The new region T is bounded by the lines

$$5s = 1, \quad 5s = 4, \quad 5t = -3 \quad \text{and} \quad 5t = 2.$$

In other words, T is the rectangle

$$1/5 \le s \le 4/5, \quad -3/5 \le t \le 2/5.$$

(c) By the change of variable formula,

$$\int_R (2x+3y)dA = \int_T (2(s+3t) + 3(s-2t))\,(|-5|)ds\,dt = \int_T 25s\,ds\,dt$$

$$= \int_{-3/5}^{2/5} \int_{1/5}^{4/5} 25s\,ds\,dt = \int_{-3/5}^{2/5} \frac{25}{2}s^2\Big|_{s=1/5}^{s=4/5} dt = \int_{-3/5}^{2/5} \frac{15}{2}dt = \frac{15}{2}.$$