CHAPTER THREE

Solutions for Section 3.1

1. $\frac{dy}{dx} = 0$

2. $\frac{dy}{dx} = 3$

3. $y' = 12x^{11}$.

4. $y' = -12x^{-13}$.

5. $y' = \frac{4}{3}x^{1/3}$.

6. $y' = 24t^2$

7. $y' = 12t^3 - 4t$

8. $\frac{dy}{dx} = 5$

9. $f'(x) = -4x^{-5}$.

10. $f'(q) = 3q^2$

11. $y' = 2x + 5$.

12. $y' = 18x^2 + 8x - 2$.

13. $\frac{dy}{dx} = 6x + 7$.

14. $\frac{dy}{dt} = 24t^2 - 8t + 12$.

15. $\frac{dy}{dq} = 8.4q - 0.5$.

16. $y' = -12x^3 - 12x^2 - 6$.

17. Since $g(t) = \frac{1}{t^5} = t^{-5}$, we have $g'(t) = -5t^{-6}$.

18. Since $f(z) = -\frac{1}{z^{6.1}} = -z^{-6.1}$, we have $f'(z) = -(-6.1)z^{-7.1} = 6.1z^{-7.1}$.

19. Since $y = \frac{1}{r^{7/2}} = r^{-7/2}$, we have $\frac{dy}{dx} = -\frac{7}{2}r^{-9/2}$.

20. Since $y = \sqrt{x} = x^{1/2}$, we have $\frac{dy}{dx} = \frac{1}{2}x^{-1/2}$.

21. Since $h(\theta) = \frac{1}{\sqrt[3]{\theta}} = \theta^{-1/3}$, we have $h'(\theta) = -\frac{1}{3}\theta^{-4/3}$.

22. Since $f(x) = \sqrt{\frac{1}{x^3}} = \frac{1}{x^{3/2}} = x^{-3/2}$, we have $f'(x) = -\frac{3}{2}x^{-5/2}$.

23. $y' = 15t^4 - \frac{5}{2}t^{-1/2} - \frac{7}{t^2}$.

24. $y' = 2z - \frac{1}{2z^2}$.

25. $y' = 6t - \frac{6}{t^{3/2}} + \frac{2}{t^3}$.

26. Since $h(t) = \frac{3}{t} + \frac{4}{t^2} = 3t^{-1} + 4t^{-2}$, we have $h'(t) = -3t^{-2} - 8t^{-3}$.

27. Since $y = \sqrt{x}(x+1) = x^{1/2}x + x^{1/2} \cdot 1 = x^{3/2} + x^{1/2}$, we have $\frac{dy}{dx} = \frac{3}{2}x^{1/2} + \frac{1}{2}x^{-1/2}$.

28. Since $h(\theta) = \theta(\theta^{-1/2} - \theta^{-2}) = \theta\theta^{-1/2} - \theta\theta^{-2} = \theta^{1/2} - \theta^{-1}$, we have $h'(\theta) = \frac{1}{2}\theta^{-1/2} + \theta^{-2}$.

29. $f'(x) = k \cdot \frac{d}{dx}(x^2) = 2kx$.

30. $y' = 2ax + b$.

31. The derivatives of P^2 and P^3 are $2P$ and $3P^2$, respectively, so

$$\frac{dQ}{dP} = 2aP + 3bP^2.$$

32. Since

$$v = at^2 + \frac{b}{t^2} = at^2 + bt^{-2}$$

we have

$$\frac{dv}{dt} = 2at + b(-2)t^{-3} = 2at - 2\frac{b}{t^3}.$$

33. Since a and b are constants, we have

$$\frac{dP}{dt} = 0 + b\frac{1}{2}t^{-1/2} = \frac{b}{2\sqrt{t}}.$$

34. Since $4/3$, π, and b are all constants, we have

$$\frac{dV}{dr} = \frac{4}{3}\pi(2r)b = \frac{8}{3}\pi rb.$$

35. Since w is a constant times q, we have $dw/dq = 3ab^2$.

36. Since $h(x) = \dfrac{ax+b}{c} = \dfrac{a}{c}x + \dfrac{b}{c}$, we have $h'(x) = \dfrac{a}{c}$.

37. **(a)**

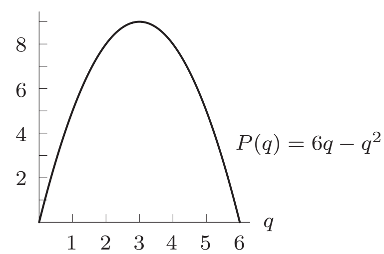


Figure 3.1

Since P is increasing when $q = 1$, $P'(1)$ is positive. At $(3, 9)$ we observe that the function has a horizontal tangent line, and horizontal lines have slope zero. Thus $P'(3) = 0$. Finally, at $q = 4$ the function is decreasing; therefore the derivative $P'(4)$ is negative.

(b) The derivative of $P(q) = 6q - q^2$ is $P'(q) = 6 - 2q$. Therefore,

$$\begin{aligned} P'(1) &= 6 - 2 = 4 \\ P'(3) &= 6 - 2(3) = 6 - 6 = 0 \\ P'(4) &= 6 - 2(4) = 6 - 8 = -2 \end{aligned}$$

38. $f'(x) = 3x^2 - 8x + 7$, so $f'(0) = 7$, $f'(2) = 3$, and $f'(-1) = 18$.

39. **(a)** $f'(t) = 2t - 4$.

(b) $f'(1) = 2(1) - 4 = -2$
$f'(2) = 2(2) - 4 = 0$

(c) We see from part (b) that $f'(2) = 0$. This means that the slope of the line tangent to the curve at $x = 2$ is zero. From Figure 3.2, we see that indeed the tangent line is horizontal at the point $(2, 1)$. The fact that $f'(1) = -2$ means that the slope of the line tangent to the curve at $x = 1$ is -2. If we draw a line tangent to the graph at $x = 1$ (the point $(1, 2)$) we see that it does indeed have a slope of -2.

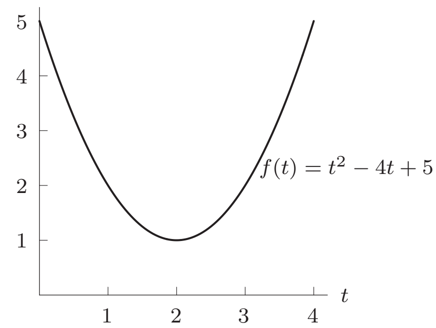


Figure 3.2

40. The rate of change of the population is given by the derivative. For $P(t) = t^3 + 4t + 1$ the derivative is $P'(t) = 3t^2 + 4$. At $t = 2$, the rate of change of the population is $3(2)^2 + 4 = 12 + 4 = 16$, meaning the population is growing by 16 units per unit of time.

41. Since $f(t) = 700 - 3t^2$, we have $f(5) = 700 - 3(25) = 625$ cm. Since $f'(t) = -6t$, we have $f'(5) = -30$ cm/year. In the year 2010, the sand dune will be 625 cm high and eroding at a rate of 30 centimeters per year.

42. After four months, there are $300(4)^2 = 4800$ mussels in the bay. The population is growing at the rate $Z'(4)$ mussels per month, where $Z'(t) = 600t$, so the rate of increase is 2400 mussels per month.

43. When $t = 10$, we have $Q = 3(10^2) + 100 = 400$ tons. Since $f'(t) = 6t$, we have $f'(10) = 6(10) = 60$ tons per year. Then

$$\text{Relative rate of change } = \frac{f'(10)}{f(10)} = \frac{60}{400} = 0.15 = 15\% \text{ per year.}$$

In 2010, there were 400 tons of waste at the site. The quantity was growing at a rate of 60 tons per year, which is 15% per year.

44. When $t = 9$, we have $N = 120\sqrt{9} = 360$ acres. Since $f'(t) = 120(1/2)t^{-1/2} = 60/\sqrt{t}$, we have $f'(9) = 60/\sqrt{9} = 60/3 = 20$ acres per year. Then

$$\text{Relative rate of change } = \frac{f'(9)}{f(9)} = \frac{20}{360} = 0.0555 = 5.55\% \text{ per year.}$$

Nine years after farming started in the region, there are 360 acres being harvested. The number of acres being harvested is increasing at a rate of 20 acres per year, or 5.55% per year.

45. $f'(t) = 6t^2 - 8t + 3$ and $f''(t) = 12t - 8$.

46. The derivative of $f(t)$ is $f'(t) = 4t^3 - 6t + 5$. The second derivative is the derivative of the derivative, and thus $f''(t) = 12t^2 - 6$.

47. **(a)** $f(100) = 1.111\sqrt{100} = 11.11$ seconds. This tells us that it takes approximately 11.11 seconds for one complete oscillation of a 100 foot long pendulum.

(b) To find $f'(100)$, we first rewrite our equation with a fractional exponent. Thus, $f(L) = 1.111\sqrt{L} = 1.111L^{1/2}$. Differentiating, we get $f'(L) = \frac{1}{2}(1.111)L^{-1/2} = \frac{0.5555}{L^{1/2}} = \frac{0.5555}{\sqrt{L}}$.
Therefore, $f'(100) = \frac{0.5555}{\sqrt{100}} = 0.05555$ seconds per foot.
This tells us, when a pendulum is 100 feet long, an increase of one foot in the length of the pendulum results in an increase of about 0.05555 seconds in the time for one complete oscillation.

48. **(a)** We have

$$C(w) = 42w^{0.75}.$$

Using the derivative rules, we find

$$C'(w) = 42(0.75w^{0.75-1}) = 31.5w^{-0.25}.$$

(b) (i) Substituting $w = 10$, we have

$$C(10) = 42(10^{0.75}) = 236.183 \quad \text{and} \quad C'(10) = 31.5(10^{-0.25}) = 17.714.$$

A mammal weighing 10 pounds needs about 236 calories a day. If weight increases by one pound, calorie consumption increases by about 17.7 calories a day.

(ii) Substituting $w = 100$, we have

$$C(100) = 42(100^{0.75}) = 1328.157 \quad \text{and} \quad C'(100) = 31.5(100^{-0.25}) = 9.961.$$

A mammal weighing 100 pounds needs about 1328 calories a day. If weight increases by one pound, calorie consumption increases by about 10 calories a day.

(iii) Substituting $w = 1000$, we have

$$C(1000) = 42(1000^{0.75}) = 7468.774 \quad \text{and} \quad C'(1000) = 31.5(1000^{-0.25}) = 5.602.$$

A mammal weighing 1000 pounds needs about 7,469 calories a day. If weight increases by one pound, calorie consumption increases by about 5.6 calories a day.

As expected, as weight increases, calorie requirements also increase. However, the derivative decreases as weight goes up, so an additional pound has a greater impact on calorie requirements for a small mammal than it does for a large mammal.

49. Differentiating gives

$$f'(x) = 6x^2 - 4x \quad \text{so} \quad f'(1) = 6 - 4 = 2.$$

Thus the equation of the tangent line is $(y - 1) = 2(x - 1)$ or $y = 2x - 1$.

50. **(a)** We have $f(2) = 8$, so a point on the tangent line is $(2, 8)$. Since $f'(x) = 3x^2$, the slope of the tangent is given by

$$m = f'(2) = 3(2)^2 = 12.$$

Thus, the equation is

$$y - 8 = 12(x - 2) \quad \text{or} \quad y = 12x - 16.$$

(b) See Figure 3.3. The tangent line lies below the function $f(x) = x^3$, so estimates made using the tangent line are underestimates.

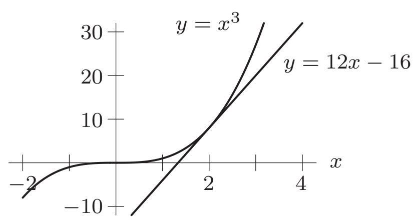


Figure 3.3

51. To find the equation of a line we need to have a point on the line and its slope. We know that this line is tangent to the curve $f(t) = 6t - t^2$ at $t = 4$. From this we know that both the curve and the line tangent to it will share the same point and the same slope. At $t = 4$, $f(4) = 6(4) - (4)^2 = 24 - 16 = 8$. Thus we have the point $(4, 8)$. To find the slope, we need to find the derivative. The derivative of $f(t)$ is $f'(t) = 6 - 2t$. The slope of the tangent line at $t = 4$ is $f'(4) = 6 - 2(4) = 6 - 8 = -2$. Now that we have a point and the slope, we can find an equation for the tangent line:

$$\begin{aligned} y &= b + mt \\ 8 &= b + (-2)(4) \\ b &= 16. \end{aligned}$$

Thus, $y = -2t + 16$ is the equation for the line tangent to the curve at $t = 4$. See Figure 3.4.

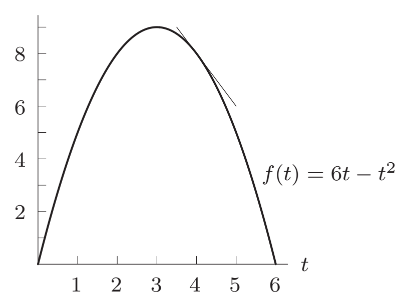


Figure 3.4

52. **(a)** If the air temperature is 20°F, and the wind is blowing at 40 mph, we substitute $v = 40$ into the formula $W(v) = 48.17 - 27.2v^{0.16}$, giving

$$W(40) = 48.17 - 27.2(40)^{0.16} = -0.909.$$

The windchill temperature is approximately -1°F.

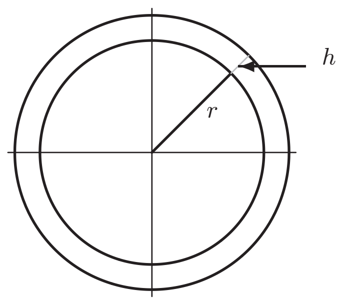
(b) To find $W'(40)$,we first determine $W'(v) = -0.16 \cdot 27.2v^{-0.84} = -4.352v^{-0.84}$.

We substitute 40 for v to get $W'(40) = -4.352(40)^{-0.84} = -0.196$°F/mph. This tells us that, when the temperature is 20°F and the wind is blowing at 40 mph, for every 1 mph the wind speed increases, the windchill temperature decreases by approximately 0.196°F.

53. **(a)** $A = \pi r^2$
$\frac{dA}{dr} = 2\pi r.$

(b) This is the formula for the circumference of a circle.

(c) $A'(r) \approx \frac{A(r+h)-A(r)}{h}$ for small h. When $h > 0$, the numerator of the difference quotient denotes the area of the region contained between the inner circle (radius r) and the outer circle (radius $r + h$). See figure below. As h approaches 0, this area can be approximated by the product of the circumference of the inner circle and the "width" of the region, i.e., h. Dividing this by the denominator, h, we get A' = the circumference of the circle with radius r.



We can also think about the derivative of A as the rate of change of area for a small change in radius. If the radius increases by a tiny amount, the area will increase by a thin ring whose area is simply the circumference at that radius times the small amount. To get the rate of change, we divide by the small amount and obtain the circumference.

54. Since W is proportional to r^3, we have $W = kr^3$ for some constant k. Thus, $dW/dr = k(3r^2) = 3kr^2$. Thus, dW/dr is proportional to r^2.

55. The marginal cost of producing the 25th item is $C'(25)$, where $C'(q) = 4q$, so the marginal cost is \$100. This means that the cost of production increases by about \$100 when we add one unit to a production level of 25 units.

56. **(a)** We have $R(p) = pq = p(300 - 3p) = 300p - 3p^2$

(b) Since $R'(p) = 300 - 6p$, we have $R'(10) = 300 - 6 \cdot 10 = 240$. This means that revenues are increasing at a rate of \$240 per dollar of price increase when the price is \$10.

(c) $R'(p) = 300 - 6p$ is positive for $p < 50$ and negative for $p > 50$.

57. **(a)** The yield is $f(5) = 320 + 140(5) - 10(5)^2 = 770$ bushels per acre.

(b) $f'(x) = 140 - 20x$, so $f'(5) = 40$ bushels per acre per pound of fertilizer. For each acre, yield will go up by about 40 bushels if an additional pound of fertilizer is used.

(c) More should be used, because at this level of use, more fertilizer will result in a higher yield. Fertilizer's use should be increased until an additional unit results in a decrease in yield. i.e. until the derivative at that point becomes negative.

58. **(a)** The p-intercept is the value of p when $q = 0$.

$$p = f(0) = 50 - 0.03(0)^2 = 50.$$

The p-intercept occurs at $p = 50$.

The q-intercept is the value of q such that $p = f(q) = 0$.

$$\begin{aligned} p = f(q) = 50 - 0.03q^2 &= 0 \\ -0.03q^2 &= -50 \\ q^2 &= \frac{50}{0.03} \quad \text{and} \quad q \geq 0 \\ q &= \sqrt{\frac{50}{0.03}} \approx 40.825 \end{aligned}$$

The q-intercept occurs at $q \approx 40.825$.

The p-intercept represents the price at which demand is zero. That is, when the price reaches 50 dollars, demand for the product will be zero. The q-intercept represents the demand for the product if the product were being given away free of charge. In this case, 40.825 units of the product would be consumed if the product were free ($p = 0$).

(b)

$$f(20) = 50 - 0.03(20)^2 = 50 - 0.03(400) = 50 - 12 = 38 \text{ dollars}.$$

This tells us that if the price per unit is \$38, then a total of 20 units are demanded.

(c) To find $f'(20)$ we first find $f'(q) = 2(-0.03)q = -0.06q$. Therefore, $f'(20) = -0.06(20) = -1.2$ dollars per unit demanded. This tells us given a demand of 20 units, which, according to our answer to part (b), occurs when the unit price is \$38, an increase of \$1.20 in the price will result in the reduction of consumption by approximately 1 unit while a decrease in price by the same amount will lead to an increase of approximately 1 unit in sales.

59. **(a)** The marginal cost function equals $C'(q) = 0.08(3q^2) + 75 = 0.24q^2 + 75$.

(b)

$$C(50) = 0.08(50)^3 + 75(50) + 1000 = \$14{,}750.$$

$C(50)$ tells us how much it costs to produce 50 items. From above we can see that the company spends \$14,750 to produce 50 items. The units for $C(q)$ are dollars.

$$C'(50) = 0.24(50)^2 + 75 = \$675 \text{ per item}.$$

$C'(q)$ tells us the approximate change in cost to produce one additional item of product. Thus at $q = 50$ costs will increase by about \$675 for one additional item of product produced. The units are dollars/item.

60. **(a)** Velocity $v(t) = \frac{dy}{dt} = \frac{d}{dt}(1250 - 16t^2) = -32t$.
Since $t \geq 0$, the ball's velocity is negative. This is reasonable, since its height y is decreasing.

(b) The ball hits the ground when its height $y = 0$. This gives

$$1250 - 16t^2 = 0$$
$$t \approx \pm 8.84 \text{ seconds}$$

We discard $t = -8.84$ because time t is nonnegative. So the ball hits the ground about 8.84 seconds after its release, at which time its velocity is

$$v(8.84) = -32(8.84) = -282.88 \text{ feet/sec} = -192.87 \text{ mph}.$$

61. Since $f(x) = x^3 - 6x^2 - 15x + 20$, we have $f'(x) = 3x^2 - 12x - 15$. To find when $f'(x) = 0$, we solve

$$3x^2 - 12x - 15 = 0$$
$$3(x^2 - 4x - 5) = 0$$
$$3(x + 1)(x - 5) = 0.$$

We see that $f'(x) = 0$ at $x = -1$ and at $x = 5$. We see that the graph of $f(x)$ in Figure 3.5 is horizontal at $x = -1$ and at $x = 5$, which confirms what we found using the derivative.

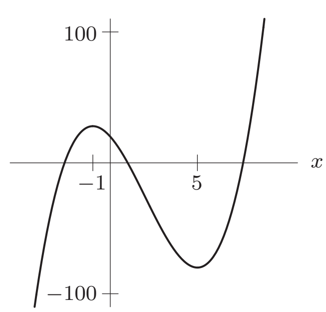


Figure 3.5: $f(x) = x^3 - 6x^2 - 15x + 20$

62. If $f(x) = x^n$, then $f'(x) = nx^{n-1}$. This means $f'(1) = n \cdot 1^{n-1} = n \cdot 1 = n$, because any power of 1 equals 1.

63. **(a)** $R(q) = q(b + mq) = bq + mq^2$.

(b) $R'(q) = b + 2mq$.

Solutions for Section 3.2

1. $f'(x) = 2e^x + 2x$.

2. $\dfrac{dP}{dt} = 9t^2 + 2e^t$.

3. $y' = 10t + 4e^t$.

4. $f'(x) = 3x^2 + 3^x \ln 3$

5. Since $y = 2^x + \dfrac{2}{x^3} = 2^x + 2x^{-3}$, we have $\dfrac{dy}{dx} = (\ln 2)2^x - 6x^{-4}$.

6. $\dfrac{dy}{dx} = 5 \cdot 5^t \ln 5 + 6 \cdot 6^t \ln 6$

7. $f'(x) = (\ln 2)2^x + 2(\ln 3)3^x$.

8. $\dfrac{dy}{dx} = 4(\ln 10)10^x - 3x^2$.

9. $\dfrac{dy}{dx} = 3 - 2(\ln 4)4^x$.

10. $\dfrac{dy}{dx} = 5(\ln 2)(2^x) - 5$.

11. $f'(t) = e^{3t} \cdot 3 = 3e^{3t}$.

12. $\dfrac{dy}{dt} = e^{0.7t} \cdot (0.7) = 0.7e^{0.7t}$.

13. $y' = -4e^{-4t}$.

14. $P' = -0.2e^{-0.2t}$.

15. $P' = 50(-0.6)e^{-0.6t} = -30e^{-0.6t}$.

16. $\dfrac{dP}{dt} = 200(0.12)e^{0.12t} = 24e^{0.12t}$.

17. $P'(t) = 3000(\ln 1.02)(1.02)^t$.

18. $P'(t) = 12.41(\ln 0.94)(0.94)^t$.

19. $P'(t) = Ce^t$.

20. $y' = Ae^t$

21. $f'(x) = Ae^x - 2Bx$.

22. Since $y = 10^x + 10x^{-1}$, we have

$$\frac{dy}{dx} = (\ln 10)10^x - 10x^{-2} = (\ln 10)10^x - \frac{10}{x^2}.$$

23. $R' = \frac{3}{q}$.

24. $D' = -1/p$.

25. $y' = 2t + 5/t$

26. $R'(q) = 2q - 2/q$.

27. $\dfrac{dy}{dx} = 2x + 4 - 3/x$.

28. $Ae^t + \dfrac{B}{t}$.

29. $f(t) = 4 - 2e^t$, so $f'(t) = -2e^t$ and $f'(-1) = -2e^{-1} \approx -0.736$. $f'(0) = -2e^0 = -2$. $f'(1) = -2e^1 \approx -5.437$. As expected, the slopes of the line segments do match the derivatives found. See Figure 3.6.

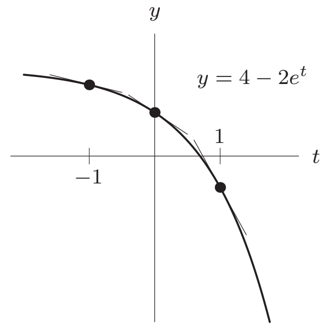


Figure 3.6

30. See Figure 3.7. The slope of the line tangent to the graph of the function at $x = 1$ will be the derivative of the function evaluated at $x = 1$. Since $y = 3^x$, $\frac{dy}{dx} = (\ln 3)3^x$. At $x = 1$, the derivative is $3 \ln 3 \approx 3.3$; this is the slope of the line tangent to the curve at $x = 1$. The function evaluated at $x = 1$ yields the point $(1, 3)$. Thus the equation of the line is

$$\begin{aligned} y - 3 &= 3.3(x - 1) \\ y &= 3.3x - 0.3 \end{aligned}$$

Thus the equation of the line is $y = 3.3x - 0.3$.

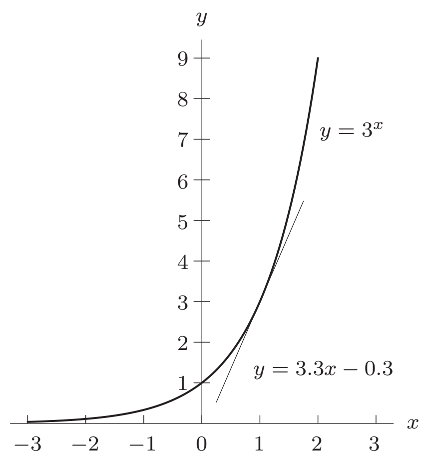


Figure 3.7

31. $y = e^{-2t}$, $y' = -2e^{-2t}$. At $t = 0$, $y = 1$ and $y' = -2$. Thus the tangent line at $(0, 1)$ is $y = -2t + 1$.

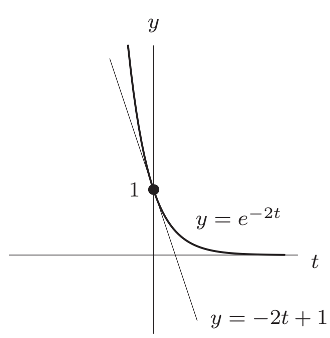


Figure 3.8

32. To find the equation of the line, we need a point and the slope. Since $f(4) = 10e^{-0.2(4)} = 4.493$, the point on the line is $(4, 4.493)$. We use the derivative to find the slope:

$$f'(x) = 10e^{-0.2x}(-0.2) = -2e^{-0.2x}.$$

Substituting $x = 4$, we see that the slope is

$$m = f'(4) = -2e^{-0.2(4)} = -0.899.$$

Using $y - y_0 = m(x - x_0)$, we find that the equation for the tangent line is:

$$y - 4.493 = -0.899(x - 4)$$

Simplifying, we have

$$y = -0.899x + 8.089.$$

33. **(a)** $P(12) = 10e^{0.6(12)} = 10e^{7.2} \approx 13{,}394$ fish. There are 13,394 fish in the area after 12 months.
(b) We differentiate to find $P'(t)$, and then substitute in to find $P'(12)$:

$$P'(t) = 10(e^{0.6t})(0.6) = 6e^{0.6t}$$
$$P'(12) = 6e^{0.6(12)} \approx 8037 \text{ fish/month}.$$

The population is growing at a rate of approximately 8037 fish per month.

34. We have $f(0) = 6.8$ and $f(10) = 6.8e^{0.012(10)} = 7.667$. The derivative of $f(t)$ is

$$f'(t) = 6.8e^{0.012t} \cdot 0.012 = 0.0816e^{0.012t},$$

and so $f'(0) = 0.0816$ and $f'(10) = 0.092$.

These values tell us that in 2009 (at $t = 0$), the population of the world was 6.8 billion people and the population was growing at a rate of 0.0816 billion people per year. In the year 2019 (at $t = 10$), this model predicts that the population of the world will be 7.667 billion people and growing at a rate of 0.092 billion people per year.

35. $f(p) = 10{,}000e^{-0.25p}$, $f(2) = 10{,}000e^{-0.5} = 6065$. If the product sells for \$2, then 6065 units can be sold.

$$f'(p) = 10{,}000e^{-0.25p}(-0.25) = -2500e^{-0.25p}$$

$$f'(2) = -2500e^{-0.5} = -1516.$$

Thus, at a price of \$2, a \$1 increase in price results in a decrease in quantity sold of about 1516 units .

36. We have $f(t) = 1040(1.3)^t$, so

$$f(0) = 1040(1.3)^0 = 1040 \text{ megawatts},$$

and

$$f(15) = 1040(1.3)^{15} = 53{,}233 \text{ megawatts}.$$

Since $f'(t) = 1040\ln(1.3)(1.3)^t$, we have

$$f'(0) = 1040\ln(1.3)(1.3)^0 = 273 \text{ megawatts/year},$$

and

$$f'(15) = 1040\ln(1.3)(1.3)^{15} = 13{,}967 \text{ megawatts/year}.$$

In 2000, solar power production was 1040 megawatts and production was increasing at 273 megawatts per year. In 2005, solar power production was 53,233 megawatts and production was increasing at about 14,000 megawatts per year.

37. When $t = 5$, we have $R = 350\ln 5 = \$563.30$. Since $f'(t) = 350/t$, we have $f'(5) = 350/5 = 70$ dollars per week. Then

$$\text{Relative rate of change } = \frac{f'(5)}{f(5)} = \frac{70}{563.30} = 0.124 = 12.4\% \text{ per week}.$$

Five weeks after the DVD was released, the revenue from sales is \$563.30, and is increasing at a rate of \$70 per week, or 12.4% per week.

38. **(a)** $P = 9.906(0.997)^{11} = 9.584$ million.

(b) Differentiating, we have

$$\frac{dP}{dt} = 9.906(\ln 0.997)(0.997)^t$$

$$\text{so} \quad \left.\frac{dP}{dt}\right|_{t=11} = 9.906(\ln 0.997)(0.997)^{11} = -0.0288 \text{ million/year.}$$

Thus in 2020, Hungary's population will be decreasing by about 28,800 people per year.

39. Since $P = 35{,}000(0.98)^t$, the rate of change of the population is given by

$$\frac{dP}{dt} = 35{,}000 \cdot (\ln 0.98)(0.98^t).$$

On January 1, 2023, we have $t = 13$. At $t = 13$, the rate of change is $35{,}000(\ln 0.98)(0.98^{13}) = -544$ people/year. The negative sign indicates that the population is decreasing.

40. Differentiating gives

$$\text{Rate of change of price } = \frac{dV}{dt} = 75(1.35)^t \ln 1.35 \approx 22.5(1.35)^t \text{ dollar/yr.}$$

41. Since $y = 2^x$, $y' = (\ln 2)2^x$. At $(0,1)$, the tangent line has slope $\ln 2$ so its equation is $y = (\ln 2)x + 1$. At c, $y = 0$, so $0 = (\ln 2)c + 1$, thus $c = -\frac{1}{\ln 2}$.

42. The concentration of the drug in the body after 4 hours is

$$f(4) = 27e^{-0.14(4)} = 15.4 \text{ ng/ml.}$$

The rate of change of the concentration is the derivative

$$f'(t) = 27e^{-0.14t}(-0.14) = -3.78e^{-0.14t}.$$

At $t = 4$, the concentration is changing at a rate of

$$f'(4) = -3.78e^{-0.14(4)} = -2.16 \text{ ng/ml per hour.}$$

43.

$$C(q) = 1000 + 30e^{0.05q}$$
$$C(50) = 1000 + 30e^{2.5} \approx 1365$$

so it costs about $1365 to produce 50 units.

$$C'(q) = 30(0.05)e^{0.05q} = 1.5e^{0.05q}$$
$$C'(50) = 1.5e^{2.5} \approx 18.27$$

It costs about $18.27 to produce an additional unit when the production level is 50 units.

44. The rate of change of temperature is

$$\frac{dH}{dt} = 16(-0.02)e^{-0.02t} = -0.32e^{-0.02t}.$$

When $t = 0$,

$$\frac{dH}{dt} = -0.32e^0 = -0.32^\circ\text{C/min.}$$

When $t = 10$,

$$\frac{dH}{dt} = -0.32e^{-0.02(10)} = -0.262^\circ\text{C/min.}$$

45. **(a)** We first substitute 185 for A_0, differentiate $A(t)$, and substitute $t = 500$.

$$\begin{aligned} A(t) &= 185e^{-0.000121t} \\ A'(t) &= 185e^{-0.000121t}(-0.000121) = -0.022385e^{-0.000121t} \\ A'(500) &= -0.022385e^{-0.000121 \cdot 500} = -0.021. \end{aligned}$$

This tell us that, the quantity of carbon-14 in the tree is decaying by approximately 0.021 micrograms per year.

(b) We are told that $A(t) = 0.91A_0$, so

$$\begin{aligned} 0.91A_0 &= A_0 e^{-0.000121t} \\ 0.91 &= e^{-0.000121t} \\ \ln 0.91 &= -0.000121t \\ t &= -\frac{\ln 0.91}{0.000121} = 779.4. \end{aligned}$$

The shroud was approximately 779.4 years old in 1988.

46. $C(500) = 1000 + 300\ln(500) \approx 2864.38$; it costs about \$2864 to produce 500 units. $C'(q) = \frac{300}{q}$, $C'(500) = \frac{300}{500} = 0.6$. When the production level is 500, each additional unit costs about \$0.60 to produce.

47. **(a)** Since the initial population (at $t = 0$) is 1.166 and the growth rate is 1.5%, we have

$$P = 1.166(1 + 0.015)^t = 1.166(1.015)^t \text{ billion.}$$

(b) Differentiating gives

$$\begin{aligned} \frac{dP}{dt} &= 1.166\frac{d}{dt}(1.015)^t = 1.166(1.015)^t(\ln 1.015). \\ \left.\frac{dP}{dt}\right|_{t=0} &= 1.166(1.015)^0 \ln 1.015 = 0.017 \text{ billion people per year.} \\ \left.\frac{dP}{dt}\right|_{t=25} &= 1.166(1.015)^{25} \ln 1.015 = 0.025 \text{ billion people per year.} \end{aligned}$$

The derivative $\frac{dP}{dt}$ is the rate of growth of India's population; $\left.\frac{dP}{dt}\right|_{t=0}$ and $\left.\frac{dP}{dt}\right|_{t=25}$ are the rates of growth in the years 2009 and 2034, respectively.

48. For t in years since 2009, the population of Mexico is given by the formula

$$M = 111(1 + 0.0113)^t = 111(1.0113)^t \text{ million}$$

and that of the US by

$$U = 307(1 + 0.00975)^t = 307(1.00975)^t \text{ million,}$$

The rate of change of each population, in people/year is given by

$$\left.\frac{dM}{dt}\right|_{t=0} = 111\frac{d}{dt}(1.0113)^t\Big|_{t=0} = 111(1.0113)^t \ln(1.0113)\Big|_{t=0} = 1.25 \text{ million people per year}$$

and $$\left.\frac{dU}{dt}\right|_{t=0} = 307\frac{d}{dt}(1.00975)^t\Big|_{t=0} = 307(1.00975)^t \ln(1.00975)\Big|_{t=0} = 2.98 \text{ million per people per year.}$$

Since $\left.\frac{dU}{dt}\right|_{t=0} > \left.\frac{dM}{dt}\right|_{t=0}$, the population of the US was growing faster in 2009.

49. **(a)** For $y = \ln x$, we have $y' = 1/x$, so the slope of the tangent line is $f'(1) = 1/1 = 1$. The equation of the tangent line is $y - 0 = 1(x - 1)$, so, on the tangent line, $y = g(x) = x - 1$.

(b) Using a value on the tangent line to approximate $\ln(1, 1)$, we have

$$\ln(1.1) \approx g(1.1) = 1.1 - 1 = 0.1.$$

Similarly, $\ln(2)$ is approximated by

$$\ln(2) \approx g(2) = 2 - 1 = 1.$$

(c) From Figure 3.9, we see that $f(1.1)$ and $f(2)$ are below $g(x) = x - 1$. Similarly, $f(0.9)$ and $f(0.5)$ are also below $g(x)$. This is true for any approximation of this function by a tangent line since f is concave down ($f''(x) = -\frac{1}{x^2} < 0$ for all $x > 0$). Thus, for a given x-value, the y-value given by the function is always below the value given by the tangent line.

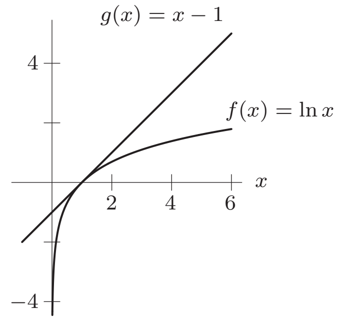


Figure 3.9

50.

$$g(x) = ax^2 + bx + c \qquad f(x) = e^x$$
$$g'(x) = 2ax + b \qquad f'(x) = e^x$$
$$g''(x) = 2a \qquad f''(x) = e^x$$

So, using $g''(0) = f''(0)$, etc., we have $2a = 1$, $b = 1$, and $c = 1$, and thus $g(x) = \frac{1}{2}x^2 + x + 1$, as shown in Figure 3.10.

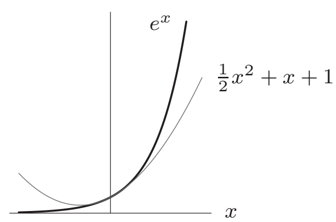


Figure 3.10

The two functions do look very much alike near $x = 0$. They both increase for large values of x, but e^x increases much more quickly. For very negative values of x, the quadratic goes to ∞ whereas the exponential goes to 0. By choosing a function whose first few derivatives agreed with the exponential when $x = 0$, we got a function which looks like the exponential for x-values near 0.

Solutions for Section 3.3

1. $\dfrac{d}{dx}\left((4x^2+1)^7\right) = 7(4x^2+1)^6 \dfrac{d}{dx}(4x^2+1) = 7(4x^2+1)^6 \cdot 8x = 56x(4x^2+1)^6.$

2. $f'(x) = 99(x+1)^{98} \cdot 1 = 99(x+1)^{98}.$

3. $\dfrac{dR}{dq} = 4(q^2+1)^3 \cdot 2q = 8q(q^2+1)^3.$

4. $w' = 100(t^2+1)^{99}(2t) = 200t(t^2+1)^{99}.$

5. $w' = 100(t^3+1)^{99}(3t^2) = 300t^2(t^3+1)^{99}$.

6. $\dfrac{dw}{dr} = 3(5r-6)^2 \cdot 5 = 15(5r-6)^2$.

7. $y' = \dfrac{3s^2}{2\sqrt{s^3+1}}$.

8. $\dfrac{dy}{dx} = -3(2x) + 2\left(3e^{3x}\right) = -6x + 6e^{3x}$.

9. By the chain rule, $\dfrac{dC}{dq} = 12 \cdot 3(3q^2-5)^2 \cdot 6q = 216q(3q^2-5)^2$.

10. $f'(x) = 6(e^{5x})(5) + (e^{-x^2})(-2x) = 30e^{5x} - 2xe^{-x^2}$.

11. $\dfrac{dy}{dt} = 5(5e^{5t+1}) = 25e^{5t+1}$.

12. $\dfrac{dw}{dt} = -6te^{-3t^2}$.

13. $w' = \dfrac{1}{2\sqrt{s}}e^{\sqrt{s}}$.

14. $\dfrac{dy}{dt} = \dfrac{5}{5t+1}$.

15. $f'(x) = \dfrac{-1}{1-x} = \dfrac{1}{x-1}$.

16. $f'(t) = \dfrac{2t}{t^2+1}$.

17. $f'(x) = \dfrac{1}{1-e^{-x}} \cdot (-e^{-x})(-1) = \dfrac{e^{-x}}{1-e^{-x}}$.

18. $f'(x) = \dfrac{1}{e^x+1} \cdot e^x$.

19. $f'(t) = 5 \cdot \dfrac{1}{5t+1} \cdot 5 = \dfrac{25}{5t+1}$.

20. $g'(t) = \dfrac{1}{4t+9} \cdot 4 = \dfrac{4}{4t+9}$.

21. $\dfrac{dy}{dx} = \dfrac{1}{3t+2} \cdot 3 = \dfrac{3}{3t+2}$.

22. $\dfrac{dQ}{dt} = 100(0.5)(t^2+5)^{-0.5} \cdot 2t = 100t(t^2+5)^{-0.5}$.

23. $\dfrac{dy}{dx} = 5 + \dfrac{1}{x+2} \cdot 1 = 5 + \dfrac{1}{x+2}$.

24. $\dfrac{dy}{dx} = 2(5+e^x)e^x$.

25. $\dfrac{dP}{dx} = 0.5(1+\ln x)^{-0.5}\left(\dfrac{1}{x}\right) = \dfrac{0.5}{x(1+\ln x)^{0.5}}$.

26. $\dfrac{d}{dx}(\sqrt{e^x+1}) = \dfrac{d}{dx}(e^x+1)^{1/2} = \dfrac{1}{2}(e^x+1)^{-1/2}\dfrac{d}{dx}(e^x+1) = \dfrac{e^x}{2\sqrt{e^x+1}}$.

27. $f'(x) = \dfrac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x) = \dfrac{-x}{\sqrt{1-x^2}}$.

28. The power and chain rules give

$$f'(\theta) = -1(e^\theta + e^{-\theta})^{-2} \cdot \frac{d}{d\theta}(e^\theta + e^{-\theta}) = -(e^\theta + e^{-\theta})^{-2}(e^\theta + e^{-\theta}(-1)) = -\left(\frac{e^\theta - e^{-\theta}}{(e^\theta + e^{-\theta})^2}\right).$$

29. Since $f'(t) = 10$, we have

$$\frac{f'(t)}{f(t)} = \frac{10}{10t+5}.$$

30. Since $f'(t) = 15$, we have
$$\frac{f'(t)}{f(t)} = \frac{15}{15t+12}.$$

31. Since $f'(t) = 5 \cdot 8e^{5t}$, we have
$$\frac{f'(t)}{f(t)} = \frac{5 \cdot 8e^{5t}}{8e^{5t}} = 5.$$

32. Since $f'(t) = -7 \cdot 30e^{-7t}$, we have
$$\frac{f'(t)}{f(t)} = \frac{-7 \cdot 30e^{-7t}}{30e^{-7t}} = -7.$$

33. Since $f'(t) = 2 \cdot 6t = 12t$, we have
$$\frac{f'(t)}{f(t)} = \frac{12t}{6t^2} = \frac{2}{t}.$$

34. Since $f'(t) = -4 \cdot 35t^{-5}$, we have
$$\frac{f'(t)}{f(t)} = \frac{-4 \cdot 35t^{-5}}{35t^{-4}} = -4t^{-1} = -\frac{4}{t}.$$

35. Since $\ln f(t) = \ln(5e^{1.5t}) = \ln 5 + 1.5t$ we have
$$\frac{d}{dt}\ln f(t) = \frac{d}{dt}(\ln 5 + 1.5t) = 0 + 1.5 = 1.5.$$

36. Since $\ln f(t) = \ln(6.8e^{-0.5t}) = \ln 6.8 - 0.5t$ we have
$$\frac{d}{dt}\ln f(t) = \frac{d}{dt}(\ln 6.8 - 0.5t) = 0 - 0.5 = -0.5.$$

37. Since $\ln f(t) = \ln 3t^2 = \ln 3 + 2\ln t$ we have
$$\frac{d}{dt}\ln f(t) = \frac{d}{dt}(\ln 3 + 2\ln t) = 0 + \frac{2}{t} = \frac{2}{t}.$$

38. Since $\ln f(t) = \ln 4.5t^{-4} = \ln 4.5 - 4\ln t$ we have
$$\frac{d}{dt}\ln f(t) = \frac{d}{dt}(\ln 4.5 - 4\ln t) = 0 - \frac{4}{t} = -\frac{4}{t}.$$

39. We have $f(2) = (2-1)^3 = 1$, so $(2, 1)$ is a point on the tangent line. Since $f'(x) = 3(x-1)^2$, the slope of the tangent line is
$$m = f'(2) = 3(2-1)^2 = 3.$$
The equation of the line is
$$y - 1 = 3(x-2) \quad \text{or} \quad y = 3x - 5.$$

40. The marginal revenue, MR, is obtained by differentiating the total revenue function, R. We use the chain rule so
$$MR = \frac{dR}{dq} = 2000q \cdot \frac{1}{1+1000q^2}.$$
When $q = 10$,
$$\text{Marginal Revenue} = \frac{2000(10)}{1+1000(10)^2} = 0.2 \text{ \$/unit}.$$

41. If the distance $s(t) = 20e^{\frac{t}{2}}$, then the velocity, $v(t)$, is given by
$$v(t) = s'(t) = \left(20e^{\frac{t}{2}}\right)' = \left(\frac{1}{2}\right)\left(20e^{\frac{t}{2}}\right) = 10e^{\frac{t}{2}}.$$

42. **(a)** $\frac{dB}{dt} = P\left(1+\frac{r}{100}\right)^t \ln\left(1+\frac{r}{100}\right)$. The expression $\frac{dB}{dt}$ tells us how fast the amount of money in the bank is changing with respect to time for fixed initial investment P and interest rate r.

(b) $\frac{dB}{dr} = Pt\left(1+\frac{r}{100}\right)^{t-1}\frac{1}{100}$. The expression $\frac{dB}{dr}$ indicates how fast the amount of money changes with respect to the interest rate r , assuming fixed initial investment P and time t.

43. When $t = 10$, we have $D = \sqrt{10^3 + 1} = 31.640$ feet. Since

$$f'(t) = \frac{1}{2}(t^3 + 1)^{-1/2}(3t^2),$$

we have

$$f'(10) = \frac{1}{2}(10^3 + 1)^{-1/2}(3 \cdot 10^2) = 4.741 \text{ feet per second.}$$

Then

$$\text{Relative rate of change } = \frac{f'(10)}{f(10)} = \frac{4.741}{31.640} = 0.150 = 15\% \text{ per second.}$$

Alternately, we could have found the relative rate by finding the derivative of $\ln(f(t))$. At $t = 10$ seconds, distance traveled is 31.460 feet and the distance is increasing at a rate of 4.741 ft/sec, which is 15% per second.

44. The chain rule gives

$$\left.\frac{d}{dx}f(g(x))\right|_{x=30} = f'(g(30))g'(30) = f'(55)g'(30) = (1)(\frac{1}{2}) = \frac{1}{2}.$$

45. The chain rule gives

$$\left.\frac{d}{dx}f(g(x))\right|_{x=70} = f'(g(70))g'(70) = f'(60)g'(70) = (1)(0) = 0.$$

46. The chain rule gives

$$\left.\frac{d}{dx}g(f(x))\right|_{x=30} = g'(f(30))f'(30) = g'(20)f'(30) = (1/2)(-2) = -1.$$

47. The chain rule gives

$$\left.\frac{d}{dx}g(f(x))\right|_{x=70} = g'(f(70))f'(70) = g'(30)f'(70) = (1)(\frac{1}{2}) = \frac{1}{2}.$$

48. Estimates may vary. From the graphs, we estimate $g(1) \approx 2$, $g'(1) \approx 1$, and $f'(2) \approx 0.8$. Thus, by the chain rule,

$$h'(1) = f'(g(1)) \cdot g'(1) \approx f'(2) \cdot g'(1) \approx 0.8 \cdot 1 = 0.8.$$

49. Estimates may vary. From the graphs, we estimate $f(1) \approx -0.4$, $f'(1) \approx 0.5$, and $g'(-0.4) \approx 2$. Thus, by the chain rule,

$$k'(1) = g'(f(1)) \cdot f'(1) \approx g'(-0.4) \cdot 0.5 \approx 2 \cdot 0.5 = 1.$$

50. Estimates may vary. From the graphs, we estimate $g(2) \approx 1.6$, $g'(2) \approx -0.5$, and $f'(1.6) \approx 0.8$. Thus, by the chain rule,

$$h'(2) = f'(g(2)) \cdot g'(2) \approx f'(1.6) \cdot g'(2) \approx 0.8(-0.5) = -0.4.$$

51. Estimates may vary. From the graphs, we estimate $f(2) \approx 0.3$, $f'(2) \approx 1.1$, and $g'(0.3) \approx 1.7$. Thus, by the chain rule,

$$k'(2) = g'(f(2)) \cdot f'(2) \approx g'(0.3) \cdot f'(2) \approx 1.7 \cdot 1.1 \approx 1.9.$$

52. **(a)** With four additional year of education, your wages are

$$10(1.14)^4 = 16.89 \text{ dollars per hour.}$$

(b) In 20 years time,

$$\text{Wages without additional education} = 10e^{0.035(20)} = 20.14 \text{ dollars/hour.}$$
$$\text{Wages with additional education} = 16.89e^{0.035(20)} = 34.01 \text{ dollars/hour.}$$

Thus, the difference is $34.01 - 20.14 = 13.87$ dollars per hour.

(c) The difference has increased from \$6.89 now (using the answer to part (a)) to \$13.87 in 20 years time. Thus, the difference is increasing. To calculate the rate, we first find a formula for the difference.

$$\text{Wages in year } t \text{ without additional education} = 10e^{0.035t} \text{ dollars/hour}.$$
$$\text{Wages in year } t \text{ with additional education} = 16.89e^{0.035t} \text{ dollars/hour}.$$
$$\text{Difference} = 16.89e^{0.035t} - 10e^{0.035t} = 6.89e^{0.035t} \text{ dollars/hour}.$$

Differentiating to find the rate of change, we have

$$\frac{d}{dt}(6.89e^{0.035t}) = 6.89(0.035)e^{0.035t} = 0.2412e^{0.035t} \text{ dollars per hour per year.}$$

When $t = 20$,

$$\text{Rate} = 0.2412e^{0.035(20)} = 0.486 \text{ dollars per hour per year.}$$

Thus, your wages are increasing at about \$0.50 per hour per year.

53. Since the graphs of $f(t)$ and $h(t)$ are tangent at $t = a$, we have $f(a) = h(a)$ and $f'(a) = h'(a)$. From $h(t) = Ae^{kt}$ we get

$$\text{Relative rate of change of } f = \frac{f'(a)}{f(a)} = \frac{h'(a)}{h(a)} = \frac{kAe^{kt}}{Ae^{kt}} = k.$$

Thus the relative rate of change of f at a point equals the continuous rate of change of the exponential function that is tangent to f at that point.

Solutions for Section 3.4

1. By the product rule,

$$f'(x) = 2(3x-2) + (2x+1)\cdot 3 = 12x - 1.$$

Alternatively,

$$f'(x) = (6x^2 - x - 2)' = 12x - 1.$$

The two answers match.

2. By the product rule, $f'(x) = 2x(x^3+5) + x^2(3x^2) = 2x^4 + 3x^4 + 10x = 5x^4 + 10x$. Alternatively, $f'(x) = (x^5 + 5x^2)' = 5x^4 + 10x$. The two answers should, and do, match.

3. $f'(x) = x \cdot e^x + e^x \cdot 1 = e^x(x+1)$.

4. $f'(t) = (1)e^{-2t} + t(-2e^{-2t}) = e^{-2t} - 2te^{-2t}$.

5. Differentiating with respect to x, we have

$$\begin{aligned}\frac{dy}{dx} = \frac{d}{dx}(5xe^{x^2}) &= \left(\frac{d}{dx}(5x)\right)e^{x^2} + 5x\frac{d}{dx}(e^{x^2}) \\ &= (5)e^{x^2} + 5x(e^{x^2}\cdot 2x) \\ &= 5e^{x^2} + 10x^2e^{x^2}.\end{aligned}$$

6. Differentiating with respect to t, we have

$$\begin{aligned}\frac{dy}{dt} = \frac{d}{dt}(t^2(3t+1)^3) &= \left(\frac{d}{dt}(t^2)\right)(3t+1)^3 + t^2\frac{d}{dt}((3t+1)^3) \\ &= (2t)(3t+1)^3 + t^2(3(3t+1)^2\cdot 3) \\ &= 2t(3t+1)^3 + 9t^2(3t+1)^2\end{aligned}$$

7. Differentiating with respect to x, we have

$$\begin{aligned}\frac{dy}{dx} = \frac{d}{dx}(x\ln x) &= \left(\frac{d}{dx}(x)\right)\ln x + x\frac{d}{dx}(\ln x)\\ &= 1\cdot \ln x + x\cdot\frac{1}{x}\\ &= \ln x + 1.\end{aligned}$$

8. $\dfrac{dy}{dt} = 2te^t + (t^2+3)e^t = e^t(t^2+2t+3)$.

9. $\dfrac{dz}{dt} = (3t+1)5 + 3(5t+2) = 15t+5+15t+6 = 30t+11$.
We could have started by multiplying the factors to obtain $15t^2+11t+2$, and then taken the derivative of the result.

10. $y' = (3t^2-14t)e^t + (t^3-7t^2+1)e^t = (t^3-4t^2-14t+1)e^t$.

11. $\dfrac{dP}{dt} = (t^2)(\frac{1}{t}) + (2t)(\ln t) = t + 2t\ln t$.

12. $\dfrac{dR}{dq} = (3q)(-e^{-q}) + (e^{-q})(3) = -3qe^{-q} + 3e^{-q}$.

13. Since $f(t) = 5t^{-1} + 6t^{-2}$, we have

$$f'(t) = -5t^{-2} - 12t^{-3} = -\frac{5}{t^2} - \frac{12}{t^3}.$$

14. Divide and then differentiate

$$\begin{aligned}f(x) &= x + \frac{3}{x}\\ f'(x) &= 1 - \frac{3}{x^2}.\end{aligned}$$

15. $y' = 1\cdot e^{-t^2} + te^{-t^2}(-2t)$

16. $f'(z) = \dfrac{1}{2\sqrt{z}}e^{-z} - \sqrt{z}e^{-z}$.

17. $g'(p) = p\left(\dfrac{2}{2p+1}\right) + \ln(2p+1)(1) = \dfrac{2p}{2p+1} + \ln(2p+1)$.

18. $f'(t) = 1\cdot e^{5-2t} + te^{5-2t}(-2) = e^{5-2t}(1-2t)$.

19.

$$\begin{aligned}f'(w) &= (e^{w^2})(10w) + (5w^2+3)(e^{w^2})(2w)\\ &= 2we^{w^2}(5+5w^2+3)\\ &= 2we^{w^2}(5w^2+8).\end{aligned}$$

20. $y' = 2^x + x(\ln 2)2^x = 2^x(1 + x\ln 2)$.

21. $w' = (3t^2+5)(t^2-7t+2) + (t^3+5t)(2t-7)$.

22. Using the product and chain rules, we have

$$\begin{aligned}\frac{dz}{dt} &= 9(te^{3t}+e^{5t})^8\cdot\frac{d}{dt}(te^{3t}+e^{5t}) = 9(te^{3t}+e^{5t})^8(1\cdot e^{3t} + t\cdot e^{3t}\cdot 3 + e^{5t}\cdot 5)\\ &= 9(te^{3t}+e^{5t})^8(e^{3t}+3te^{3t}+5e^{5t}).\end{aligned}$$

23. $f'(x) = \dfrac{e^x\cdot 1 - x\cdot e^x}{(e^x)^2} = \dfrac{e^x(1-x)}{(e^x)^2} = \dfrac{1-x}{e^x}$.

24. Using the quotient rule, we have

$$\frac{dw}{dz} = \frac{d}{dz}\left(\frac{3z}{1+2z}\right) = \frac{3(1+2z) - 3z(2)}{(1+2z)^2} = \frac{3}{(1+2z)^2}.$$

25. Using the quotient rule gives

$$\frac{dz}{dt} = \frac{d}{dt}\left(\frac{1-t}{1+t}\right) = \frac{-1\cdot(1+t)-(1-t)\cdot 1}{(1+t)^2} = \frac{-1-t-1+t}{(1+t)^2} = \frac{-2}{(1+t)^2}.$$

26. Using the quotient rule,

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{e^x}{1+e^x}\right) = \frac{e^x(1+e^x)-e^x(e^x)}{(1+e^x)^2} = \frac{e^x+e^{2x}-e^{2x}}{(1+e^x)^2} = \frac{e^x}{(1+e^x)^2}.$$

27. Using the quotient rule,

$$\begin{aligned}\frac{dw}{dy} = \frac{d}{dy}\left(\frac{3y+y^2}{5+y}\right) &= \frac{(3+2y)(5+y)-(3y+y^2)\cdot 1}{(5+y)^2}\\ &= \frac{15+13y+2y^2-3y-y^2}{(5+y)^2} = \frac{15+10y+y^2}{(5+y)^2}.\end{aligned}$$

28. Using the quotient rule, we have

$$\frac{dy}{dz} = \frac{d}{dz}\left(\frac{1+z}{\ln z}\right) = \frac{1\cdot\ln z-(1+z)(1/z)}{(\ln z)^2} = \frac{z\ln z-1-z}{z(\ln z)^2}.$$

29. We use the quotient rule. We have

$$f'(x) = \frac{(cx+k)a-(ax+b)c}{(cx+k)^2} = \frac{acx+ak-acx-bc}{(cx+k)^2} = \frac{ak-bc}{(cx+k)^2}.$$

30. Since a and b are constants, we have $f'(x) = 3(ax^2+b)^2(2ax) = 6ax(ax^2+b)^2$.

31. We use the product rule. We have $f'(x) = (ax)(e^{-bx}(-b)) + (a)(e^{-bx}) = -abxe^{-bx} + ae^{-bx}$.

32. Since a and b are constants, we have $f'(t) = ae^{bt}(b) = abe^{bt}$.

33. Using the product and chain rules, we have

$$\begin{aligned}g'(\alpha) &= e^{\alpha e^{-2\alpha}}\cdot\frac{d}{dx}(\alpha e^{-2\alpha}) = e^{\alpha e^{-2\alpha}}(1\cdot e^{-2\alpha}+\alpha e^{-2\alpha}(-2))\\ &= e^{\alpha e^{-2\alpha}}(e^{-2\alpha}-2\alpha e^{-2\alpha})\\ &= (1-2\alpha)e^{-2\alpha}e^{\alpha e^{-2\alpha}}.\end{aligned}$$

34.

$$\begin{aligned}f'(x) &= 3(2x-5)+2(3x+8) = 12x+1\\ f''(x) &= 12.\end{aligned}$$

35. $f(x) = x^2e^{-x}$, $f(0) = 0$

$f'(x) = 2xe^{-x} + x^2e^{-x}\cdot(-1) = e^{-x}(2x-x^2)$, so $f'(0) = 0$. Thus the tangent line is $y = 0$ (the x-axis). See Figure 3.11.

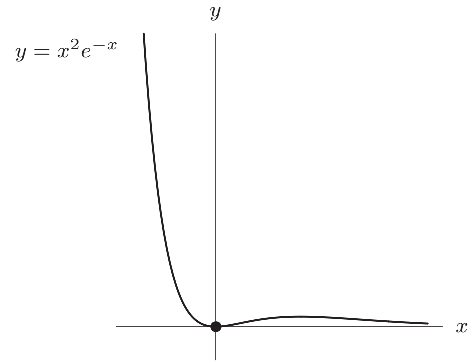


Figure 3.11

36. Since $f(0) = -5/1 = -5$, the tangent line passes through the point $(0, -5)$, so its vertical intercept is -5. To find the slope of the tangent line, we find the derivative of $f(x)$ using the quotient rule:

$$f'(x) = \frac{(x+1)\cdot 2 - (2x-5)\cdot 1}{(x+1)^2} = \frac{7}{(x+1)^2}.$$

At $x = 0$, the slope of the tangent line is $m = f'(0) = 7$. The equation of the tangent line is $y = 7x - 5$.

37. We have $f(t) = 100te^{-0.5t}$ so

$$f(1) = 100(1)e^{-0.5} = 60.65 \text{ mg},$$

and

$$f(5) = 100(5)e^{-0.5(5)} = 41.04 \text{ mg}.$$

We use the product rule to find $f'(t)$:

$$f'(t) = 100e^{-0.5t} + 100t(e^{-0.5t}(-0.5)) = 100e^{-0.5t} - 50te^{-0.5t}.$$

Therefore,

$$f'(1) = 100e^{-0.5} - 50e^{-0.5} = 30.33 \text{ mg/hour},$$

and

$$f'(5) = 100e^{-2.5} - 250e^{-2.5} = -12.31 \text{ mg/hour}.$$

One hour after the drug was administered, the quantity of drug in the body is 60.65 mg and the quantity is increasing at a rate of 30.33 mg per hour. Five hours after the drug was administered, the quantity in the body is 41.04 mg and the quantity is decreasing at a rate of 12.31 mg per hour.

38. **(a)** See Figure 3.12.

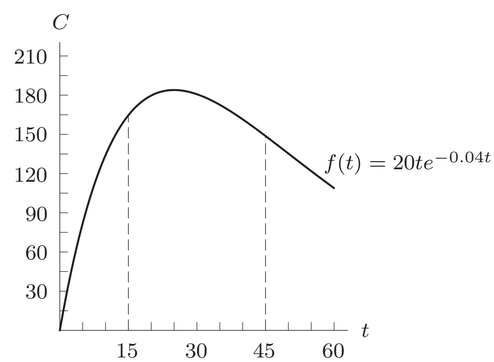


Figure 3.12

Looking at the graph of C, we can see that the see that at $t = 15$, C is increasing. Thus, the slope of the curve at that point is positive, and so $f'(15)$ is also positive. At $t = 45$, the function is decreasing, i.e. the slope of the curve is negative, and thus $f'(45)$ is negative.

(b) We begin by differentiating the function:

$$f'(t) = (20t)(-0.04e^{-0.04t}) + (e^{-0.04t})(20)$$
$$f'(t) = e^{-0.04t}(20 - 0.8t).$$

At $t = 30$,

$$f(30) = 20(30)e^{-0.04\cdot(30)} = 600e^{-1.2} \approx 181 \text{ mg/ml}$$
$$f'(30) = e^{-1.2}(20 - (0.8)(30)) = e^{-1.2}(-4) \approx -1.2 \text{ mg/ml/min}.$$

These results mean the following: At $t = 30$, or after 30 minutes, the concentration of the drug in the body ($f(30)$) is about 181 mg/ml. The rate of change of the concentration ($f'(30)$) is about -1.2 mg/ml/min, meaning that the concentration of the drug in the body is dropping by 1.2 mg/ml each minute at $t = 30$ minutes.

39. We use the quotient rule:

$$\frac{dP}{dr} = \frac{(k+r)\cdot c - cr\cdot 1}{(k+r)^2} = \frac{kc}{(k+r)^2}.$$

The derivative dP/dr gives the approximate change in the size of the population given a one unit increase in the available quantity of the resource.

40. **(a)** $q(10) = 5000e^{-0.8} \approx 2247$ units.
(b) $q' = 5000(-0.08)e^{-0.08p} = -400e^{-0.08p}$, $q'(10) = -400e^{-0.8} \approx -180$. This means that at a price of \$10, a \$1 increase in price will result in a decrease in quantity demanded by 180 units.

41. $R(p) = p\cdot q = 5000pe^{-0.08p}$

$R(10) = 50{,}000e^{-0.8} \approx 22{,}466$; revenues of about \$22,466 can be expected when the selling price is \$10.
$R'(p) = 5000e^{-0.08p} + 5000p(-0.08)e^{-0.08p} = (5000 - 400p)e^{-0.08p}$

$R'(10) = 1000e^{-0.8} \approx 449$; if price is increased by one dollar over \$10, revenue will increase by about \$449.

42. **(a)** $R(p) = p\cdot 1000e^{-0.02p} = 1000pe^{-0.02p}$.
(b) $R'(p) = 1000e^{-0.02p} + 1000pe^{-0.02p}(-0.02) = e^{-0.02p}(1000 - 20p)$
(c) $R(10) = 10,000e^{-0.2} \approx 8187$; you will have about 8187 dollars in revenue if you sell the product for \$10.
$R'(10) = e^{-0.2}(1000 - 200) \approx 655$; a one dollar increase in price over \$10 will generate about \$655 in additional revenue.

43. By the product rule, $\dfrac{d}{dt}tf(t) = f(t) + tf'(t)$. Thus, using the information given in the problem, we have

$$f(t) + tf'(t) = 1 + f(t).$$

Subtracting $f(t)$ from both sides gives $tf'(t) = 1$, so $f'(t) = 1/t$.

44. **(a)** $f(140) = 15{,}000$ says that 15,000 skateboards are sold when the cost is \$140 per board.
$f'(140) = -100$ means that if the price is increased from \$140, roughly speaking, every dollar of increase will decrease the total sales by 100 boards.
(b) $\dfrac{dR}{dp} = \dfrac{d}{dp}(p\cdot q) = \dfrac{d}{dp}(p\cdot f(p)) = f(p) + pf'(p)$.
So,

$$\left.\frac{dR}{dp}\right|_{p=140} = f(140) + 140f'(140)$$
$$= 15{,}000 + 140(-100) = 1000.$$

(c) From (b) we see that $\left.\dfrac{dR}{dp}\right|_{p=140} = 1000 > 0$. This means that the revenue will increase by about \$1000 if the price is raised by \$1.

45. This problem can be solved by using either the product rule or the fact that

$$\frac{f'}{f} = \frac{d}{dx}(\ln f) \quad \text{and} \quad \frac{g'}{g} = \frac{d}{dx}(\ln g).$$

We use the second method. The relative rate of change of fg is $(fg)'/(fg)$, so

$$\frac{(fg)'}{fg} = \frac{d}{dx}\ln(fg) = \frac{d}{dx}(\ln f + \ln g) = \frac{d}{dx}(\ln f) + \frac{d}{dx}(\ln g) = \frac{f'}{f} + \frac{g'}{g}.$$

Thus, the relative rate of change of fg is the sum of the relative rates of change of f and g.

46. This problem can be solved by using either the quotient rule or the fact that

$$\frac{f'}{f} = \frac{d}{dx}(\ln f) \quad \text{and} \quad \frac{g'}{g} = \frac{d}{dx}(\ln g).$$

We use the second method. The relative rate of change of f/g is $(f/g)'/(f/g)$, so

$$\frac{(f/g)'}{f/g} = \frac{d}{dx}\ln\left(\frac{f}{g}\right) = \frac{d}{dx}(\ln f - \ln g) = \frac{d}{dx}(\ln f) - \frac{d}{dx}(\ln g) = \frac{f'}{f} - \frac{g'}{g}.$$

Thus, the relative rate of change of f/g is the difference between the relative rates of change of f and g.

47. The chain rule gives

$$(f^n)' = nf^{n-1}f'.$$

Dividing by f^n yields

$$\frac{(f^n)'}{f^n} = \frac{nf^{n-1}f'}{f^n} = n\frac{f'}{f}.$$

Solutions for Section 3.5

1. $\dfrac{dy}{dx} = 5\cos x$.

2. $\dfrac{dP}{dt} = -\sin t$.

3. $\dfrac{dy}{dt} = 2t - 5\sin t$.

4. $\dfrac{dy}{dt} = A\cos t$.

5. $R'(q) = 2q + 2\sin q$.

6. $\dfrac{dy}{dx} = 5\cos x - 5$.

7. $f'(x) = \cos(3x)\cdot 3 = 3\cos(3x)$.

8. $R' = 5\cos(5t)$.

9. $\dfrac{dW}{dt} = 4(-\sin(t^2))\cdot 2t = -8t\sin(t^2)$.

10. $\dfrac{dy}{dt} = 2(-\sin(5t))(5) = -10\sin(5t)$.

11. $\dfrac{dy}{dx} = 2x\cos(x^2)$.

12. $\dfrac{dy}{dt} = A(\cos(Bt))\cdot B = AB\cos(Bt)$.

13. $z' = -4\sin(4\theta)$.

14. $\dfrac{dy}{dt} = 6\cdot 2\cos(2t) + (-4\sin(4t)) = 12\cos(2t) - 4\sin(4t)$.

15. $f'(x) = (2x)(\cos x) + x^2(-\sin x) = 2x\cos x - x^2\sin x$.

16. $f'(x) = 2\cdot[\sin(3x)] + 2x[\cos(3x)]\cdot 3 = 2\sin(3x) + 6x\cos(3x)$

17. $f'(\theta) = 3\theta^2\cos\theta - \theta^3\sin\theta$.

18. Using the quotient and chain rules

$$\begin{aligned}\frac{dz}{dt} &= \frac{\frac{d}{dt}(e^{t^2}+t)\cdot\sin(2t)-(e^{t^2}+t)\frac{d}{dt}(\sin(2t))}{(\sin(2t))^2}\\ &= \frac{\left(e^{t^2}\cdot\frac{d}{dt}(t^2)+1\right)\sin(2t)-(e^{t^2}+t)\cos(2t)\frac{d}{dt}(2t)}{\sin^2(2t)}\\ &= \frac{(2te^{t^2}+1)\sin(2t)-(e^{t^2}+t)2\cos(2t)}{\sin^2(2t)}.\end{aligned}$$

19. Using the quotient rule, we get

$$\begin{aligned}\frac{d}{dt}\left(\frac{t^2}{\cos t}\right) &= \frac{2t\cos t - t^2(-\sin t)}{(\cos t)^2}\\ &= \frac{2t\cos t + t^2\sin t}{(\cos t)^2}.\end{aligned}$$

20. Using the quotient rule, we have

$$\frac{d}{d\theta}\left(\frac{\sin\theta}{\theta}\right) = \frac{(\cos\theta)(\theta)-(\sin\theta)(1)}{\theta^2} = \frac{\theta\cos\theta-\sin\theta}{\theta^2}.$$

This problem can also be done by writing $(\sin\theta)/\theta = (\sin\theta)\theta^{-1}$ and using the product rule.

21. At $x=\pi$, $y=\sin\pi=0$, and the slope $\left.\frac{dy}{dx}\right|_{x=\pi} = \left.\cos x\right|_{x=\pi} = -1$. Therefore the equation of the tangent line is $y=-(x-\pi)=-x+\pi$. See Figure 3.13.

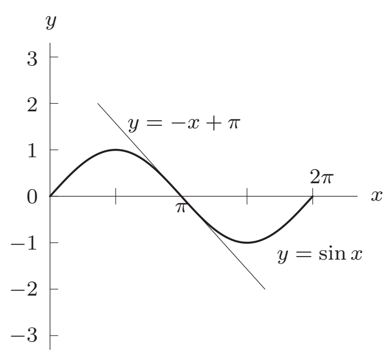


Figure 3.13

22. (a) See Figure 3.14. The number of bird species is highest in June and lowest in December. We see that $f'(1)$ is negative since the function is decreasing there, and $f'(10)$ is positive since the function is increasing there.

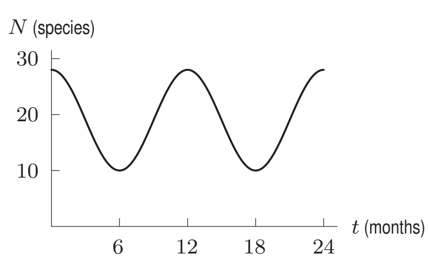


Figure 3.14: $N=f(t)=19+9\cos(\pi t/6)$

(b) We have

$$f'(t) = 0 + 9\left(-\sin\left(\frac{\pi}{6}t\right)\right)\frac{\pi}{6} = -4.712\sin\left(\frac{\pi}{6}t\right).$$

(c) We have $f(1) = 19 + 9\cos(\pi/6) = 26.8$ and $f'(1) = -4.712\sin(\pi/6) = -2.36$. In July, there are about 26.8 bird species, and the number of species is decreasing at a rate of 2.36 species per month.

We have $f(10) = 19 + 9\cos(10\pi/6) = 23.5$ and $f'(10) = -4.712\sin(10\pi/6) = 4.08$. In April, there are about 23.5 bird species, and the number of species is increasing at a rate of 4.08 species per month.

23. We begin by taking the derivative of $y = \sin(x^4)$ and evaluating at $x = 10$:

$$\frac{dy}{dx} = \cos(x^4)\cdot 4x^3.$$

Evaluating $\cos(10{,}000)$ on a calculator (in radians) we see $\cos(10{,}000) < 0$, so we know that $dy/dx < 0$, and therefore the function is decreasing.

Next, we take the second derivative and evaluate it at $x = 10$, giving $\sin(10{,}000) < 0$:

$$\frac{d^2y}{dx^2} = \underbrace{\cos(x^4)\cdot(12x^2)}_{\text{negative}} + \underbrace{4x^3\cdot(-\sin(x^4))(4x^3)}_{\substack{\text{positive, but much}\\ \text{larger in magnitude}}}.$$

From this we can see that $d^2y/dx^2 > 0$, thus the graph is concave up.

24. The tangent lines to $f(x) = \sin x$ have slope $\dfrac{d}{dx}(\sin x) = \cos x$. The tangent line at $x = 0$ has slope $f'(0) = \cos 0 = 1$ and goes through the point $(0,0)$. Consequently, its equation is $y = g(x) = x$. The approximate value of $\sin(\pi/6)$ given by this equation is $g(\pi/6) = \pi/6 \approx 0.524$.

Similarly, the tangent line at $x = \frac{\pi}{3}$ has slope

$$f'\left(\frac{\pi}{3}\right) = \cos\frac{\pi}{3} = \frac{1}{2}$$

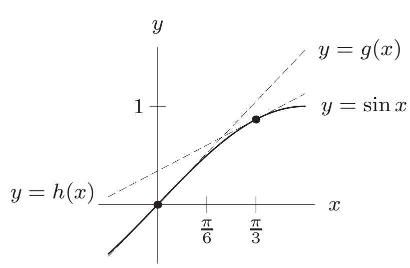
and goes through the point $(\pi/3, \sqrt{3}/2)$. Consequently, its equation is

$$y = h(x) = \frac{1}{2}x + \frac{3\sqrt{3}-\pi}{6}.$$

The approximate value of $\sin(\pi/6)$ given by this equation is then

$$h\left(\frac{\pi}{6}\right) = \frac{6\sqrt{3}-\pi}{12} \approx 0.604.$$

The actual value of $\sin(\pi/6)$ is $\frac{1}{2}$, so the approximation from 0 is better than that from $\pi/3$. This is because the slope of the function changes less between $x = 0$ and $x = \pi/6$ than it does between $x = \pi/6$ and $x = \pi/3$. This is illustrated by the following figure.



25. **(a)** Looking at the graph in Figure 3.15, we see that the maximum is \$2600 per month and the minimum is \$1400 per month. If $t = 0$ is January 1, then the sales are highest on April 1.

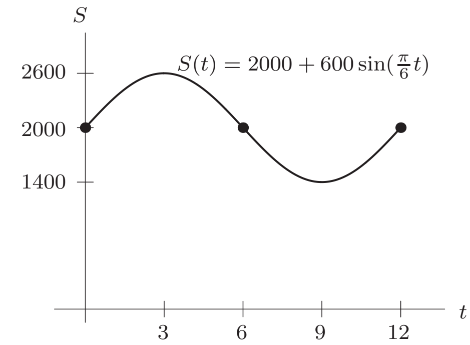


Figure 3.15

(b) $S(2)$ is the monthly sales on March 1,

$$\begin{aligned} S(2) &= 2000 + 600\sin(\frac{\pi}{3}) \\ &= 2000 + 600\sqrt{3}/2 \approx 2519.62 \quad \text{dollars/month} \end{aligned}$$

$S'(2)$ is the rate of change of monthly sales on March 1, and since

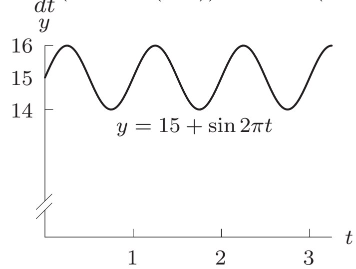
$$\begin{aligned} S'(t) &= 600[\cos(\frac{\pi}{6}t)](\frac{\pi}{6}) \\ &= 100\pi\cos(\frac{\pi}{6}t), \end{aligned}$$

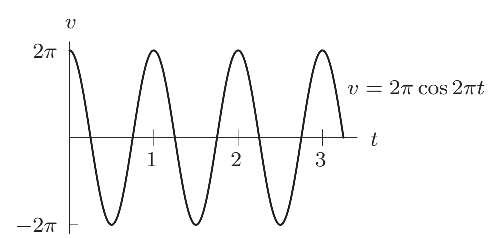
We have,

$$S'(2) = 100\pi\cos(\frac{\pi}{3}) = 50\pi \approx 157.08$$

26. **(a)** $v(t) = \dfrac{dy}{dt} = \dfrac{d}{dt}(15 + \sin(2\pi t)) = 2\pi\cos(2\pi t)$.

(b)





27.

$$\frac{dy}{dt} = -7.5(0.507)\sin(0.507t) = -3.80\sin(0.507t)$$

(a) When $t = 6$, we have $\dfrac{dy}{dt} = -3.80\sin(0.507 \cdot 6) = -0.38$ meters/hour. So the tide is falling at 0.38 meters/hour.

(b) When $t = 9$, we have $\dfrac{dy}{dt} = -3.80\sin(0.507 \cdot 9) = 3.76$ meters/hour. So the tide is rising at 3.76 meters/hour.

(c) When $t = 12$, we have $\dfrac{dy}{dt} = -3.80\sin(0.507 \cdot 12) = 0.75$ meters/hour. So the tide is rising at 0.75 meters/hour.

(d) When $t = 18$, we have $\dfrac{dy}{dt} = -3.80\sin(0.507 \cdot 18) = -1.12$ meters/hour. So the tide is falling at 1.12 meters/hour.

28. **(a)** One cycle is completed in $60/12 = 5$ seconds.

(b) We differentiate to get

$$A'(t) = 2\left[\sin\left(\frac{2\pi}{5}t\right)\right]\left(\frac{2\pi}{5}\right) = \frac{4\pi}{5}\sin\left(\frac{2\pi}{5}t\right).$$

Substitute $t = 1$ to get

$$A'(1) = \frac{4\pi}{5}\sin\left(\frac{2\pi}{5}(1)\right) = 2.390 \text{ hundred cubic centimeters/second}.$$

This tell us that, one second after the cycle begins, the patient is inhaling at a rate of approximately 2.39 hundred cubic centimeters/second; that is 239 cubic centimeters/second.

29. (a) We differentiate to get

$$H'(t) = -50\left[\sin\left(\frac{\pi}{15}t\right)\right]\left(\frac{\pi}{15}\right) = -\frac{10\pi}{3}\sin\left(\frac{\pi}{15}t\right).$$

The derivative gives the rate of change in the percent of the moon that is illuminated. The units are percent/day.

(b) We set $H'(t) = 0$ and solve for $0 \le t \le 30$.

$$\begin{aligned}
-\frac{10\pi}{3}\sin\left(\frac{\pi}{15}t\right) &= 0\\
\sin\left(\frac{\pi}{15}t\right) &= 0\\
\frac{\pi}{15}t &= \arcsin(0)\\
\frac{\pi}{15}t &= 0, \pi, 2\pi\\
t &= 0, 15, 30 \text{ days}.
\end{aligned}$$

Thus $H'(t) = 0$ at $t = 0, 15, 30$ days.

When $H'(t) = 0$, the percentage of the moon that is illuminated is not changing. Therefore, there is either a full moon, when the moon is 100% illuminated (the maximum), or a new moon, when the moon is 0% illuminated (the minimum).

(c) We can graph $H'(t) = -10\pi/3\sin(\pi t/15)$ to see that it is negative for $0 < t < 15$. A negative derivative tells us that the percentage of the moon illuminated is decreasing.

We see that $H'(t) = -10\pi/3\sin(\pi t/15)$ is positive for $15 < t < 30$. A positive derivative tells us that the percentage of the moon illuminated is increasing.

This makes sense because after the moon is full at $t = 0$, the percentage that is illuminated decreases until the new moon, approximately 15 days later. Then, the percentage of the moon that is illuminated increases until the next full moon, in approximately 15 more days.

30. (a) We substitute $t = 40$:

$$\begin{aligned}
D(t) &= 4\cos\left(\frac{2\pi}{365}(t-172)\right) + 12\\
D(40) &= 4\cos\left(\frac{2\pi}{365}(40-172)\right) + 12 = 9.4186 \text{ hours}.
\end{aligned}$$

This tells us that, on the 40^{th} day of the year, February 9, 2009, Paris had approximately 9.4 hours of daylight.

We differentiate to get

$$\begin{aligned}
D'(t) &= -4\left[\sin\left(\frac{2\pi}{365}(t-172)\right)\right]\left(\frac{2\pi}{365}\right)\\
&= -\frac{8\pi}{365}\sin\left(\frac{2\pi}{365}(t-172)\right).
\end{aligned}$$

Substitute $t = 40$ to get

$$D'(40) = -\frac{8\pi}{365}\sin\left(\frac{2\pi}{365}(40-172)\right) = 0.053 \text{ hours/day}.$$

This tell us that, on February 9th, the number of hours of daylight in Paris was increasing at a rate of about 0.053 hours/day. (This is approximately 3.2 minutes per day.)

(b) We substitute $t = 172$ into $D(t)$ to get

$$D(172) = 4\cos\left(\frac{2\pi}{365}(172-172)\right) + 12 = 4\cos(0) + 12 = 16 \text{ hours.}$$

This tells us that, on the 172^{nd} day of the year, June 21, 2009, Paris had approximately 16 hours of daylight.

Substitute $t = 172$ into $D'(t)$ to get

$$D'(172) = -\frac{8\pi}{365}\sin\left(\frac{2\pi}{365}(172-172)\right) = -\frac{8\pi}{365}\sin(0) = 0 \text{ hours/day.}$$

This tell us that, on June 21st, the rate of change of the number of hours of daylight in Paris was zero. June 21st was the summer solstice (the longest day of the year), so the maximum number of hours of daylight in Paris in 2009 was about 16 hours.

Solutions for Chapter 3 Review

1. $f'(t) = 24t^3$.

2. $f'(x) = 3x^2 - 6x + 5$.

3. $P'(t) = 2e^{2t}$.

4. $\dfrac{dW}{dr} = 3r^2 + 5$.

5. $\dfrac{dC}{dq} = 0.08e^{0.08q}$.

6. $\dfrac{dy}{dt} = 5(-0.2)e^{-0.2t} = -e^{-0.2t}$.

7. $\dfrac{dy}{dx} = e^{3x} + x\cdot 3e^{3x} = e^{3x}(1+3x)$.

8. $s'(t) = 2t(5t-1) + (t^2+4)\cdot 5 = 15t^2 - 2t + 20$.

9. $\dfrac{d}{dt}e^{(1+3t)^2} = e^{(1+3t)^2}\dfrac{d}{dt}(1+3t)^2 = e^{(1+3t)^2}\cdot 2(1+3t)\cdot 3 = 6(1+3t)e^{(1+3t)^2}$.

10. $f'(x) = 2x + \frac{3}{x}$.

11. $Q'(t) = 5 + 3.6e^{1.2t}$.

12. $g'(z) = 3(z^2+5)^2\cdot\dfrac{d}{dz}(z^2+5) = 3(z^2+5)^2(2z) = 6z(z^2+5)^2$.

13. $f'(x) = 6(3(5x-1)^2)\cdot\dfrac{d}{dx}(5x-1) = 18(5x-1)^2(5) = 90(5x-1)^2$.

14. $f'(z) = \frac{1}{z^2+1}(2z) = \frac{2z}{z^2+1}$.

15. We use the chain rule with $z = g(x) = 1 + e^x$ as the inside function and $f(z) = z^{10}$ as the outside function. Since $g'(x) = e^x$ and $f'(z) = 10z^9$ we have

$$h'(x) = \frac{d}{dx}\left((1+e^x)^{10}\right) = 10z^9e^x = 10e^x(1+e^x)^9.$$

16. $\dfrac{dq}{dp} = 100(-0.05)e^{-0.05p} = -5e^{-0.05p}$.

17. $\dfrac{dy}{dx} = 2x\ln x + x^2\cdot\dfrac{1}{x} = x(2\ln x + 1)$.

18. $s'(t) = 2t + \frac{2}{t}$.

19. $\dfrac{dP}{dt} = 8t + 7\cos t$.

20. $R'(t) = 5(\sin t)^4\cdot\dfrac{d}{dt}(\sin t) = 5(\sin t)^4(\cos t)$.

21. $h'(t) = \dfrac{1}{e^{-t} - t}\left(-e^{-t} - 1\right).$

22. $f'(x) = 2\cos(2x).$

23. $g'(x) = \dfrac{50xe^x - 25x^2e^x}{e^{2x}} = \dfrac{50x - 25x^2}{e^x}.$

24. $h'(t) = \dfrac{(1)(t-4) - (1)(t+4)}{(t-4)^2} = \dfrac{t-4-t-4}{(t-4)^2} = \dfrac{-8}{(t-4)^2}.$

25. $\dfrac{dy}{dx} = 2x\cos x + x^2(-\sin x) = 2x\cos x - x^2\sin x.$

26. We use the chain rule with $z = g(x) = 1 + e^x$ as the inside function and $f(z) = \ln z$ as the outside function. Since $g'(x) = e^x$ and $f'(z) = 1/z$ we have

$$h'(x) = \frac{d}{dx}\left(\ln(1+e^x)\right) = \frac{1}{z}e^x = \frac{e^x}{1+e^x}.$$

27. We use the chain rule with $z = g(w) = \ln w + 1$ as the inside function and $f(z) = e^z$ as the outside function. Since $g'(w) = 1/w$ and $f'(z) = e^z$ we have

$$h'(w) = \frac{d}{dw}\left(e^{\ln w + 1}\right) = \frac{1}{w}(e^z) = \frac{e^{\ln w+1}}{w} = \frac{e^{\ln w}\cdot e}{w} = \frac{w\cdot e}{w} = e.$$

Alternatively, note that $e^{\ln w+1} = e^{\ln w}e^1 = we$ so that its derivative with respect to w is just e.

28. We use the chain rule with $z = g(x) = x^3 + x$ as the inside function and $f(z) = \ln(z)$as the outside function. Since $g'(x) = 3x^2 + 1$ and $f'(z) = 1/z$ we

$$h'(x) = \frac{d}{dx}\left(\ln(x^3+x)\right) = \frac{1}{z}\cdot(3x^2+1) = \frac{3x^2+1}{x^3+x}.$$

29. This is a quotient where $u(x) = 1 + e^x$ and $v(x) = 1 - e^{-x}$ so that $q(x) = u(x)/v(x)$.

Using the quotient rule the derivative is

$$q'(x) = \frac{vu' - uv'}{v^2},$$

where $u' = e^x$ and $v' = e^{-x}$. Therefore

$$q'(x) = \frac{(1-e^{-x})e^x - e^{-x}(1+e^x)}{(1-e^{-x})^2} = \frac{e^x - 2 - e^{-x}}{(1-e^{-x})^2}.$$

30. This is a quotient where $u(x) = x$ and $v(x) = 1 + x$ so that $q(x) = u(x)/v(x)$. Using the quotient rule the derivative is

$$q'(x) = \frac{vu' - uv'}{v^2},$$

where $u' = 1$ and $v' = 1$. Therefore

$$q'(x) = \frac{(1+x)\cdot 1 - x\cdot 1}{(1+x)^2} = \frac{1}{(1+x)^2}.$$

31. Using the product rule, the derivative is

$$f'(x) = x\frac{d}{dx}\left(e^x\right) + \left(\frac{d}{dx}(x)\right)e^x = xe^x + e^x.$$

32. We use the chain rule with $z = g(x) = e^x$ as the inside function and $f(z) = \sin z$ as the outside function. Since $g'(x) = e^x$ and $f'(z) = \cos z$, we have

$$h'(x) = \frac{d}{dx}\left(\sin(e^x)\right) = (\cos z)e^x = e^x\cos(e^x).$$

33. We use the chain rule with $z = g(x) = x^3$ as the inside function and $f(z) = \cos z$ as the outside function. Since $g'(x) = 3x^2$ and $f'(z) = -\sin z$, we have

$$h'(x) = \frac{d}{dx}\left(\cos(x^3)\right) = -\sin z \cdot (3x^2) = -3x^2 \sin(x^3).$$

34. $\dfrac{dz}{dt} = \dfrac{3(5t+2)-(3t+1)5}{(5t+2)^2} = \dfrac{15t+6-15t-5}{(5t+2)^2} = \dfrac{1}{(5t+2)^2}.$

35. $z' = \dfrac{(2t+5)(t+3)-(t^2+5t+2)}{(t+3)^2} = \dfrac{t^2+6t+13}{(t+3)^2}.$

36. $h'(p) = \dfrac{2p(3+2p^2)-4p(1+p^2)}{(3+2p^2)^2} = \dfrac{6p+4p^3-4p-4p^3}{(3+2p^2)^2} = \dfrac{2p}{(3+2p^2)^2}.$

37. $\dfrac{dy}{dx} = \dfrac{1}{3}(\ln 3)3^x - \dfrac{33}{2}(x^{-\frac{3}{2}}).$

38. Using the chain rule twice:

$$f'(t) = \cos\sqrt{e^t+1}\frac{d}{dt}\sqrt{e^t+1} = \cos\sqrt{e^t+1}\frac{1}{2\sqrt{e^t+1}}\cdot\frac{d}{dt}(e^t+1) = \cos\sqrt{e^t+1}\frac{1}{2\sqrt{e^t+1}}e^t = e^t\frac{\cos\sqrt{e^t+1}}{2\sqrt{e^t+1}}.$$

39. Using the chain rule twice:

$$g'(y) = e^{2e^{(y^3)}}\frac{d}{dy}\left(2e^{(y^3)}\right) = 2e^{2e^{(y^3)}}e^{(y^3)}\frac{d}{dy}(y^3) = 6y^2e^{(y^3)}e^{2e^{(y^3)}}.$$

40. Using the quotient rule gives

$$g'(t) = \frac{\left(\frac{k}{kt}+1\right)(\ln(kt)-t)-(\ln(kt)+t)\left(\frac{k}{kt}-1\right)}{(\ln(kt)-t)^2}$$

$$g'(t) = \frac{\left(\frac{1}{t}+1\right)(\ln(kt)-t)-(\ln(kt)+t)\left(\frac{1}{t}-1\right)}{(\ln(kt)-t)^2}$$

$$g'(t) = \frac{\ln(kt)/t-1+\ln(kt)-t-\ln(kt)/t-1+\ln(kt)+t}{(\ln(kt)-t)^2}$$

$$g'(t) = \frac{2\ln(kt)-2}{(\ln(kt)-t)^2}.$$

41. Figure 3.16 shows the graph of $f(x) = x^2+1$.

We have $f'(x) = \frac{d}{dx}(x^2+1) = 2x$, thus, $f'(0) = 2(0) = 0$. We check this by seeing in that Figure 3.16 the tangent line at $x = 0$ has slope 0.

We have $f'(1) = 2(1) = 2$, $f'(2) = 2(2) = 4$. and $f'(-1) = 2(-1) = -2$. Thus, the slope is positive at $x = 2$ and $x = 1$, and negative at $x = -1$.

Moreover, it is greater at $x = 2$ than at $x = 1$. This agrees with the graph in Figure 3.16.

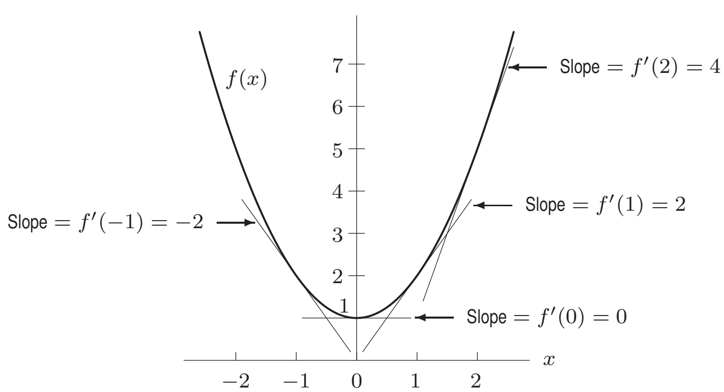


Figure 3.16: Using slopes to check values for derivatives

42. $f'(x) = 2x + 3$, so $f'(0) = 3$, $f'(3) = 9$, and $f'(-2) = -1$.

43. Since $f(1) = 2(1^3) - 5(1^2) + 3(1) - 5 = -5$, the point $(1, -5)$ is on the line. We use the derivative to find the slope. Differentiating gives

$$f'(x) = 6x^2 - 10x + 3,$$

and so the slope at $x = 1$ is

$$f'(1) = 6(1^2) - 10(1) + 3 = -1.$$

The equation of the tangent line is

$$\begin{aligned} y - (-5) &= -1(x - 1) \\ y + 5 &= -x + 1 \\ y &= -4 - x. \end{aligned}$$

44. We have $f'(t) = 3$, so the relative rate of change when $t = 5$ is

$$\frac{f'(5)}{f(5)} = \frac{3}{3(5) + 2} = \frac{3}{17} = 0.176 = 17.6\% \text{ per year.}$$

Alternately, we could use logs to find the relative rate of change.

45. We have $f'(t) = 2e^{0.3t}(0.3)$, so the relative rate of change is

$$\frac{f'(t)}{f(t)} = \frac{2e^{0.3t}(0.3)}{2e^{0.3t}} = 0.3 = 30\% \text{ per year.}$$

In particular, since the answer does not depend on the value of t, the relative rate of change at any value of t, including $t = 7$, is 30% per year. Alternately, we could use logs to find the relative rate of change.

46. We have $f'(t) = 6t^2$, so the relative rate of change at $t = 4$ is

$$\frac{f'(4)}{f(4)} = \frac{6(4^2)}{2(4^3) + 10} = \frac{96}{138} = 0.696 = 69.6\% \text{ per year.}$$

Alternately, we could use logs to find the relative rate of change.

47. We have $f(2) = \ln(2^2 + 1) = \ln(5) = 1.609$. We have $f'(t) = (2t)/(t^2 + 1)$, so $f'(2) = 4/5 = 0.8$. The relative rate of change at $t = 2$ is

$$\frac{f'(2)}{f(2)} = \frac{0.8}{1.609} = 0.497 = 49.7\% \text{ per year.}$$

Alternately, we could use logs to find the relative rate of change.

48. Since $P = 1 \cdot (1.05)^t$, $\frac{dP}{dt} = \ln(1.05)1.05^t$. When $t = 10$,

$$\frac{dP}{dt} = (\ln 1.05)(1.05)^{10} \approx \$0.07947/\text{year} \approx 7.95¢/\text{year}.$$

49. The rate of growth, in billions of people per year, was

$$\frac{dP}{dt} = 6.8(0.012)e^{0.012t}.$$

On January 1, 2009, we have $t = 0$, so

$$\frac{dP}{dt} = 6.8(0.012)e^0 = 0.0816 \text{ billion/year } = 81.6 \text{ million people/year}.$$

50. Since $f'(x) = 2x - 4$, we have $f'(x) = 0$ when $x = 2$.

51. **(a)** Figure 3.17 shows the height of the ball at time t.

(b) Velocity $v(t) = \dfrac{dh}{dt} = \dfrac{d}{dt}(32t - 16t^2) = 32 - 32t$.

(c) We substitute 1 into $v(t) = 32 - 32t$ to get $v(1) = 32 - 32 \cdot 1 = 0$ ft/sec.

Since the velocity at this time is 0 ft/sec, the ball is not going up or down. When we look at the graph of $h(t)$ from part (a), we see that the football has reached its maximum height when $t = 1$.

Since $h(t) = 32t - 16t^2$, when $t = 1$, the height of the ball is $h(1) = 32 \cdot 1 - 16 \cdot 1^2 = 16$ feet.

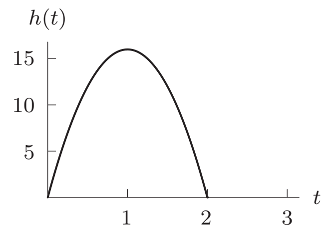


Figure 3.17

52.

$$\begin{aligned} y' &= 3x^2 - 18x - 16 \\ 5 &= 3x^2 - 18x - 16 \\ 0 &= 3x^2 - 18x - 21 \\ 0 &= x^2 - 6x - 7 \\ 0 &= (x+1)(x-7) \\ x &= -1 \text{ or } x = 7. \end{aligned}$$

When $x = -1$, $y = 7$; when $x = 7$, $y = -209$.
Thus, the two points are $(-1, 7)$ and $(7, -209)$.

53. **(a)** $f(x) = 1 - e^x$ crosses the x-axis where $0 = 1 - e^x$, which happens when $e^x = 1$, so $x = 0$. Since $f'(x) = -e^x$, $f'(0) = -e^0 = -1$.

(b) $y = -x$

54. To find the equation of the line tangent to the graph of $P(t) = t \ln t$ at $t = 2$ we must find the point $(2, P(2))$ as well as the slope of the tangent line at $t = 2$. $P(2) = 2(\ln 2) \approx 1.386$. Thus we have the point $(2, 1.386)$. To find the slope, we must first find $P'(t)$:

$$P'(t) = t\frac{1}{t} + \ln t(1) = 1 + \ln t.$$

At $t = 2$ we have

$$P'(2) = 1 + \ln 2 \approx 1.693$$

Since we now have the slope of the line and a point, we can solve for the equation of the line:

$$\begin{aligned} Q(t) - 1.386 &= 1.693(t - 2) \\ Q(t) - 1.386 &= 1.693t - 3.386 \\ Q(t) &= 1.693t - 2. \end{aligned}$$

The equation of the tangent line is $Q(t) = 1.693t - 2$. We see our results displayed graphically in Figure 3.18.

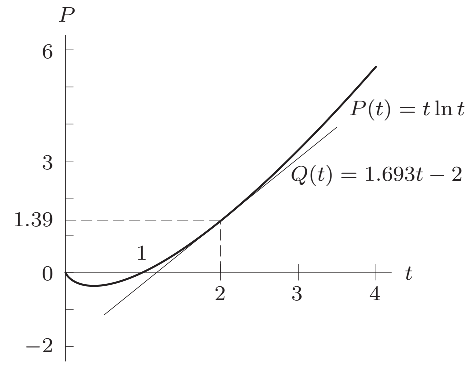


Figure 3.18

55. We can find the rate the balance changes by differentiating B with respect to time: $B'(t) = 5000e^{0.08t} \cdot 0.08 = 400e^{0.08t}$. Calculating B' at time $t = 5$, we have $B'(5) = \$596.73$/yr. After 5 years, the account generates about \$597 interest in the next year.

56. (a) When the coffee was first left on the counter, $t = 0$. Thus,

$$C(0) = 74 + 103e^0 = 74 + 103 \cdot 1 = 177.$$

The temperature was $177°$F.

(b) Since $e^{-0.033t} = \dfrac{1}{e^{0.033t}}$, as t gets larger, $\dfrac{1}{e^{0.033t}}$ gets smaller and tends to zero. Thus, $103e^{-0.033t}$ gets very small, so the temperature tends to $74 + 0 = 74°$F. This is room temperature.

(c) To find $C(5)$, we substitute $t = 5$.

$$C(5) = 74 + 103e^{-0.033 \cdot 5} = 161.333.$$

This tells us that, after the coffee sits on the counter for five minutes, its temperature is approximately $161°$F

To find $C'(5)$, we differentiate $C(t)$ and substitute.

$$C'(t) = 103e^{-0.033t}(-0.033) = -3.399e^{-0.033t}.$$
$$C'(5) = -3.399e^{-0.033 \cdot 5} = -2.882.$$

This tell us that, after the coffee has sat on the counter for 5 minutes, it is cooling at a rate of $2.882°$F/minute.

(d) The magnitude of $C'(t)$ is the rate at which the coffee is cooling at time t. We expect the magnitude of $C'(50)$ to be less than the magnitude of $C'(5)$ because, when the coffee is first put on the counter (at $t = 5$), it cools fast. When the coffee has been on the counter for some time (at $t = 50$), it is cooling more slowly.

57. (a) The rate of change of the period is given by

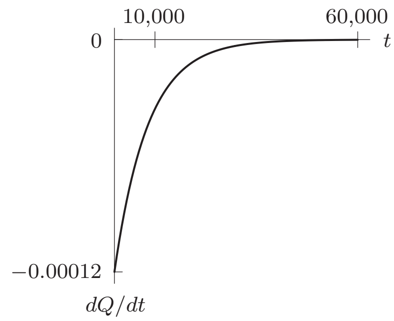
$$\frac{dT}{dl} = \frac{2\pi}{\sqrt{9.8}} \frac{d}{dl}(\sqrt{l}) = \frac{2\pi}{\sqrt{9.8}} \cdot \frac{1}{2} l^{-1/2} = \frac{\pi}{\sqrt{9.8}} \cdot \frac{1}{\sqrt{l}} = \frac{\pi}{\sqrt{9.8l}}.$$

(b) The rate decreases since $\sqrt{l}$ is in the denominator.

58. (a) Differentiating using the chain rule gives

$$\frac{dQ}{dt} = \frac{d}{dt} e^{-0.000121t} = -0.000121e^{-0.000121t}.$$

(b) The following graph shows the rate, dQ/dt, as a function of time.



59. **(a)**

$$\frac{dH}{dt} = \frac{d}{dt}(40 + 30e^{-2t}) = 30(-2)e^{-2t} = -60e^{-2t}.$$

(b) Since e^{-2t} is always positive, $\frac{dH}{dt} < 0$; this makes sense because the temperature of the soda is decreasing.

(c) The magnitude of $\frac{dH}{dt}$ is

$$\left|\frac{dH}{dt}\right| = \left|-60e^{-2t}\right| = 60e^{-2t} \le 60 = \left|\frac{dH}{dt}_{t=0}\right|,$$

since $e^{-2t} \le 1$ for all $t \ge 0$ and $e^0 = 1$. This is just saying that at the moment that the can of soda is put in the refrigerator (at $t = 0$), the temperature difference between the soda and the inside of the refrigerator is the greatest, so the temperature of the soda is dropping the quickest.

60. We are interested in when the derivative $\frac{d(a^x)}{dx}$ is positive and when it is negative. The quantity a^x is always positive. However $\ln a > 0$ for $a > 1$ and $\ln a < 0$ for $0 < a < 1$. Thus the function a^x is increasing for $a > 1$ and decreasing for $a < 1$.

61. **(a)** Substituting $t = 4$ gives $V(4) = 25(0.85)^4 = 25(0.522) = 13.050$. Thus the value of the car after 4 years is \$13,050.

(b) We have a function of the form $f(t) = Ca^t$. We know that such functions have a derivative of the form $(C \ln a) \cdot a^t$. Thus, $V'(t) = 25(0.85)^t \cdot \ln 0.85 = -4.063(0.85)^t$. The units are the change in value (in thousands of dollars) with respect to time (in years), or thousands of dollars/year.

(c) Substituting $t = 4$ gives $V'(4) = -4.063(0.85)^4 = -4.063(0.522) = -2.121$. This means that at the end of the fourth year, the value of the car is decreasing by \$2121 per year.

(d) The function $V(t)$ is positive and decreasing, so that the value of the automobile is positive and decreasing. The function $V'(t)$ is negative, and its magnitude is decreasing, meaning the value of the automobile is always dropping, but the yearly loss of value decreases as time goes on. The graphs of $V(t)$ and $V'(t)$ confirm that the value of the car decreases with time. What they do not take into account are the *costs* associated with owning the vehicle. At some time, t, it is likely that the yearly costs of owning the vehicle will outweigh its value. At that time, it may no longer be worthwhile to keep the car.

62. **(a)** To find the temperature of the yam when it was placed in the oven, we need to evaluate the function at $t = 0$. In this case, the temperature of the yam to begin with equals $350(1 - 0.7e^0) = 350(0.3) = 105°$.

(b) By looking at the function we see that the temperature which the yam is approaching is $350°$. That is, if the yam were left in the oven for a long period of time (i.e. as $t \to \infty$) the temperature would move closer and closer to $350°$ (because $e^{-0.008t}$ would approach zero, and thus $1 - 0.7e^{-0.008t}$ would approach 1). Thus, the temperature of the oven is $350°$.

(c) The yam's temperature will reach $175°$ when $Y(t) = 175$. Thus, we must solve for t:

$$\begin{aligned}
Y(t) &= 175 \\
175 &= 350(1 - 0.7e^{-0.008t}) \\
\frac{175}{350} &= 1 - 0.7e^{-0.008t} \\
0.7e^{-0.008t} &= 0.5 \\
e^{-0.008t} &= 5/7 \\
\ln e^{-0.008t} &= \ln 5/7
\end{aligned}$$

$$
\begin{aligned}
-0.008t &= \ln 5/7 \\
t &= \frac{\ln 5/7}{-0.008} \approx 42 \text{ minutes.}
\end{aligned}
$$

Thus the yam's temperature will be 175° approximately 42 minutes after it is put into the oven.

(d) The rate at which the temperature is increasing is given by the derivative of the function.

$$Y(t) = 350(1 - 0.7e^{-0.008t}) = 350 - 245e^{-0.008t}.$$

Therefore,

$$Y'(t) = 0 - 245(-0.008e^{-0.008t}) = 1.96e^{-0.008t}.$$

At $t = 20$, the rate of change of the temperature of the yam is given by $Y'(20)$:

$$Y'(20) = 1.96e^{-0.008(20)} = 1.96e^{-.16} = 1.96(0.8521) \approx 1.67 \text{ degrees/minute.}$$

Thus, at $t = 20$ the yam's temperature is increasing by about 1.67 degrees each minute.

63. (a) We substitute $k = 1.65864 \cdot 10^{-19}$ and take the square root of both sides to give us $P(d) = 4.07264 \cdot 10^{-10} d^{3/2}$. Since P must positive, we do not choose $P = -4.07264 \cdot 10^{-10} d^{3/2}$.

(b) We substitute 36 million for d.

$$P(36{,}000{,}000) = 4.07264 \cdot 10^{-10}(36{,}000{,}000)^{3/2} = 87.969 \text{ days}$$

Thus, it takes Mercury approximately 88 days to orbit the sun.

(c) We differentiate to get

$$
\begin{aligned}
P'(d) &= 4.07264 \cdot 10^{-10} d^{3/2} \\
&= \frac{3}{2} \cdot 4.07264 \cdot 10^{-10} d^{1/2} \\
&= 6.10896 \cdot 10^{-10} d^{1/2} \text{ days per mile}
\end{aligned}
$$

Since $d^{1/2}$ is positive, $P'(d)$ is always positive. This tells us that as the planet gets further from the sun, the time it takes for the planet to orbit the sun increases.

This makes sense, because as a planet's distance from the sun increases, the length of its orbit also increases. Thus, it is not surprising that the length of time it takes to complete this orbit also increases.

64. All of the functions go through the origin. They will look the same if they have the same tangent line, or equivalently, the same slope at $x = 0$. Therefore for each function we find the derivative and evaluate it at $x = 0$:

$$
\begin{array}{lll}
\text{For } y = x, & y' = 1, & \text{so } y'(0) = 1. \\
\text{For } y = \sqrt{x}, & y' = \frac{1}{2\sqrt{x}}, & \text{so } y'(0) \text{ is undefined.} \\
\text{For } y = x^2, & y' = 2x, & \text{so } y'(0) = 0. \\
\text{For } y = x^3 + \frac{1}{2}x^2, & y' = 3x^2 + x, & \text{so } y'(0) = 0. \\
\text{For } y = x^3, & y' = 3x^2, & \text{so } y'(0) = 0. \\
\text{For } y = \ln(x+1), & y' = \frac{1}{x+1}, & \text{so } y'(0) = 1. \\
\text{For } y = \frac{1}{2}\ln(x^2+1), & y' = \frac{x}{x^2+1}, & \text{so } y'(0) = 0. \\
\text{For } y = \sqrt{2x - x^2}, & y' = \frac{1-x}{\sqrt{2x-x^2}}, & \text{so } y'(0) \text{ is undefined.}
\end{array}
$$

So near the origin, functions with $y'(0) = 1$ will all be indistinguishable resembling the line $y = x$. These functions are:

$$y = x \qquad \text{and} \qquad y = \ln(x+1).$$

Functions with $y'(0) = 0$ will be indistinguishable near the origin and resemble the line $y = 0$ (a horizontal line). These functions are:

$$y = x^2, \qquad y = x^3 + \frac{1}{2}x^2, \qquad y = x^3, \qquad \text{and} \qquad y = \frac{1}{2}\ln(x^2+1).$$

Functions that have undefined derivatives at $x = 0$ look like vertical lines at the origin. These functions are

$$y = \sqrt{x} \qquad \text{and} \qquad y = \sqrt{2x - x^2}.$$

65. Since $f(x) = ax^n$, $f'(x) = anx^{n-1}$. We know that $f'(2) = (an)2^{n-1} = 3$, and $f'(4) = (an)4^{n-1} = 24$. Therefore,

$$\frac{f'(4)}{f'(2)} = \frac{24}{3}$$
$$\frac{(an)4^{n-1}}{(an)2^{n-1}} = \left(\frac{4}{2}\right)^{n-1} = 8$$
$$2^{n-1} = 8, \text{ and thus } n = 4.$$

Substituting $n = 4$ into the expression for $f'(2)$, we get $3 = a(4)(8)$, or $a = 3/32$.

66. **(a)** $H'(2) = r'(2) + s'(2) = -1 + 3 = 2$.
(b) $H'(2) = 5s'(2) = 5(3) = 15$.
(c) $H'(2) = r'(2)s(2) + r(2)s'(2) = -1 \cdot 1 + 4 \cdot 3 = 11$.
(d) $H'(2) = \dfrac{r'(2)}{2\sqrt{r(2)}} = \dfrac{-1}{2\sqrt{4}} = -\dfrac{1}{4}$.

67. **(a)** $H(x) = F(G(x))$
$H(4) = F(G(4)) = F(2) = 1$
(b) $H(x) = F(G(x))$
$H'(x) = F'(G(x)) \cdot G'(x)$
$H'(4) = F'(G(4)) \cdot G'(4) = F'(2) \cdot 6 = 5 \cdot 6 = 30$
(c) $H(x) = G(F(x))$
$H(4) = G(F(4)) = G(3) = 4$
(d) $H(x) = G(F(x))$
$H'(x) = G'(F(x)) \cdot F'(x)$
$H'(4) = G'(F(4)) \cdot F'(4) = G'(3) \cdot 7 = 8 \cdot 7 = 56$
(e) $H(x) = \frac{F(x)}{G(x)}$
$H'(x) = \frac{G(x)\cdot F'(x) - F(x)\cdot G'(x)}{[G(x)]^2}$
$H'(4) = \frac{G(4)\cdot F'(4) - F(4)\cdot G'(4)}{[G(4)]^2} = \frac{2\cdot 7 - 3\cdot 6}{2^2} = \frac{14-18}{4} = \frac{-4}{4} = -1$

68. **(a)** The rate of change of temperature change is

$$\frac{dT}{dD} = \frac{d}{dD}\left(\frac{C}{2}D^3 - \frac{D^4}{3}\right) = \frac{3C}{2}D^2 - \frac{4D^3}{3}.$$

(b) We want to know for what values of D the value of dT/dD is positive. This occurs when

$$\frac{dT}{dD} = \left(\frac{3C}{2} - \frac{4D}{3}\right)D^2 > 0$$

Since $D^2 \geq 0$ for all D, we have

$$\frac{3C}{2} - \frac{4}{3}D > 0 \quad \text{so} \quad \frac{3C}{2} > \frac{4}{3}D \quad \text{so} \quad D < \frac{9C}{8}.$$

So the rate of change of temperature change is positive for doses less than $9C/8$.

69. Since we're given that the instantaneous rate of change of T at $t = 30$ is 2, we want to choose a and b so that the derivative of T agrees with this value. Differentiating, $T'(t) = ab \cdot e^{-bt}$. Then we have

$$2 = T'(30) = abe^{-30b} \text{ or } e^{-30b} = \frac{2}{ab}$$

We also know that at $t = 30$, $T = 120$, so

$$120 = T(30) = 200 - ae^{-30b} \text{ or } e^{-30b} = \frac{80}{a}$$

Thus $\dfrac{80}{a} = e^{-30b} = \dfrac{2}{ab}$, so $b = \frac{1}{40} = 0.025$ and $a = 169.36$.

70. Estimates may vary. From the graphs, we estimate $f(1) \approx -0.4$, $f'(1) \approx 0.5$, $g(1) \approx 2$, and $g'(1) \approx 1$. By the product rule,

$$h'(1) = f'(1) \cdot g(1) + f(1) \cdot g'(1) \approx (0.5)2 + (-0.4)1 = 0.6.$$

71. Estimates may vary. From the graphs, we estimate $f(1) \approx -0.4$, $f'(1) \approx 0.5$, $g(1) \approx 2$, and $g'(1) \approx 1$. By the quotient rule, to one decimal place

$$k'(1) = \frac{f'(1) \cdot g(1) - f(1) \cdot g'(1)}{(g(1))^2} \approx \frac{(0.5)2 - (-0.4)1}{2^2} = 0.4.$$

72. Estimates may vary. From the graphs, we estimate $f(2) \approx 0.3$, $f'(2) \approx 1.1$, $g(2) \approx 1.6$, and $g'(2) \approx -0.5$. By the product rule, to one decimal place

$$h'(2) = f'(2) \cdot g(2) + f(2) \cdot g'(2) \approx 1.1(1.6) + 0.3(-0.5) = 1.6.$$

73. Estimates may vary. From the graphs, we estimate $f(2) \approx 0.3$, $f'(2) \approx 1.1$, $g(2) \approx 1.6$, and $g'(2) \approx -0.5$. By the quotient rule, to one decimal place

$$k'(2) = \frac{f'(2) \cdot g(2) - f(2) \cdot g'(2)}{(g(2))^2} \approx \frac{1.1(1.6) - 0.3(-0.5)}{(1.6)^2} = 0.7.$$

74. Estimates may vary. From the graphs, we estimate $f(1) \approx -0.4$, $f'(1) \approx 0.5$, $g(1) \approx 2$, and $g'(1) \approx 1$. By the quotient rule, to one decimal place

$$l'(1) = \frac{g'(1) \cdot f(1) - g(1) \cdot f'(1)}{(f(1))^2} \approx \frac{1(-0.4) - 2(0.5)}{(-0.4)^2} = -8.8.$$

75. Estimates may vary. From the graphs, we estimate $f(2) \approx 0.3$, $f'(2) \approx 1.1$, $g(2) \approx 1.6$, and $g'(2) \approx -0.5$. By the quotient rule, to one decimal place

$$l'(2) = \frac{g'(2) \cdot f(2) - g(2) \cdot f'(2)}{(f(2))^2} \approx \frac{(-0.5)0.3 - 1.6(1.1)}{(0.3)^2} = -21.2.$$

76. Decreasing means $f'(x) < 0$:

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3),$$

so $f'(x) < 0$ when $x < 3$ and $x \neq 0$. Concave up means $f''(x) > 0$:

$$f''(x) = 12x^2 - 24x = 12x(x-2)$$

so $f''(x) > 0$ when

$$\begin{aligned} 12x(x-2) &> 0 \\ x < 0 \quad \text{or} \quad x &> 2. \end{aligned}$$

So, both conditions hold for $x < 0$ or $2 < x < 3$.

77. **(a)** We have $p(x) = x^2 - x$. We see that $p'(x) = 2x - 1 < 0$ when $x < \frac{1}{2}$. So p is decreasing when $x < \frac{1}{2}$.

(b) We have $p(x) = x^{1/2} - x$, so

$$\begin{aligned} p'(x) = \frac{1}{2}x^{-1/2} - 1 &< 0 \\ \frac{1}{2}x^{-1/2} &< 1 \\ x^{-1/2} &< 2 \\ x^{1/2} &> \frac{1}{2} \\ x &> \frac{1}{4}. \end{aligned}$$

Thus $p(x)$ is decreasing when $x > \frac{1}{4}$.

(c) We have $p(x) = x^{-1} - x$, so

$$\begin{aligned} p'(x) = -1x^{-2} - 1 &< 0 \\ -x^{-2} &< 1 \\ x^{-2} &> -1, \end{aligned}$$

which is always true where x^{-2} is defined since $x^{-2} = 1/x^2$ is always positive. Thus $p(x)$ is decreasing for $x < 0$ and for $x > 0$.

78. The first and second derivatives of e^x are e^x. Thus, the graph of $y = e^x$ is concave up. The tangent line at $x = 0$ has slope $e^0 = 1$ and equation $y = x + 1$. A graph that is always concave up is always above any of its tangent lines. Thus $e^x \geq x + 1$ for all x, as shown in Figure 3.19.

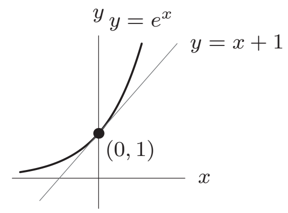


Figure 3.19

79. (a) Differentiating gives

$$\frac{dy}{dt} = -\frac{4.4\pi}{6} \sin\left(\frac{\pi}{6}t\right).$$

The derivative represents the rate of change of the depth of the water in feet/hour.

(b) The derivative, dy/dt, is zero where the tangent line to the curve y is horizontal. This occurs when $dy/dt = 0$, so $\sin(\pi t/6) = 0$, that is when $t = 0$, 6, 12, 18 and 24 (midnight, 6 am, noon, 6 pm, and midnight). When $dy/dt = 0$, the depth of the water is no longer changing. Therefore, it has either just finished rising or just finished falling, and we know that the harbor's level is at a maximum or a minimum.

80. The slopes of the tangent lines to $y = x^2 - 2x + 4$ are given by $y' = 2x - 2$. A line through the origin has equation $y = mx$. So, at the tangent point, $x^2 - 2x + 4 = mx$ where $m = y' = 2x - 2$.

$$\begin{aligned} x^2 - 2x + 4 &= (2x-2)x \\ x^2 - 2x + 4 &= 2x^2 - 2x \\ -x^2 + 4 &= 0 \\ -(x+2)(x-2) &= 0 \\ x &= 2, -2. \end{aligned}$$

Thus, the points of tangency are $(2, 4)$ and $(-2, 12)$. The lines through these points and the origin are $y = 2x$ and $y = -6x$, respectively. Graphically, this can be seen in Figure 3.20.

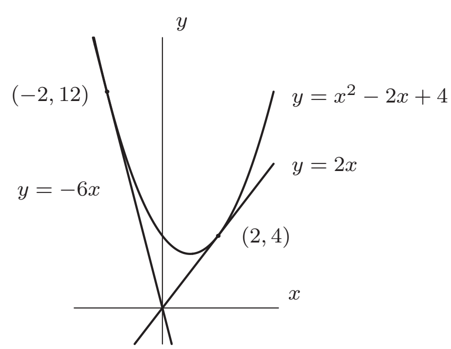


Figure 3.20

81. (a) If the museum sells the painting and invests the proceeds $P(t)$ at time t, then t years have elapsed since 2000, and the time span up to 2020 is $20 - t$. This is how long the proceeds $P(t)$ are earning interest in the bank. Each year the money is in the bank it earns 5% interest, which means the amount in the bank is multiplied by a factor of 1.05. So, at the end of $(20 - t)$ years, the balance is given by

$$B(t) = P(t)(1 + 0.05)^{20-t} = P(t)(1.05)^{20-t}.$$

(b)

$$B(t) = P(t)(1.05)^{20}(1.05)^{-t} = (1.05)^{20}\frac{P(t)}{(1.05)^t}.$$

(c) By the quotient rule,

$$B'(t) = (1.05)^{20}\left[\frac{P'(t)(1.05)^t - P(t)(1.05)^t \ln 1.05}{(1.05)^{2t}}\right].$$

So,

$$\begin{aligned} B'(10) &= (1.05)^{20}\left[\frac{5000(1.05)^{10} - 150{,}000(1.05)^{10}\ln 1.05}{(1.05)^{20}}\right] \\ &= (1.05)^{10}(5000 - 150{,}000 \ln 1.05) \\ &\approx -3776.63. \end{aligned}$$

82. **(a)** With μ and σ constant, differentiating $m(t) = e^{\mu t + \sigma^2 t^2/2}$ with respect to t gives

$$m'(t) = e^{ut+\sigma^2 t^2/2} \cdot \left(\mu + \frac{2\sigma^2 t}{2}\right) = e^{\mu t + \sigma^2 t^2/2}(\mu + \sigma^2 t).$$

Thus,

$$\text{Mean } = m'(0) = e^0(\mu + 0) = \mu.$$

(b) Differentiating $m'(t) = e^{\mu t+\sigma^2 t^2/2}(\mu + \sigma^2 t)$, we have

$$m''(t) = e^{\mu t+\sigma^2 t^2/2}(\mu + \sigma^2 t)^2 + e^{\mu t + \sigma^2 t^2/2}\sigma^2.$$

Thus

$$\text{Variance } = m''(0) - (m'(0))^2 = e^0\mu^2 + e^0\sigma^2 - \mu^2 = \sigma^2.$$

83. If $a = e$, the only solution is (0,1). See Figure 3.21. If $1 < a < e$, there are two solutions as illustrated in Figure 3.22. If $a > e$, there are also two solutions. See See Figure 3.23.

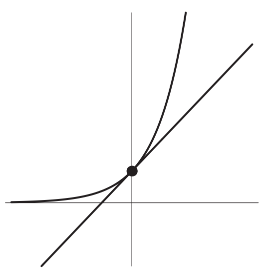


Figure 3.21

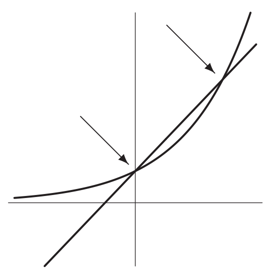


Figure 3.22

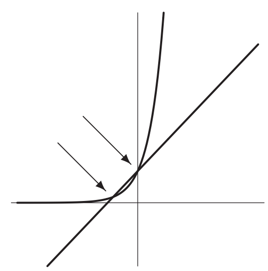


Figure 3.23

One way to justify these results is to compare the slopes of the lines. For example, e^x will have slope greater than 1 for all $x > 0$ and less than 1 for all $x < 0$, so it cannot meet the line $1 + x$ at any other points. Similar arguments can be made for the other cases.

84. We have

$$\begin{aligned} h(t_0) &= Ae^{kt_0} = f(t_0) = P_0 \\ h(t_1) &= Ae^{kt_1} = f(t_1) = P_1. \end{aligned}$$

Division gives

$$\frac{Ae^{kt_1}}{Ae^{kt_0}} = e^{k(t_1 - t_0)} = \frac{P_1}{P_0}.$$

Taking natural logarithms of both sides of the equation yields

$$\begin{aligned} k(t_1 - t_0) &= \ln(P_1/P_0) = \ln P_1 - \ln P_0 \\ k &= \frac{\ln P_1 - \ln P_0}{t_1 - t_0}. \end{aligned}$$

85. **(a)** On the interval $0 < M < 70$, we have

$$\text{Slope } = \frac{\Delta G}{\Delta M} = \frac{2.8}{70} = 0.04 \text{ gallons per mile.}$$

On the interval $70 < M < 100$, we have

$$\text{Slope } = \frac{\Delta G}{\Delta M} = \frac{4.6 - 2.8}{100 - 70} = \frac{1.8}{30} = 0.06 \text{ gallons per mile.}$$

(b) Gas consumption, in miles per gallon, is the reciprocal of the slope, in gallons per mile. On the interval $0 < M < 70$, gas consumption is $1/(0.04) = 25$ miles per gallon. On the interval $70 < M < 100$, gas consumption is $1/(0.06) = 16.667$ miles per gallon.

(c) In Figure 3.25 in the text, we see that the velocity for the first hour of this trip is 70 mph and the velocity for the second hour is 30 mph. The first hour may have been spent driving on an interstate highway and the second hour may have been spent driving in a city. The answers to part (b) would then tell us that this car gets 25 miles to the gallon on the highway and about 16 miles to the gallon in the city.

(d) Since $M = h(t)$, we have $G = f(M) = f(h(t)) = k(t)$. The function k gives the total number of gallons of gas used t hours into the trip. We have

$$G = k(0.5) = f(h(0.5)) = f(35) = 1.4 \text{ gallons.}$$

The car consumes 1.4 gallons of gas during the first half hour of the trip.

(e) Since $k(t) = f(h(t))$, by the chain rule, we have

$$\frac{dG}{dt} = k'(t) = f'(h(t)) \cdot h'(t).$$

Therefore:

$$\left.\frac{dG}{dt}\right|_{t=0.5} = k'(0.5) = f'(h(0.5)) \cdot h'(0.5) = f'(35) \cdot 70 = 0.04 \cdot 70 = 2.8 \text{ gallons per hour,}$$

and

$$\left.\frac{dG}{dt}\right|_{t=1.5} = k'(1.5) = f'(h(1.5)) \cdot h'(1.5) = f'(85) \cdot 30 = 0.06 \cdot 30 = 1.8 \text{ gallons per hour.}$$

Gas is being consumed at a rate of 2.8 gallons per hour at time $t = 0.5$ and is being consumed at a rate of 1.8 gallons per hour at time $t = 1.5$. Notice that gas is being consumed more quickly on the highway, even though the gas mileage is significantly better there.

86. Since the population is 300 million on October 17, 2006 and growing exponentially, $P = 300e^{kt}$, where P is the population in millions and t is time in years since October 17, 2006. Then

$$\frac{dP}{dt} = 300ke^{kt},$$

so, since the population is growing at 2.9 million/year on October 17, 2006,

$$\left.\frac{dP}{dt}\right|_{t=0} = 300ke^{k\cdot 0} = 300k = 2.9$$
$$k = \frac{2.9}{300} = 0.0097,$$

so $P = 300e^{0.0097t}$.

87. We have

$$\begin{aligned} f(0) &= 331.3 \\ f'(T) &= \frac{1}{2} \cdot 331.3 \left(1 + \frac{T}{273.15}\right)^{-1/2} \frac{1}{273.15} \\ f'(0) &= 0.606. \end{aligned}$$

Thus, for temperatures, T, near zero, we have

$$\text{Speed of sound} = f(T) \approx f(0) + f'(0)T = 331.3 + 0.606T \text{ meters/second.}$$

CHECK YOUR UNDERSTANDING

1. True. If $f(x) = 5x^2 + 1$ then $f'(x) = 10x$ so $f'(-1) = 10(-1) = -10$.

2. True. We see that $f'(x) = g'(x) = 15x^4$.

3. False. Since $h(t) = (3t^2 + 1)(2t) = 6t^3 + 2t$, we see that $h'(t) = 18t^2 + 2$. In a later section, we learn a product rule which gives another way to find h' without multiplying the factors of h together.

4. False. We write $k(s) = \sqrt{s^3} = s^{3/2}$. Then $k'(s) = (3/2)s^{1/2} = 1.5\sqrt{s}$.

5. False. The slope of the tangent line is given by the derivative. We have $f'(x) = 5x^4$ so $f'(1) = 5$. The slope of the tangent line at $x = 1$ is 5 (not 9) so the statement is false.

6. False. We first write $f(r)$ as a power function: $f(r) = r^{-5}$. Then we see $f'(r) = -5r^{-4} = -5/r^4$.

7. True. Since $g'(w) = 3w^2 - 3 = 3(w^2 - 1) = 3(w+1)(w-1)$, when we set $g'(w) = 0$, there are exactly two solutions, at $w = 1$ and $w = -1$.

8. False. Since $f'(x) = 9x^2 - 2x + 2$, we see that $f'(1) = 9$. Since the derivative is positive at $x = 1$, the function is increasing at $x = 1$.

9. True. We see that $f'(x) = 9x^2 - 2x + 2$ and $f''(x) = 18x - 2$. Then $f''(1) = 16$. Since the second derivative is positive at $x = 1$, the graph of f is concave up at $x = 1$.

10. True. Since $g'(t) = \pi t^{\pi-1}$, we have $g'(1) = \pi(1^{\pi-1}) = \pi$.

11. False. The derivative of e^x is e^x.

12. True. We have $g'(s) = 5(1/s)$ so $g'(2) = 5(1/2) = 5/2$.

13. True. We have $f'(x) = 3e^x + 1$ so the slope of the tangent line at $x = 0$ is $f'(0) = 3e^0 + 1 = 3 + 1 = 4$. Since $f(0) = 3e^0 + 0 = 3$, the vertical intercept is 3.

14. False. The derivative is $h'(x) = 2/x - 2x$ so $h'(1) = 2/1 - 2(1) = 2 - 2 = 0$. The graph of $h(x)$ is horizontal (since the slope is 0) at $x = 1$.

15. True. We find the second derivative at $x = 1$. We have $f'(x) = 2/x - 2x$ so $f''(x) = -2/x^2 - 2$. Substituting $x = 1$, we have $f''(1) = -2/(1^2) - 2 = -2 - 2 = -4$. Since the second derivative is negative at $x = 1$, the graph of the function is concave down at $x = 1$.

16. False. Since $\ln 2$ is a constant, the derivative is zero.

17. False. We see that $f'(x) = 2e^{2x}$.

18. True.

19. False. Since the derivative of e^p is e^p, we have $k'(p) = 5e^p$.

20. True. We have $f'(x) = 3(5e^{5x}) = 15e^{5x}$.

21. False. The derivative is $f'(t) = 2te^{t^2}$.

22. False. The derivative is $dy/dx = 5(x + x^2)^4(1 + 2x)$.

23. True.

24. False. If $y = (1 - 2t)^{1/2}$ then $y' = (1/2)(1 - 2t)^{-1/2} \cdot (-2) = -1/\sqrt{1 - 2t}$.

25. True.

26. False. The chain rule says $d/dt(f(g(t))) = f'(g(t)) \cdot g'(t)$.

27. False. By the chain rule, we see that $g'(x) = \dfrac{1}{x^2 + 3x}(2x + 3)$.

28. True. The derivative is $f'(x) = -e^{1-x}$ so $f'(1) = -e^0 = -1$. Since the derivative is negative at $x = 1$, the function is decreasing at $x = 1$.

29. True. The derivative is $f'(x) = -e^{1-x}$ so $f''(x) = e^{1-x}$ and $f''(1) = e^0 = 1$. Since the second derivative is positive at $x = 1$, the function is concave up at $x = 1$.

30. False. By the chain rule, we need to multiply by the derivative of the inside function. The derivative is $dB/dr = 150(1 + 2r)^4 \cdot 2$.

31. False. Since y is the product of two functions, we use the product rule to find the derivative: $y' = e^x/x + e^x \ln x$.

32. False. Since y is the product of two functions, we need to use the product rule to find the derivative.

33. True. The derivative is found using the quotient rule.

34. True.

35. False. We need to use the product rule to find the derivative.

36. False. We need to use the chain rule when we differentiate $\ln(q^2+1)$. We have $P' = q/(q^2+1)\cdot(2q) + \ln(q^2+1)$.

37. True. We use the chain rule and then the product rule to find the derivative of the exponent. Notice that $e^{x\ln x} = x^x$ so this method shows us how to find the derivative of x^x.

38. True. Since $x^2 \cdot x^2 = x^4$, both ways of finding the derivative give the same answer. Try it to see!

39. False. The product rule must be used to differentiate the product of two functions.

40. False. The quotient rule must be used to find the derivative of the quotient of two functions.

41. True.

42. False, the answer is off by a minus sign. We have $f'(t) = \cos t$ so $f''(t) = -\sin t$.

43. True. We have $g'(t) = -\sin t$ so $g''(t) = -\cos t$.

44. False. By the chain rule, we need to multiply by the derivative of the inside function. We have $y' = 2\cos 2t$.

45. False. We use the chain rule to get $y' = -2t\sin t^2$.

46. False. We use the product rule to see that $z' = (\sin 2t)(-3\sin 3t) + (2\cos 2t)\cos(3t)$.

47. False. We use the chain rule to find this derivative. We do not use the product rule since this is not the product of two functions. Using the chain rule, we have $y' = \cos(\cos t)\cdot(-\sin t)$.

48. False. By the chain rule, the derivative of $(\sin q)^{-1}$ is $-(\sin q)^{-2}(\cos q)$.

49. True.

50. True.

PROJECTS FOR CHAPTER THREE

1. (a) Assuming that $T(1) = 98.6 - 2 = 96.6$, we get

$$\begin{aligned} 96.6 &= 68 + 30.6e^{-k\cdot 1} \\ 28.6 &= 30.6e^{-k} \\ 0.935 &= e^{-k}. \end{aligned}$$

So

$$k = -\ln(0.935) \approx 0.067.$$

(b) We're looking for a value of t which gives $T'(t) = -1$. First we find $T'(t)$:

$$\begin{aligned} T(t) &= 68 + 30.6e^{-0.067t} \\ T'(t) &= (30.6)(-0.067)e^{-0.067t} \approx -2e^{-0.067t}. \end{aligned}$$

Setting this equal to -1 per hour gives

$$\begin{aligned} -1 &= -2e^{-0.067t} \\ \ln(0.5) &= -0.067t \\ t &= -\frac{\ln(0.5)}{0.067} \approx 10.3. \end{aligned}$$

Thus, when $t \approx 10.3$ hours, we have $T'(t) \approx -1^\circ$F per hour.

(c) The coroner's rule of thumb predicts that in 24 hours the body temperature will decrease 25°F, to about 73.6°F. The formula predicts a temperature of

$$T(24) = 68 + 30.6e^{-0.067\cdot 24} \approx 74.1^\circ\text{F}.$$

2. (a) See Figure 3.24.

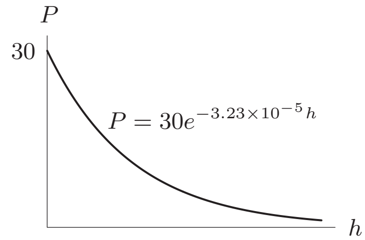


Figure 3.24

(b) Using the chain rule, we have

$$\frac{dP}{dh} = 30e^{-3.23\times 10^{-5}h}(-3.23\times 10^{-5})$$

so

$$\left.\frac{dP}{dh}\right|_{h=0} = -30(3.23\times 10^{-5}) = -9.69\times 10^{-4}$$

Hence, at $h = 0$, the slope of the tangent line is -9.69×10^{-4}, so the equation of the tangent line is

$$\begin{aligned} y - 30 &= (-9.69\times 10^{-4})(h-0) \\ y &= (-9.69\times 10^{-4})h + 30 = 30 - 0.000969h. \end{aligned}$$

(c) The rule of thumb says

$$\begin{array}{c}\text{Drop in pressure from} \\ \text{sea level to height } h\end{array} = \frac{h}{1000}$$

But since the pressure at sea level is 30 inches of mercury, this drop in pressure is also $(30 - P)$, so

$$30 - P = \frac{h}{1000}$$

giving

$$P = 30 - 0.001h.$$

(d) The equations in (b) and (c) are almost the same: both have P intercepts of 30, and the slopes are almost the same $(9.69\times 10^{-4} \approx 0.001)$. The rule of thumb calculates values of P which are very close to the tangent lines, and therefore yields values very close to the curve.

(e) The tangent line is slightly below the curve, and the rule of thumb line, having a slightly more negative slope, is slightly below the tangent line (for $h > 0$). Thus, the rule of thumb values are slightly smaller.

3. (a) (i) The GDP per capita is Y/P, where Y is GDP and P is population. Since

$$\ln\left(\frac{Y}{P}\right) = \ln Y - \ln P,$$

differentiating this relationship tells us that

Relative rate of change of Y/P = Relative rate of change of Y − Relative rate of change of P,

so we have

Relative rate of change of P = Relative rate of change of Y − Relative rate of change of Y/P.

This equation expresses the relative rate of change of population as the vertical distance between the graphs of the relative rates of change of GDP and of GDP per capita.

(ii) In 1970 the relative growth rate of GDP was 5.1% and the relative growth rate of GDP per capita was 2.9%. Thus the relative growth rate of population was $5.1\% - 2.9\% = 2.2\%$ per year.

In 2000 the relative growth rate of GDP was 4.4% and of GDP per capita was 3.1%. Thus the relative growth rate of population was $4.4\% - 3.1\% = 1.3\%$ per year.

(b) The per capita GDP is the quotient GDP$/P$, where P is the world population. Since

$$\text{Relative rate of change of GDP}/P = \text{Relative rate of change of GDP} - \text{Relative rate of change of } P$$

we have

$$\text{Relative rate of change of GDP}/P = 4.5\% - 1.2\% = 3.3\%.$$

In 2006 the per capita GDP in developing countries grew at a relative rate of 3.3% per year.

(c) The world's per capita production is the quotient Y/P, where Y is the world's annual production and P is the world population. Since

$$\text{Relative rate of change of } Y/P = \text{Relative rate of change of } Y - \text{Relative rate of change of } P$$

we have

$$2.6\% = 3.8\% - \text{Relative rate of change of } P.$$

In 2006 the world population grew at a relative rate of $3.8\% - 2.6\% = 1.2\%$ per year.

Solutions to Problems on Establishing the Derivative Formulas

1. Using the definition of the derivative, we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{2(x+h) + 1 - (2x+1)}{h} \\ &= \lim_{h\to 0} \frac{2x + 2h + 1 - 2x - 1}{h} \\ &= \lim_{h\to 0} \frac{2h}{h}. \end{aligned}$$

As long as h is very close to, but not actually equal to, zero we can say that $\lim_{h\to 0} \frac{2h}{h} = 2$, and thus conclude that $f'(x) = 2$.

2. Using the definition of the derivative, we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{5(x+h)^2 - 5x^2}{h} \\ &= \lim_{h\to 0} \frac{5(x^2 + 2xh + h^2) - 5x^2}{h} = \lim_{h\to 0} \frac{5x^2 + 10xh + 5h^2 - 5x^2}{h} \\ &= \lim_{h\to 0} \frac{10xh + 5h^2}{h} = \lim_{h\to 0} \frac{h(10x + 5h)}{h}. \end{aligned}$$

As h gets close to zero, but not equal to zero, we can cancel the h's in the numerator and denominator to obtain the following limit which is equal to $f'(x)$: $\lim_{h\to 0}(10x + 5h) = 10x$. Thus, $f'(x) = 10x$.

3. Using the definition of the derivative, we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{2(x+h)^2 + 3 - (2x^x + 3)}{h} \end{aligned}$$

$$= \lim_{h \to 0} \frac{2x^2 + 4xh + 2h^2 + 3 - 2x^2 - 3}{h}$$
$$= \lim_{h \to 0} \frac{4xh + 2h^2}{h}$$
$$= \lim_{h \to 0} \frac{h(4x + 2h)}{h}$$

As h gets close to zero (but not equal to zero), we can cancel the h in the numerator and denominator to obtain the following:

$$f'(x) = \lim_{h \to 0} (4x + 2h) = 4x$$

Thus, we get $f'(x) = 4x$.

4. Using the definition of the derivative, we have

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 + (x+h) - (x^2 + x)}{h}$$
$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + x + h - x^2 - x}{h}$$
$$= \lim_{h \to 0} \frac{2xh + h^2 + h}{h} = \lim_{h \to 0} \frac{h(2x + h + 1)}{h}.$$

As h approaches, but does not equal, zero we can cancel h's in the numerator and denominator to obtain the following limit equal to $f'(x)$:

$$\lim_{h \to 0} (2x + h + 1) = 2x + 1.$$

Thus, $f'(x) = 2x + 1$.

5. The definition of the derivative states that

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Using this definition, we have

$$f'(x) = \lim_{h \to 0} \frac{4(x+h)^2 + 1 - (4x^2 + 1)}{h}$$
$$= \lim_{h \to 0} \frac{4x^2 + 8xh + 4h^2 + 1 - 4x^2 - 1}{h}$$
$$= \lim_{h \to 0} \frac{8xh + 4h^2}{h}$$
$$= \lim_{h \to 0} \frac{h(8x + 4h)}{h}.$$

As long as h approaches, but does not equal, zero we can cancel h in the numerator and denominator. The derivative now becomes

$$\lim_{h \to 0} (8x + 4h) = 8x.$$

Thus, $f'(x) = 6x$ as we stated above.

6. Using the definition of the derivative, we have

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^4 - x^4}{h}$$
$$= \lim_{h \to 0} \frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^4}{h}$$
$$= \lim_{h \to 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h}$$
$$= \lim_{h \to 0} \frac{h(4x^3 + 6x^2h + 4xh^2 + h^3)}{h}.$$

We can cancel a factor of h as long as $h \neq 0$; thus we get $\lim_{h \to 0} (4x^3 + 6x^2h + 4xh^2 + h^3)$ which goes to $4x^3$ as $h \to 0$. Thus, $f'(x) = 4x^3$.

7. Using the definition of the derivative, we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^5 - x^5}{h} \\ &= \lim_{h\to 0} \frac{x^5 + 5x^4h + 10x^3h^2 + 10x^2h^3 + 5xh^4 + h^5 - x^5}{h} \\ &= \lim_{h\to 0} \frac{5x^4h + 10x^3h^2 + 10x^2h^3 + 5xh^4 + h^5}{h} \\ &= \lim_{h\to 0} \frac{h(5x^4 + 10x^3h + 10x^2h^2 + 5xh^3 + h^4)}{h}. \end{aligned}$$

As $h \to 0$ but does not equal it, we can safely factor h out of the numerator and denominator and cancel, leaving us with the following limit which equals $f'(x)$:

$$\lim_{h\to 0}(5x^4 + 10x^3h + 10x^2h^2 + 5xh^3 + h^4) = 5x^4.$$

Thus, $f'(x) = 5x^4$.

8. **(a)**

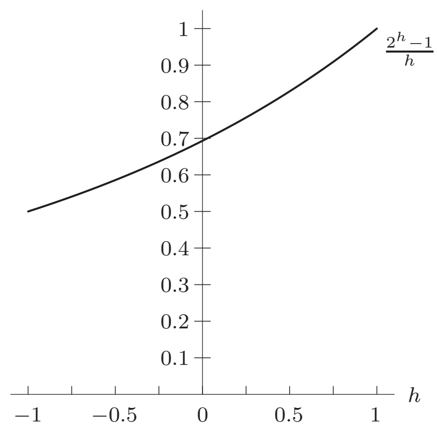


Figure 3.25

The graph of the function $\dfrac{2^h-1}{h}$ is given in Figure 3.25. We can see that as h gets closer to zero, the value of the function approaches 0.6931. Thus, we can conclude that

$$\lim_{h\to 0} \frac{2^h-1}{h} \approx 0.6931.$$

You may also want to consider plugging some values of h close to zero into your calculator so that you can observe that the function is indeed approaching 0.6931 as the values of h get closer and closer to zero.

(b) Using the definition of the derivative and the results from part (a), we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{2^{x+h}-2^x}{h} \\ &= \lim_{h\to 0} \frac{2^x \cdot 2^h - 2^x}{h} \\ &= \lim_{h\to 0} \frac{2^x(2^h-1)}{h} \\ &= 2^x \cdot \left(\lim_{h\to 0} \frac{2^h-1}{h}\right). \end{aligned}$$

From part (a) we know that $\lim_{h\to 0} \dfrac{2^h-1}{h} \approx 0.6931$, and thus

$$f'(x) = 2^x \cdot \left(\lim_{h\to 0} \frac{2^h-1}{h}\right) \approx (0.6931)2^x.$$

9. Since $f(x) = C$ for all x, we have $f(x+h) = C$. Using the definition of the derivative, we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{C - C}{h} \\ &= \lim_{h\to 0} \frac{0}{h}. \end{aligned}$$

As h gets very close to zero without actually equaling zero, we have $0/h = 0$, so

$$f'(x) = \lim_{h\to 0}(0) = 0.$$

10. Using the definition of the derivative, we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(b + m(x+h)) - (b + mx)}{h} \\ &= \lim_{h\to 0} \frac{b + mx + mh - b - mx}{h} \\ &= \lim_{h\to 0} \frac{mh}{h}. \end{aligned}$$

Provided $h \neq 0$, we can cancel the h in the numerator and denominator. Thus, as h gets very close to zero without actually equaling zero, we obtain

$$f'(x) = \lim_{h\to 0}(m) = m.$$

11. Using the definition of the derivative and properties of limits, we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{k \cdot u(x+h) - k \cdot u(x)}{h} \\ &= \lim_{h\to 0} \frac{k(u(x+h) - u(x))}{h} \\ &= \lim_{h\to 0} k \cdot \frac{u(x+h) - u(x)}{h} \\ &= k \cdot \lim_{h\to 0} \frac{u(x+h) - u(x)}{h} \\ &= k \cdot u'(x). \end{aligned}$$

12. Using the definition of the derivative and properties of limits, we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(u(x+h) + v(x+h)) - (u(x) + v(x))}{h} \\ &= \lim_{h\to 0} \frac{(u(x+h) - u(x)) + (v(x+h) - v(x))}{h} \\ &= \lim_{h\to 0} \left(\frac{u(x+h) - u(x)}{h} + \frac{v(x+h) - v(x)}{h} \right) \\ &= \lim_{h\to 0} \frac{u(x+h) - u(x)}{h} + \lim_{h\to 0} \frac{v(x+h) - v(x)}{h} \\ &= u'(x) + v'(x). \end{aligned}$$

Solutions to Practice Problems on Differentiation

1. $f'(t) = 2t + 4t^3$
2. $g'(x) = 20x^3$
3. $y' = 15x^2 + 14x - 3$
4. $s'(t) = -12t^{-3} + 9t^2 - 2t^{-1/2}$
5. $f'(x) = -2x^{-3} + 5\left(\frac{1}{2}x^{-1/2}\right) = \dfrac{-2}{x^3} + \dfrac{5}{2\sqrt{x}}$
6. $P'(t) = 100e^{0.05t}(0.05) = 5e^{0.05t}$
7. $f'(x) = 10e^{2x} - 2 \cdot 3^x(\ln 3)$
8. $P'(t) = 1{,}000(1.07)^t(\ln 1.07) \approx 68(1.07)^t$
9. $D'(p) = 2pe^{p^2} + 10p$
10. $y' = t^2\left(5e^{5t}\right) + 2t\left(e^{5t}\right) = 5t^2e^{5t} + 2te^{5t}$
11. $y' = 2x\sqrt{x^2+1} + x^2\left(\frac{1}{2}(x^2+1)^{-1/2} \cdot 2x\right) = 2x\sqrt{x^2+1} + \dfrac{x^3}{\sqrt{x^2+1}}$
12. $f'(x) = \dfrac{2x}{x^2+1}$
13. $s'(t) = \dfrac{16}{2t+1}$
14. $g'(w) = 2w\ln w + w^2\left(\dfrac{1}{w}\right) = 2w\ln w + w$
15. $f'(x) = 2^x(\ln 2) + 2x$
16. $P'(t) = \frac{1}{2}(t^2+4)^{-1/2}(2t) = \dfrac{t}{\sqrt{t^2+4}}$
17. $C'(q) = 3(2q+1)^2 \cdot 2 = 6(2q+1)^2$
18. $g'(x) = 5(x+3)^2 + 5x(2(x+3)) = 5(x+3)^2 + 10x(x+3)$
19. $P'(t) = bke^{kt}$
20. $f'(x) = 2ax + b$
21. $y' = 2x\ln(2x+1) + \dfrac{2x^2}{2x+1}$
22. $f'(t) = 3(e^t+4)^2(e^t) = 3e^t(e^t+4)^2$
23. $f'(x) = 10\cos(2x)$
24. $W'(r) = 2r\cos r - r^2\sin r$
25. $g'(t) = 15\cos(5t)$
26. $y' = 3e^{3t}\sin(2t) + 2e^{3t}\cos(2t)$
27. $y' = 2e^x + 3\cos x$
28. $f'(t) = 6t - 4.$
29. $y' = 17 + 12x^{-1/2}.$
30. $g'(x) = -\dfrac{1}{2}(5x^4 + 2).$
31. The power rule gives $f'(x) = 20x^3 - \dfrac{2}{x^3}.$
32. $\dfrac{dy}{dx} = \dfrac{2e^{2x}(x^2+1) - e^{2x}(2x)}{(x^2+1)^2} = \dfrac{2e^{2x}(x^2+1-x)}{(x^2+1)^2}$
33. Either notice that $f(x) = \dfrac{x^2+3x+2}{x+1}$ can be written as $f(x) = \dfrac{(x+2)(x+1)}{x+1}$ which reduces to $f(x) = x+2,$

giving $f'(x) = 1$, or use the quotient rule which gives

$$\begin{aligned} f'(x) &= \frac{(x+1)(2x+3) - (x^2+3x+2)}{(x+1)^2} \\ &= \frac{2x^2+5x+3-x^2-3x-2}{(x+1)^2} \\ &= \frac{x^2+2x+1}{(x+1)^2} \\ &= \frac{(x+1)^2}{(x+1)^2} \\ &= 1. \end{aligned}$$

34. $y' = 2\left(\frac{x^2+2}{3}\right)\left(\frac{2x}{3}\right) = \frac{4}{9}x\left(x^2+2\right)$

35. $\frac{d}{dx}\sin(2-3x) = \cos(2-3x)\frac{d}{dx}(2-3x) = -3\cos(2-3x).$

36. $f(z) = \frac{z}{3} + \frac{1}{3}z^{-1} = \frac{1}{3}\left(z+z^{-1}\right)$, so $f'(z) = \frac{1}{3}\left(1-z^{-2}\right) = \frac{1}{3}\left(\frac{z^2-1}{z^2}\right)$.

37. $q'(r) = \frac{3(5r+2)-3r(5)}{(5r+2)^2} = \frac{15r+6-15r}{(5r+2)^2} = \frac{6}{(5r+2)^2}$

38. $\frac{dy}{dx} = \ln x + x\left(\frac{1}{x}\right) - 1 = \ln x$

39. $j'(x) = \frac{ae^{ax}}{(e^{ax}+b)}$

40. $g'(t) = \frac{(t+4)-(t-4)}{(t+4)^2} = \frac{8}{(t+4)^2}.$

41. $h'(w) = 5(w^4-2w)^4(4w^3-2)$

42. Using the product and chain rules gives $h'(w) = 3w^2\ln(10w) + w^3\frac{10}{10w} = 3w^2\ln(10w) + w^2$.

43. Using the chain rule gives $f'(x) = \frac{\cos x - \sin x}{\sin x + \cos x}$.

44. We can write $w(r) = (r^4+1)^{1/2}$, so
$w'(r) = \frac{1}{2}(r^4+1)^{-1/2}(4r^3) = \frac{2r^3}{\sqrt{r^4+1}}$.

45. $h'(w) = 6w^{-4} + \frac{3}{2}w^{-1/2}$

46. We can write $h(x) = \left(\frac{x^2+9}{x+3}\right)^{1/2}$, so

$$h'(x) = \frac{1}{2}\left(\frac{x^2+9}{x+3}\right)^{-1/2}\left[\frac{2x(x+3)-(x^2+9)}{(x+3)^2}\right] = \frac{1}{2}\sqrt{\frac{x+3}{x^2+9}}\left[\frac{x^2+6x-9}{(x+3)^2}\right].$$

47. Using the product rule gives $v'(t) = 2te^{-ct} - ce^{-ct}t^2 = (2t-ct^2)e^{-ct}$.

48. Using the quotient rule gives

$$\begin{aligned} f'(x) &= \frac{1+\ln x - x(\frac{1}{x})}{(1+\ln x)^2} \\ &= \frac{\ln x}{(1+\ln x)^2}. \end{aligned}$$

49. Using the chain rule, $g'(\theta) = (\cos\theta)e^{\sin\theta}$.

50. $p'(t) = 4e^{4t+2}$.

51. $j'(x) = \dfrac{3x^2}{a} + \dfrac{2ax}{b} - c$

52. $\dfrac{d}{dz}\left(\dfrac{z^2+1}{\sqrt{z}}\right) = \dfrac{d}{dz}(z^{\frac{3}{2}} + z^{-\frac{1}{2}}) = \dfrac{3}{2}z^{\frac{1}{2}} - \dfrac{1}{2}z^{-\frac{3}{2}} = \dfrac{\sqrt{z}}{2}(3 - z^{-2})$.

53. $h'(r) = \dfrac{d}{dr}\left(\dfrac{r^2}{2r+1}\right) = \dfrac{(2r)(2r+1) - 2r^2}{(2r+1)^2} = \dfrac{2r(r+1)}{(2r+1)^2}$.

54. $g'(x) = \dfrac{d}{dx}(2x - x^{-1/3} + 3^x - e) = 2 + \dfrac{1}{3x^{\frac{4}{3}}} + 3^x \ln 3$.

55. $f'(t) = \dfrac{d}{dt}\left(2te^t - \dfrac{1}{\sqrt{t}}\right) = 2e^t + 2te^t + \dfrac{1}{2t^{3/2}}$.

56.

$$\begin{aligned}\frac{dw}{dz} &= \frac{(-3)(5+3z) - (5-3z)(3)}{(5+3z)^2} \\ &= \frac{-15 - 9z - 15 + 9z}{(5+3z)^2} = \frac{-30}{(5+3z)^2}\end{aligned}$$

57. $f'(x) = \dfrac{3x^2}{9}(3\ln x - 1) + \dfrac{x^3}{9}\left(\dfrac{3}{x}\right) = x^2 \ln x - \dfrac{x^2}{3} + \dfrac{x^2}{3} = x^2 \ln x$

58. $g'(x) = \dfrac{d}{dx}\left(x^{\frac{1}{2}} + x^{-1} + x^{-\frac{3}{2}}\right) = \dfrac{1}{2}x^{-\frac{1}{2}} - x^{-2} - \dfrac{3}{2}x^{-\frac{5}{2}}$.

59. Using the product and chain rules, we have

$$\begin{aligned}\frac{dy}{dx} &= 3(x^2+5)^2(2x)(3x^3-2)^2 + (x^2+5)^3[2(3x^3-2)(9x^2)] \\ &= 3(2x)(x^2+5)^2(3x^3-2)[(3x^3-2) + (x^2+5)(3x)] \\ &= 6x(x^2+5)^2(3x^3-2)[6x^3+15x-2].\end{aligned}$$

60. Using the quotient rule gives

$$f'(x) = \frac{(-2x)(a^2+x^2) - (2x)(a^2-x^2)}{(a^2+x^2)^2} = \frac{-4a^2x}{(a^2+x^2)^2}.$$

61. Using the quotient rule gives

$$\begin{aligned}w'(r) &= \frac{2ar(b+r^3) - 3r^2(ar^2)}{(b+r^3)^2} \\ &= \frac{2abr - ar^4}{(b+r^3)^2}.\end{aligned}$$

62. Using the product rule gives

$$\begin{aligned}H'(t) &= 2ate^{-ct} - c(at^2+b)e^{-ct} \\ &= (-cat^2 + 2at - bc)e^{-ct}.\end{aligned}$$

63. Since $g(w) = 5(a^2 - w^2)^{-2}$, $g'(w) = -10(a^2 - w^2)^{-3}(-2w) = \dfrac{20w}{(a^2 - w^2)^3}$