CHAPTER FOUR

Solutions for Section 4.1

1. We find a critical point by noting where $f'(x) = 0$ or f' is undefined. Since the curve is smooth throughout, f' is always defined, so we look for where $f'(x) = 0$, or equivalently where the tangent line to the graph is horizontal. These points are shown in Figure 4.1.

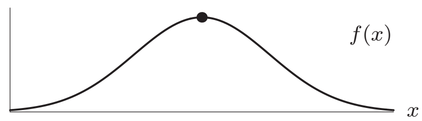


Figure 4.1

As we can see, there is one critical point. Since it is higher than nearby points, it is a local maximum.

2. We find a critical point by noting where $f'(t) = 0$ or f' is undefined. Since the curve is smooth throughout, f' is always defined, so we look for where $f'(t) = 0$, or equivalently where the tangent line to the graph is horizontal. These points are shown in Figure 4.2.

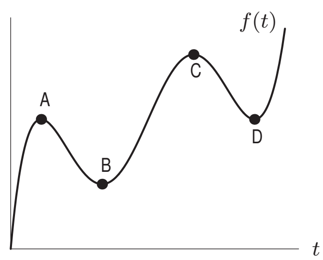


Figure 4.2

As we can see, there are four labeled critical points. Critical point A is a local maximum because points near it are all lower; similarly, point B is a local minimum, point C is a local maximum, and point D is a local minimum.

3. We find a critical point by noting where $f'(x) = 0$ or f' is undefined. Since the curve is smooth throughout, f' is always defined, so we look for where $f'(x) = 0$, or equivalently where the tangent line to the graph is horizontal. These points are shown in Figure 4.3:

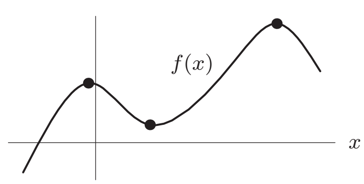


Figure 4.3

As we can see, there are three critical points. The leftmost one is a local maximum, because points near it are all lower; similarly, the middle critical point is surrounded by higher points, and is a local minimum. The critical point to the right is a local maximum.

4. We find a critical point by noting where $f'(x) = 0$ or f' is undefined. Since the curve is smooth throughout, f' is always defined, so we look for where $f'(x) = 0$, or equivalently where the tangent line to the graph is horizontal. These points are shown in Figure 4.4.

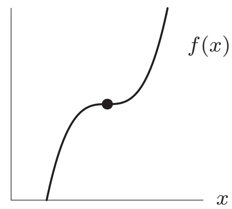


Figure 4.4

As we can see, there is one critical point. Since some nearby points (those to the left) are lower, this point is not a local minimum; since nearby points to the right are higher, it is not a local maximum. So the one critical point is neither a local minimum nor a local maximum.

5. **(a)** One possible answer is shown in Figure 4.5.
(b) One possible answer is shown in Figure 4.6

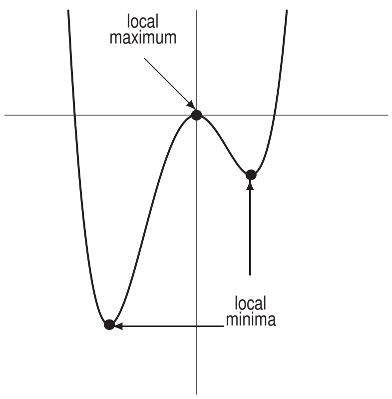


Figure 4.5

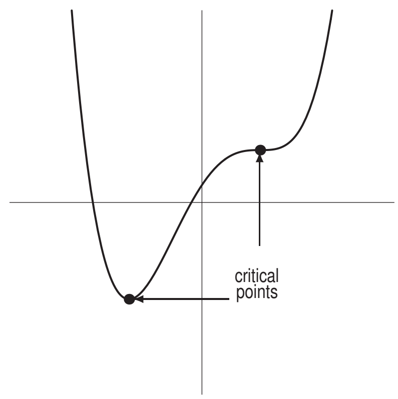
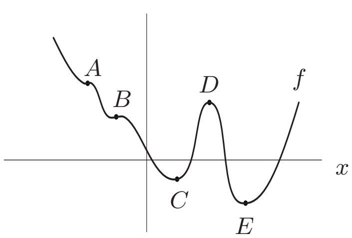


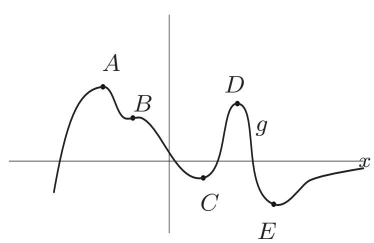
Figure 4.6

6. There was a critical point after the first eighteen hours when temperature was at its highest point, a local maximum for the temperature function.

7. **(a)**



(b)



8. A critical point of f requires $f'(x) = 0$ or f' undefined. Since f' is clearly defined over the relevant range, we find where $f'(x) = 0$, that is, where the graph of f' crosses the x-axis. These points are shown and labeled in Figure 4.7.

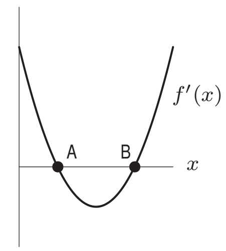


Figure 4.7

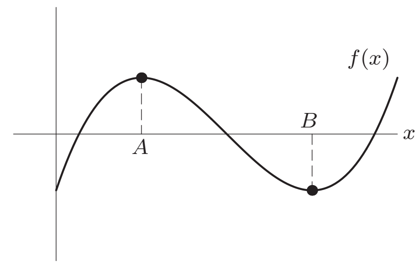


Figure 4.8

To the left of critical point A, we see that $f' > 0$ and f is increasing; to the right of the critical point, we see that $f' < 0$ and f is decreasing. So there is a local maximum at A.

To the left of critical point B, we see that $f' < 0$ and f is decreasing; to the right of the critical point, we see that $f' > 0$ and f is increasing. So there is a local minimum at B. The sketch of $f(x)$ in Figure 4.8 shows A is a local maximum and B is a local minimum.

9. A critical point of f would require $f'(x) = 0$ or f' undefined. Since f' is clearly defined over the relevant range, we wish to find where $f'(x) = 0$, or where the graph shown intersects the x-axis. These points are shown and labeled in Figure 4.9.

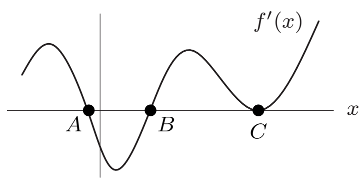


Figure 4.9: The critical points of $f(x)$

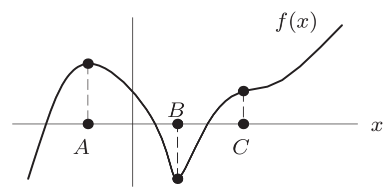


Figure 4.10: A possible graph of $f(x)$

To the left of critical point A, $f' > 0$ and f is increasing; to the right, $f' < 0$ and f is decreasing. So there is a local maximum at A.

To the left of critical point B, $f' < 0$ and f is decreasing; to the right, $f' > 0$ and f is increasing. So there is a local minimum at B.

To both the left and right of critical point C, $f' > 0$ and so f increases on both sides of point C. So, point C is neither a local maximum nor a local minimum.

A sketch of $f(x)$ is shown in Figure 4.10.

10. The graph of f in Figure 4.11 appears to be increasing for $x < -1.4$, decreasing for $-1.4 < x < 1.4$, and increasing for $x > 1.4$. There is a local maximum near $x = -1.4$ and local minimum near $x = 1.4$. The derivative of f is $f'(x) = 3x^2 - 6$. Thus $f'(x) = 0$ when $x^2 = 2$, that is $x = \pm\sqrt{2}$. This explains the critical points near $x = \pm 1.4$. Since $f'(x)$ changes from positive to negative at $x = -\sqrt{2}$, and from negative to positive at $x = \sqrt{2}$, there is a local maximum at $x = -\sqrt{2}$ and a local minimum at $x = \sqrt{2}$.

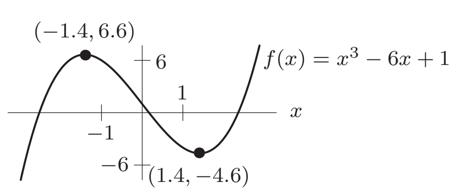


Figure 4.11

11. The graph of f in Figure 4.12 appears to be increasing for all x, with no critical points. Since $f'(x) = 3x^2 + 6$ and $x^2 \geq 0$ for all x, we have $f'(x) > 0$ for all x. That explains why f is increasing for all x.

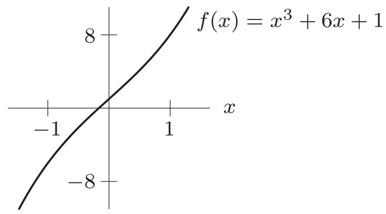


Figure 4.12

12. The graph of f in Figure 4.13 appears to be increasing for $x < -1$, decreasing for $-1 < x < 1$ although it is flat at $x = 0$, and increasing for $x > 1$. There are critical points at $x = -1$ and $x = 1$, and apparently also at $x = 0$.
Since $f'(x) = 15x^4 - 15x^2 = 15x^2(x^2 - 1)$, we have $f'(x) = 0$ at $x = 0, -1, 1$. Notice that although $f'(0) = 0$, making $x = 0$ a critical point, there is no change in sign of $f'(x)$ at $x = 0$; the only sign changes are at $x = \pm 1$. Thus the graph of f must alternate increasing/decreasing for $x < -1, -1 < x < 1, x > 1$, just as we described.

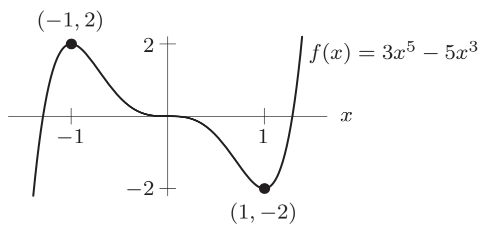


Figure 4.13

13. The graph of f in Figure 4.14 appears to be decreasing for $x < 2.3$ (almost like a straight line for $x < 0$), and increasing sharply for $x > 2.3$. Here $f'(x) = e^x - 10$, so $f'(x) = 0$ when $e^x = 10$, that is $x = \ln 10 = 2.302\ldots$ This is the only place where $f'(x)$ changes sign, and it is a minimum of f. Notice that e^x is small for $x < 0$ so $f'(x) \approx -10$ for $x < 0$, which means the graph looks like a straight line of slope -10 for $x < 0$. However, e^x gets large quickly for $x > 0$, so $f'(x)$ gets large quickly for $x > \ln 10$, meaning the graph increases sharply there.

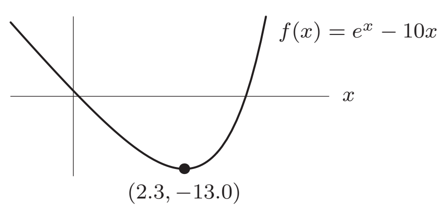
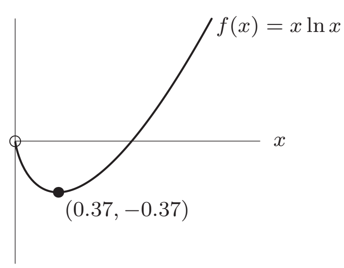


Figure 4.14

14. The graph of f below appears to be decreasing for $0 < x < 0.37$, and then increasing for $x > 0.37$. We have $f'(x) = \ln x + x(1/x) = \ln x + 1$, so $f'(x) = 0$ when $\ln x = -1$, that is, $x = e^{-1} \approx 0.37$. This is the only place where f' changes sign and $f'(1) = 1 > 0$, so the graph must decrease for $0 < x < e^{-1}$ and increase for $x > e^{-1}$. Thus, there is a local minimum at $x = e^{-1}$.



15. The graph of f in Figure 4.15 looks like a climbing sine curve, alternately increasing and decreasing, with more time spent increasing than decreasing. Here $f'(x) = 1 + 2\cos x$, so $f'(x) = 0$ when $\cos x = -1/2$; this occurs when

$$x = \pm\frac{2\pi}{3}, \pm\frac{4\pi}{3}, \pm\frac{8\pi}{3}, \pm\frac{10\pi}{3}, \pm\frac{14\pi}{3}, \pm\frac{16\pi}{3}...$$

Since $f'(x)$ changes sign at each of these values, the graph of f must alternate increasing/decreasing. However, the distance between values of x for critical points alternates between $(2\pi)/3$ and $(4\pi)/3$, with $f'(x) > 0$ on the intervals of length $(4\pi)/3$. For example, $f'(x) > 0$ on the interval $(4\pi)/3 < x < (8\pi)/3$. As a result, f is increasing on the intervals of length $(4\pi/3)$ and decreasing on the intervals of length $(2\pi/3)$.

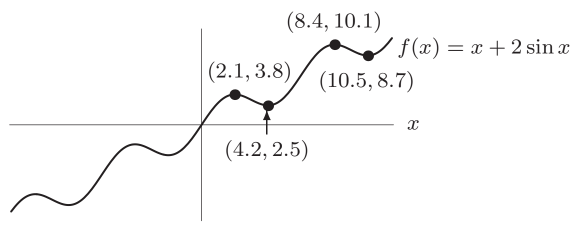


Figure 4.15

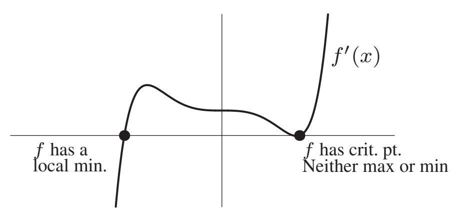
16. The function f has critical points at $x = 1, x = 3, x = 5$.

By the first derivative test, since f' is positive to the left of $x = 1$ and negative to the right, $x = 1$ is a local maximum.

Since f' is negative to the left of $x = 3$ and positive to the right, $x = 3$ is a local minimum.

Since f' does not change sign at $x = 5$, this point is neither a local maximum nor a local minimum.

17. The critical points of f occur where f' is zero. These two points are indicated in the figure below.



Note that the point labeled as a local minimum of f is not a critical point of f'.

18. Looking at the graph of the derivative function in Figure 4.16, we see that
$f(t)$ is increasing when $0 < t < 2$ or when $t > 4$, because $f'(t) > 0$ for these values of t.
$f(t)$ is decreasing when $2 < t < 4$, because $f'(t) < 0$ for these values of t.
$f(t)$ has a local maximum at $t = 2$, since $f(t)$ is increasing to the left of $t = 2$ and decreasing to the right of $t = 2$. In addition, $f(t)$ has a local maximum at $t = 5$ since $f(t)$ is increasing to the left of $t = 5$ and $t = 5$ is an endpoint. Also, $f(t)$ has a local minimum at $t = 4$, since it is decreasing to the left of $t = 4$ and increasing to the right of $t = 4$. In addition, $f(t)$ has a local minimum at $t = 0$ since it is increasing to the right of $t = 0$ and $t = 0$ is an endpoint.

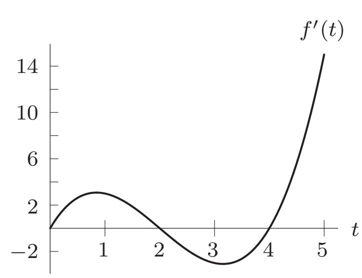


Figure 4.16

19. To find the critical points, we set the derivative equal to zero and solve for t.

$$\begin{aligned} F'(t) = Ue^t + Ve^{-t}(-1) &= 0 \\ Ue^t - \frac{V}{e^t} &= 0 \\ Ue^t &= \frac{V}{e^t} \\ Ue^{2t} &= V \\ e^{2t} &= \frac{V}{U} \\ 2t &= \ln(V/U) \\ t &= \frac{\ln(V/U)}{2}. \end{aligned}$$

The derivative $F'(t)$ is never undefined, so the only critical point is $t = 0.5\ln(V/U)$.

20. **(a)** The demand for the product is increasing when $f'(t)$ is positive, and decreasing when $f'(t)$ is negative. Inspection of the table suggests that demand is increasing during weeks 0 to 2 and weeks 6 to 10, and decreasing during weeks 3 to 5.

(b) Since $f'(t) = 4 > 0$ during week 2 and $f'(t) = -2 < 0$ during week 3, the demand for the product changes from increasing to decreasing near the end of week 2 or the beginning of week 3. Thus the demand has a local maximum during this time period. Since $f'(t) = -1 < 0$ during week 5 and $f'(t) = 3 > 0$ during week 6, the demand for the product changes from decreasing to increasing near the end of week 5 or the beginning of week 6. Thus the demand has a local minimum during this time period.

21. **(a)** A critical point occurs when $f'(x) = 0$. Since $f'(x)$ changes sign between $x = 2$ and $x = 3$, between $x = 6$ and $x = 7$, and between $x = 9$ and $x = 10$, we expect critical points at around $x = 2.5$, $x = 6.5$, and $x = 9.5$.

(b) Since $f'(x)$ goes from positive to negative at $x \approx 2.5$, a local maximum should occur there. Similarly, $x \approx 6.5$ is a local minimum and $x \approx 9.5$ a local maximum.

22. Since $f'(x) = 4x^3 - 12x^2 + 8$, we see that $f'(1) = 0$, as we expected. We apply the second derivative test to $f''(x) = 12x^2 - 24x$. Since $f''(1) = -12 < 0$, the graph is concave down at the critical point $x = 1$, making it a local maximum.

23. Differentiating using the product rule gives

$$f'(x) = 3x^2(1-x)^4 - 4x^3(1-x)^3 = x^2(1-x)^3(3(1-x) - 4x) = x^2(1-x)^3(3-7x).$$

The critical points are the solutions to

$$\begin{aligned} f'(x) = x^2(1-x)^3(3-7x) &= 0 \\ x &= 0, 1, \frac{3}{7}. \end{aligned}$$

For $x < 0$, since $1 - x > 0$ and $3 - 7x > 0$, we have $f'(x) > 0$.
For $0 < x < \frac{3}{7}$, since $1 - x > 0$ and $3 - 7x > 0$, we have $f'(x) > 0$.
For $\frac{3}{7} < x < 1$, since $1 - x > 0$ and $3 - 7x < 0$, we have $f'(x) < 0$.
For $1 < x$, since $1 - x < 0$ and $3 - 7x < 0$, we have $f'(x) > 0$.

Thus, $x = 0$ is neither a local maximum nor a local minimum; $x = 3/7$ is a local maximum; $x = 1$ is a local minimum.

24. **(a)** See Figure 4.17.

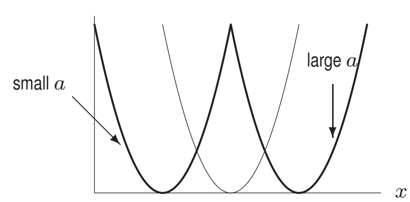


Figure 4.17

(b) We see in Figure 4.17 that in each case the graph of f is a parabola with one critical point, its vertex, on the positive x-axis. The critical point moves to the right along the x-axis as a increases.

(c) To find the critical points, we set the derivative equal to zero and solve for x.

$$\begin{aligned} f'(x) = 2(x-a) &= 0 \\ x &= a. \end{aligned}$$

The only critical point is at $x = a$. As we saw in the graph, and as a increases, the critical point moves to the right.

25. (a) See Figure 4.18.

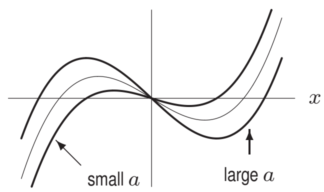


Figure 4.18

(b) We see in Figure 4.18 that in each case f has two critical points placed symmetrically about the origin, one in each of quadrants II and IV. As a increases, they appear to move farther apart, the one in quadrant II up and to the left, the one in quadrant IV down and to the right.

(c) To find the critical points, we set the derivative equal to zero and solve for x.

$$\begin{aligned} f'(x) = 3x^2 - a &= 0 \\ x^2 &= \frac{a}{3} \\ x &= \pm\sqrt{\frac{a}{3}}. \end{aligned}$$

There are two critical points, at $x = \sqrt{a/3}$ and $x = -\sqrt{a/3}$. (Since the parameter a is positive, the critical points exist.) As we saw in the graph, and as a increases, the critical points both move away from the vertical axis.

26. (a) See Figure 4.19.

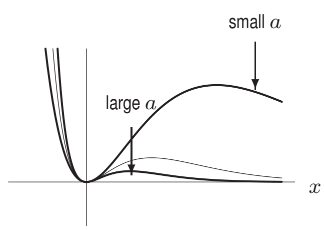


Figure 4.19

(b) We see in Figure 4.19 that in each case f appears to have two critical points. One critical point is a local minimum at the origin and the other is a local maximum in quadrant I. As the parameter a increases, the critical point in quadrant I appears to move down and to the left, closer to the origin.

(c) To find the critical points, we set the derivative equal to zero and solve for x. Using the product rule, we have:

$$\begin{aligned} f'(x) = x^2 \cdot e^{-ax}(-a) + 2x \cdot e^{-ax} &= 0 \\ xe^{-ax}(-ax+2) &= 0 \\ x = 0 \quad \text{and} \quad x &= \frac{2}{a}. \end{aligned}$$

There are two critical points, at $x = 0$ and $x = 2/a$. As we saw in the graph, as a increases the nonzero critical point moves to the left.

27. Since $f(x)$ has its minimum at $x = 3$, then $x = 3$ must be a critical point. So $f'(3) = 0$.

$$f'(x) = 2x + a \qquad \text{so} \qquad f'(3) = 6 + a.$$

Since $f'(3) = 0$, then $6 + a = 0$ or $a = -6$. Since $(3, 5)$ is a point on the graph of $f(x)$ we must have $f(3) = 5$:

$$f(3) = 3^2 + a(3) + b = 9 + 3a + b = 5.$$

Since we know that $a = -6$, we have:

$$f(3) = 9 + 3(-6) + b = b - 9 = 5 \qquad \text{so} \qquad b = 14.$$

Thus, we have found that $a = -6$, $b = 14$, giving us $f(x) = x^2 - 6x + 14$.

28. If the minimum of $f(x)$ is at $(-2, -3)$, then the derivative of f must be equal to 0 there. In other words, $f'(-2) = 0$. If

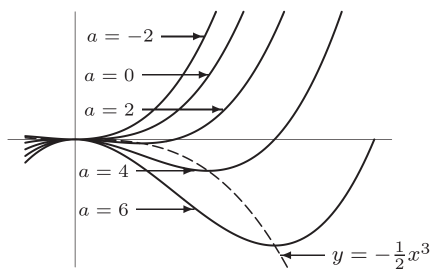
$$\begin{aligned} f(x) &= x^2 + ax + b, \quad \text{then} \\ f'(x) &= 2x + a \\ f'(-2) &= 2(-2) + a = -4 + a = 0 \end{aligned}$$

so $a = 4$. Since $(-2, -3)$ is on the graph of $f(x)$ we know that $f(-2) = -3$. So

$$\begin{aligned} f(-2) = (-2)^2 + a(-2) + b &= -3 \\ a = 4, \text{ so} \qquad (-2)^2 + 4(-2) + b &= -3 \\ 4 - 8 + b &= -3 \\ -4 + b &= -3 \\ b &= 1 \end{aligned}$$

so $a = 4$ and $b = 1$, and $f(x) = x^2 + 4x + 1$.

29.



To solve for the critical points, we set $\frac{dy}{dx} = 0$. Since $\frac{d}{dx}\left(x^3 - ax^2\right) = 3x^2 - 2ax$, we want $3x^2 - 2ax = 0$, so $x = 0$ or $x = \frac{2}{3}a$. At $x = 0$, we have $y = 0$. This first critical point is independent of a and lies on the curve $y = -\frac{1}{2}x^3$. At $x = \frac{2}{3}a$, we calculate $y = -\frac{4}{27}a^3 = -\frac{1}{2}\left(\frac{2}{3}a\right)^3$. Thus the second critical point also lies on the curve $y = -\frac{1}{2}x^3$.

30. Recall that the natural logarithm is undefined for $x \leq 0$, so the domain of f is $x > 0$. We see from looking at the graph of $f(x) = x - \ln x$ in the text that this function has one local minimum. We want to assign values to a and b so that this local minimum occurs at $x = 2$. The function must therefore have a critical point at $x = 2$. We find the derivative of $f(x) = a(x - b\ln x)$ and the critical points in terms of a and b.

$$\begin{aligned} f'(x) = a\left(1 - b\left(\frac{1}{x}\right)\right) &= 0 \\ 1 - b\left(\frac{1}{x}\right) &= 0 \\ 1 &= \frac{b}{x} \\ x &= b \end{aligned}$$

We see that $f(x)$ has only one critical point, at $x = b$. Since we want a critical point at $x = 2$, we choose $b = 2$.

Since $b = 2$, we have $f(x) = a(x - 2\ln x)$. We now use the condition that $f(2) = 5$ to find a:

$$\begin{aligned} f(2) &= 5 \\ a(2 - 2\ln 2) &= 5 \\ a &= 5/(2 - 2\ln 2) \\ a &\approx 8.147. \end{aligned}$$

We let $a = 8.147$ and $b = 2$, so the function is $f(x) = 8.147(x - 2\ln x)$. If we sketch a graph of this function, we see that this function does indeed have a local minimum approximately at the point $(2, 5)$.

31. We wish to have $f'(3) = 0$. Differentiating to find $f'(x)$ and then solving $f'(3) = 0$ for a gives:

$$\begin{aligned} f'(x) &= x(ae^{ax}) + 1(e^{ax}) = e^{ax}(ax + 1) \\ f'(3) &= e^{3a}(3a + 1) = 0 \\ 3a + 1 &= 0 \\ a &= -\frac{1}{3}. \end{aligned}$$

Thus, $f(x) = xe^{-x/3}$.

32. **(a)** This function is defined for $x > 0$. We set the derivative equal to zero and solve for x to find critical points:

$$\begin{aligned} f'(x) = 1 - b\frac{1}{x} &= 0 \\ 1 &= \frac{b}{x} \\ x &= b. \end{aligned}$$

The only critical point is at $x = b$.

(b) Since $f'(x) = 1 - bx^{-1}$, the second derivative is

$$f''(x) = bx^{-2} = \frac{b}{x^2}.$$

Since $b > 0$, the second derivative is always positive. Thus, the function is concave up everywhere and f has a local minimum at $x = b$.

33. **(a)** The function $f(x)$ is defined for $x \geq 0$.

We set the derivative equal to zero and solve for x to find critical points:

$$\begin{aligned} f'(x) = 1 - \frac{1}{2}ax^{-1/2} &= 0 \\ 1 - \frac{a}{2\sqrt{x}} &= 0 \\ 2\sqrt{x} &= a \\ x &= \frac{a^2}{4}. \end{aligned}$$

Notice that f' is undefined at $x = 0$ so there are two critical points: $x = 0$ and $x = a^2/4$.

(b) We want the critical point $x = a^2/4$ to occur at $x = 5$, so we have:

$$\begin{aligned} 5 &= \frac{a^2}{4} \\ 20 &= a^2 \\ a &= \pm\sqrt{20}. \end{aligned}$$

Since a is positive, we use the positive square root. The second derivative,

$$f''(x) = \frac{1}{4}ax^{-3/2} = \frac{1}{4}\sqrt{20}x^{-3/2}$$

is positive for all $x > 0$, so the function is concave up and $x = 5$ gives a local minimum. See Figure 4.20.

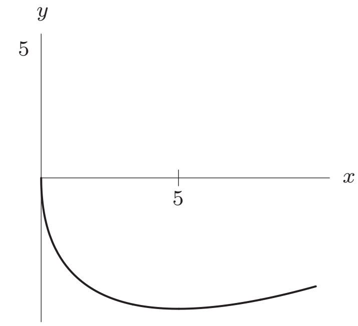


Figure 4.20

34. The function g will have a critical point when $g'(x) = 0$. Solving this equation gives

$$\begin{aligned} g'(x) = 1 - ke^x &= 0 \\ ke^x &= 1 \\ e^x &= \frac{1}{k} \\ x &= \ln \frac{1}{k} = -\ln k. \end{aligned}$$

Since the natural logarithm has a domain of all positive real numbers, such a value for x may only exist for $k > 0$.

35. The domain is all real numbers except $x = b$. The function is undefined at $x = b$ and has a vertical asymptote there. To find the critical points, we set the derivative equal to zero and solve for x. Using the quotient rule, we have:

$$\begin{aligned} f'(x) = \frac{(x-b)2ax - (ax^2)1}{(x-b)^2} &= 0 \\ \frac{2ax^2 - 2abx - ax^2}{(x-b)^2} &= 0 \\ \frac{ax^2 - 2abx}{(x-b)^2} &= 0. \end{aligned}$$

The first derivative is equal to zero if

$$\begin{aligned} ax^2 - 2abx &= 0 \\ ax(x - 2b) &= 0 \\ x = 0 \quad \text{or} \quad x &= 2b. \end{aligned}$$

There are two critical points: at $x = 0$ and $x = 2b$.

36. Since f is differentiable everywhere, f' must be zero (not undefined) at any critical points; thus, $f'(3) = 0$. Since f has exactly one critical point, f' may change sign only at $x = 3$. Thus f is always increasing or always decreasing for $x < 3$ and for $x > 3$. Using the information in parts (a) through (d), we determine whether $x = 3$ is a local minimum, local maximum, or neither.

(a) $x = 3$ is a local maximum because $f(x)$ is increasing when $x < 3$ and decreasing when $x > 3$. See Figure 4.21.

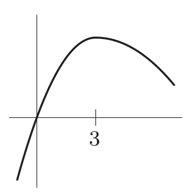


Figure 4.21

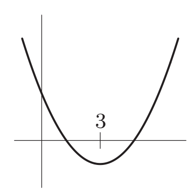


Figure 4.22

(b) $x = 3$ is a local minimum because $f(x)$ heads to infinity to either side of $x = 3$. See Figure 4.22.

(c) $x = 3$ is neither a local minimum nor maximum, as $f(1) < f(2) < f(4) < f(5)$. See Figure 4.23.

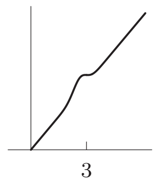


Figure 4.23

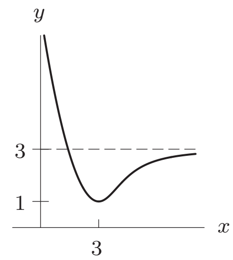


Figure 4.24

(d) $x = 3$ is a local minimum because $f(x)$ is decreasing to the left of $x = 3$ and must increase to the right of $x = 3$, as $f(3) = 1$ and eventually $f(x)$ must become close to 3. See Figure 4.24.

37. (a) In Figure 4.25, we see that $f(\theta) = \theta - \sin\theta$ has a zero at $\theta = 0$. To see if it has any other zeros near the origin, we use our calculator to zoom in. (See Figure 4.26.) No extra root seems to appear no matter how close to the origin we zoom. However, zooming can never tell you for sure that there is not a root that you have not found yet.

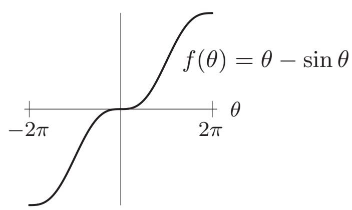


Figure 4.25: Graph of $f(\theta)$

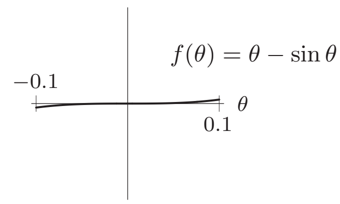


Figure 4.26: Graph of $f(\theta)$ Zoomed In

(b) Using the derivative, $f'(\theta) = 1 - \cos\theta$, we can argue that there is no other zero. Since $\cos\theta < 1$ for $0 < \theta \leq 1$, we know $f'(\theta) > 0$ for $0 < \theta \leq 1$. Thus, f increases for $0 < \theta \leq 1$. Consequently, we conclude that the only zero of f is the one at the origin. If f had another zero at x_0, with $x_0 > 0$, then f would have to "turn around", and recross the x-axis at x_0. But if this were the case, f' would be nonpositive somewhere, which we know is not the case.

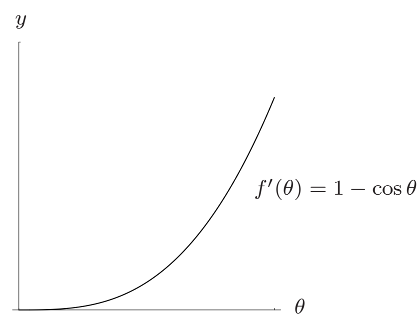


Figure 4.27: Graph of $f'(\theta)$

Solutions for Section 4.2

1. We find an inflection point by noting where the concavity changes. Such points are shown in Figure 4.28. There are three inflection points.

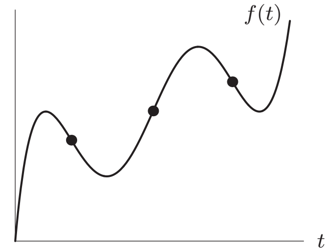


Figure 4.28

2. We find an inflection point by noting where the concavity changes. Such points are shown in Figure 4.29. There are two inflection points.

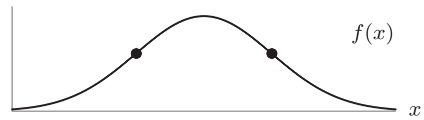


Figure 4.29

3. We find an inflection point by noting where the concavity changes. Looking at Figure 4.30, we see that in fact the concavity changes only at the critical point. So there is one inflection point, which is also a critical point.

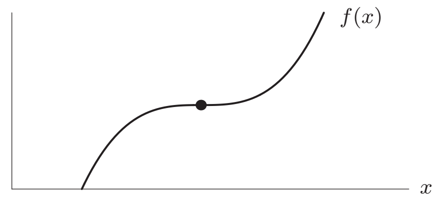


Figure 4.30

4. We find an inflection point by noting where the concavity changes. Such points are shown in Figure 4.31: There are two inflection points.

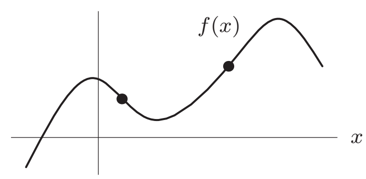


Figure 4.31

5. **(a)** One possible answer is shown in Figure 4.32.

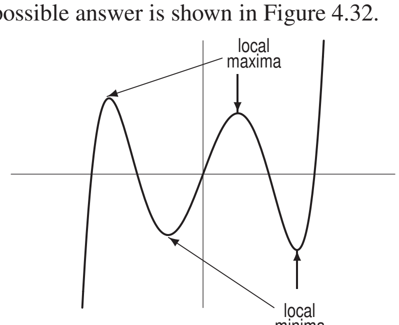


Figure 4.32

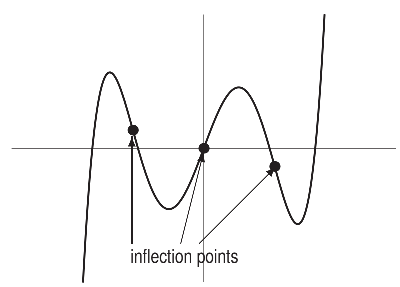


Figure 4.33

(b) This function is concave down at each local maximum and concave up at each local minimum, so it changes concavity at least three times. This function has at least 3 inflection points. See Figure 4.33

6. One possible answer is shown in Figure 4.34.

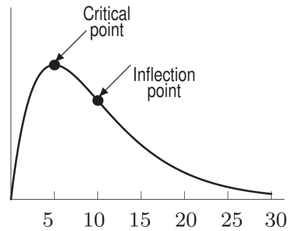


Figure 4.34

7. See Figure 4.35.

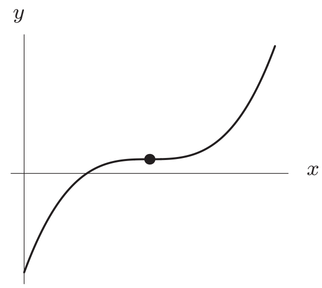


Figure 4.35

8. **(a)** Critical point.
(b) Inflection point.

9. **(a)** There was a critical point at 6 pm when the temperature was at a local minimum.
(b) The graph of temperature was decreasing but concave up in the morning. In the early afternoon the graph was decreasing but concave down. There was an inflection point at noon when the northerly wind started blowing. By 6 pm when the temperature was at a local minimum, the graph must have been concave up again so there must have been a second inflection point between noon and 6 pm. See Figure 4.36.

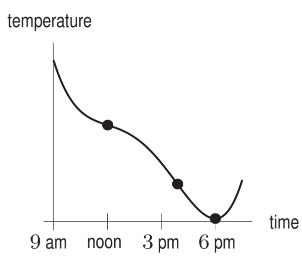


Figure 4.36

10. We have $f'(x) = 3x^2 - 36x - 10$ and $f''(x) = 6x - 36$. The inflection point occurs where $f''(x) = 0$, hence $6x - 36 = 0$. The inflection point is at $x = 6$. A graph is shown in Figure 4.37.

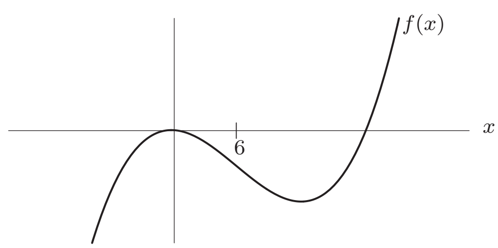


Figure 4.37

11. A critical point will occur whenever $f'(x) = 0$ or f' is undefined. Since $f'(x)$ is always defined, we set

$$\begin{aligned} f'(x) = 2x - 5 &= 0 \\ 2x &= 5 \\ x &= \frac{5}{2} \end{aligned}$$

To find the inflection points of $f(x)$, we find where $f''(x)$ goes from negative to positive or vice versa. For a point to satisfy this condition, it must have at least $f''(x) = 0$ or f'' undefined. Since $f''(x) = 2$, we know $f''(x)$ is always defined and never equal to zero, so $f(x)$ cannot have inflection points.

So $x = \frac{5}{2}$ is a critical point of $f(x)$, and there are no inflection points.

To identify the nature of the critical point $x = \frac{5}{2}$ that we have found, we can look at a graph of $f(x)$ for values of x near the critical point. Such a graph is shown in Figure 4.38. From the graph we see that $f(\frac{5}{2})$ is a local minimum of f.

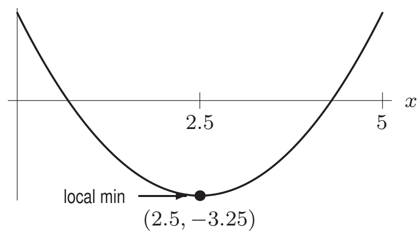


Figure 4.38

12. A critical point will occur whenever $f'(x) = 0$ or f' is undefined. Since $f'(x)$ is always defined, we set

$$f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 0.$$

Factoring, we get

$$f'(x) = 3(x - 1)(x + 1) = 0.$$

So, $x = 1$ or $x = -1$. To find the inflection points of $f(x)$, we find where $f''(x)$ goes from negative to positive or vice versa. For a point to satisfy this condition, it must have at least $f''(x) = 0$ or f'' undefined. Since $f''(x) = 6x$, we know $f''(x)$ is always defined. It is zero when $6x = 0$, so $x = 0$. Since $f''(x) = 6x$ is negative for $x < 0$ and positive for $x > 0$, $x = 0$ must be an inflection point for $f(x)$.

So $x = 1$ and $x = -1$ are critical points of $f(x)$, and $x = 0$ is an inflection point for $f(x)$.

To identify the nature of the critical points $x = 1$ and $x = -1$ that we have found, we can look at a graph of $f(x)$ for values of x near the critical points. Such a graph is shown in Figure 4.39. From the graph we see that $f(-1)$ is a local maximum of f and $f(1)$ is a local minimum of f.

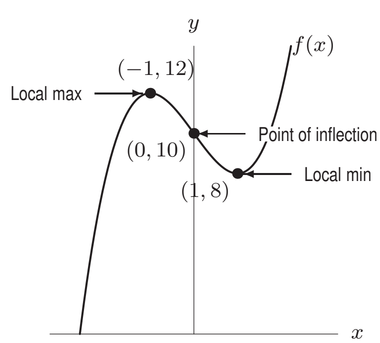


Figure 4.39

13. $f'(x) = 6x^2 + 6x - 36$. To find critical points, we set $f'(x) = 0$. Then

$$6(x^2 + x - 6) = 6(x + 3)(x - 2) = 0.$$

Therefore, the critical points of f are $x = -3$ and $x = 2$. To find the inflection points of $f(x)$ we look for the points at which $f''(x)$ goes from negative to positive or vice-versa. Since $f''(x) = 12x + 6$, $x = -1/2$ is an inflection point. From Figure 4.40, we see the critical point $x = -3$ is a local maximum and the critical point $x = 2$ is a local minimum.

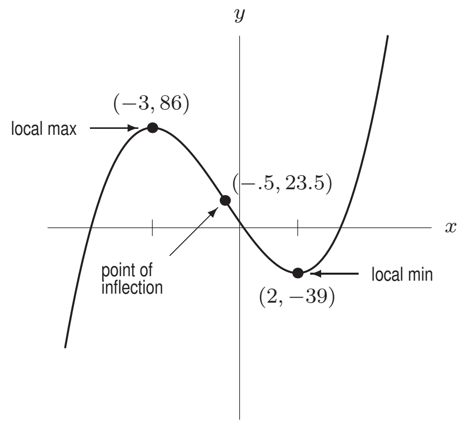


Figure 4.40

14. A critical point will occur whenever $f'(x) = 0$ or f' is undefined. Since $f'(x)$ is always defined, we set

$$f'(x) = \frac{3x^2}{6} + \frac{2x}{4} - 1 = \frac{x^2}{2} + \frac{x}{2} - 1 = 0.$$

Factoring, we get

$$f'(x) = \frac{1}{2}(x^2 + x - 2) = \frac{1}{2}(x - 1)(x + 2) = 0.$$

So $x = 1$ or $x = -2$. To find the inflection points of $f(x)$, we find where $f''(x)$ goes from negative to positive or vice versa. For a point to satisfy this condition, it must have at least $f''(x) = 0$ or f'' undefined. Since $f''(x) = x + \frac{1}{2}$, we know $f''(x)$ is always defined and is zero when $x = -\frac{1}{2}$. Since $f''(x) = x + \frac{1}{2}$ is negative for $x < -\frac{1}{2}$ and positive for $x > -\frac{1}{2}$, $x = -\frac{1}{2}$ must be an inflection point for $f(x)$.

So $x = 1$ and $x = -2$ are critical points of $f(x)$, and $x = -\frac{1}{2}$ is an inflection point for $f(x)$.

To identify the nature of the critical points $x = 1$ and $x = -2$ that we have found, we can look at a graph of $f(x)$ for values of x near the critical points. Such a graph is shown in Figure 4.41. From the graph we see that $f(-2)$ is a local maximum of f and $f(1)$ is a local minimum of f.

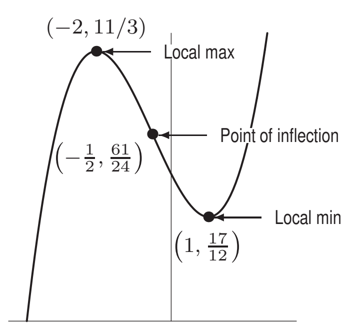


Figure 4.41

15. A critical point will occur whenever $f'(x) = 0$ or f' is undefined. Since $f'(x)$ is always defined, we set

$$f'(x) = 4x^3 - 4x = 4(x^3 - x) = 0.$$

Factoring, we get,

$$f'(x) = 4x(x^2 - 1) = 4x(x-1)(x+1) = 0.$$

So $x = -1$, $x = 0$ or $x = 1$. To find the inflection points of $f(x)$, we find where $f''(x)$ goes from negative to positive or vice versa. For a point to satisfy this condition, it must have at least $f''(x) = 0$ or f'' undefined. Since $f''(x) = 12x^2 - 4$, we know $f''(x)$ is always defined. Factoring, we get

$$f''(x) = 12x^2 - 4 = 12\left(x^2 - \frac{1}{3}\right) = 12\left(x - \frac{1}{\sqrt{3}}\right)\left(x + \frac{1}{\sqrt{3}}\right) = 0,$$

which is zero when $x = -1/\sqrt{3}$ or $x = 1/\sqrt{3}$. Putting in numbers close to $-1/\sqrt{3}$, we see that $f''(x)$ is positive for $x < -1/\sqrt{3}$ and negative for $x > -1/\sqrt{3}$, so $x = -1/\sqrt{3}$ must be an inflection point for $f(x)$. Putting in numbers close to $1/\sqrt{3}$, we see that $f''(x)$ is negative for $x < 1/\sqrt{3}$ and positive for $x > 1/\sqrt{3}$, so $x = 1/\sqrt{3}$ must be an inflection point for $f(x)$.

So $x = -1$, $x = 0$ and $x = 1$ are critical points of $f(x)$, and $x = -1/\sqrt{3}$ and $x = 1/\sqrt{3}$ are inflection points for $f(x)$.

To identify the nature of the critical points $x = -1$, $x = 0$ and $x = 1$ that we have found, we can look at a graph of $f(x)$ for values of x near the critical points. Such a graph is shown in Figure 4.42. From the graph we see that $f(-1)$ is a local minimum of f, $f(0)$ is a local maximum of f and $f(1)$ is a local minimum of f.

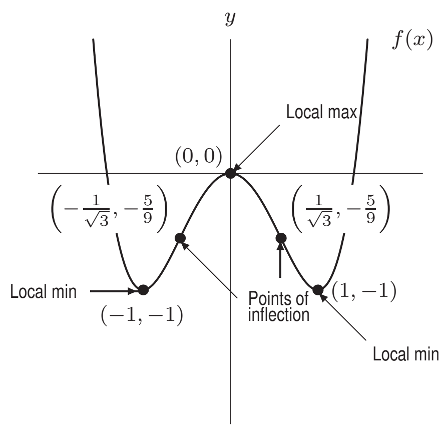


Figure 4.42

16. $f'(x) = 12x^3 - 12x^2$. To find critical points, we set $f'(x) = 0$. This implies $12x^2(x-1) = 0$. So the critical points of f are $x = 0$ and $x = 1$. To find the inflection points of $f(x)$ we look for points at which $f''(x)$ goes from negative to positive or vice-versa. At any such point $f''(x)$ is either zero or undefined. Since $f''(x) = 36x^2 - 24x = 12x(3x-2)$, our candidate points are $x = 0$ and $x = 2/3$. Both $x = 0$ and $x = 2/3$ are inflection points, since at both, $f''(x)$ changes sign.

From Figure 4.43, we see that the critical point $x = 1$ is a local minimum and the critical point $x = 0$ is neither a local maximum nor a local minimum.

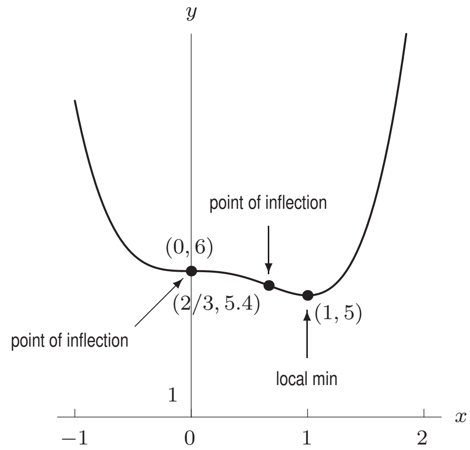


Figure 4.43

17. The derivative of $f(x)$ is $f'(x) = 4x^3 - 16x$. The critical points of $f(x)$ are points at which $f'(x) = 0$. Factoring, we get

$$\begin{aligned} 4x^3 - 16x &= 0 \\ 4x(x^2 - 4) &= 0 \\ 4x(x-2)(x+2) &= 0 \end{aligned}$$

So the critical points of f will be $x = 0, x = 2,$ and $x = -2$.

To find the inflection points of f we look for the points at which $f''(x)$ changes sign. At any such point $f''(x)$ is either zero or undefined. Since $f''(x) = 12x^2 - 16$ our candidate points are $x = \pm 2/\sqrt{3}$. At both of these points $f''(x)$ changes sign, so both of these points are inflection points.

From Figure 4.44, we see that the critical points $x = -2$ and $x = 2$ are local minima and the critical point $x = 0$ is a local maximum.

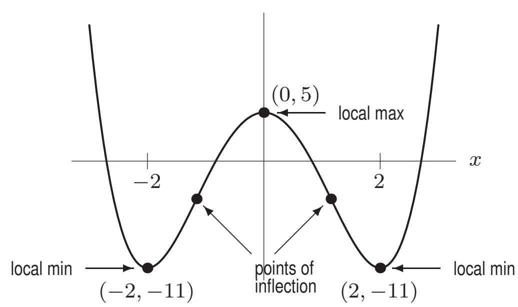


Figure 4.44

18. We first find the critical points of $f(x) = x^4 - 4x^3 + 10$. Since $f'(x) = 4x^3 - 12x^2$, setting the derivative equal to 0 and factoring yields

$$\begin{aligned} 4x^3 - 12x^2 &= 0 \\ 4x^2(x-3) &= 0 \end{aligned}$$

So $x = 0$ and $x = 3$ are the critical points of $f(x)$.

We now find the inflection points of $f(x)$. Since

$$f''(x) = 12x^2 - 24x,$$

setting the second derivative equal to 0 and factoring yields

$$12x^2 - 24x = 0$$
$$12x(x-2) = 0.$$

So $x = 0$ and $x = 2$ may be points of inflection of $f(x)$. Since $f''(x)$ changes sign at both $x = 0$ and $x = 2$, both are points of inflection for $f(x)$. From Figure 4.45, we see that the critical point $x = 3$ is a local minimum and the critical point $x = 0$ is neither a local minimum nor a local maximum.

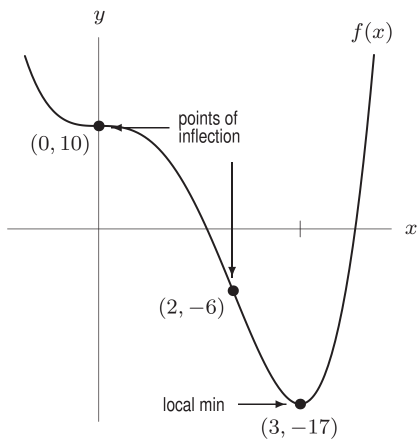


Figure 4.45

19. A critical point will occur whenever $f'(x) = 0$ or f' is undefined. Since $f'(x)$ is always defined, we set

$$f'(x) = 5x^4 - 20x^3 = 5(x^4 - 4x^3) = 0.$$

We get

$$f'(x) = 5x^3(x-4) = 0,$$

so $x = 0$ or $x = 4$. To find the inflection points of $f(x)$, we find where $f''(x)$ goes from negative to positive or vice versa. For a point to satisfy this condition, it must have at least $f''(x) = 0$ or f'' undefined. Since $f''(x) = 20x^3 - 60x^2$, we know $f''(x)$ is always defined. It is zero when $20x^3 - 60x^2 = 0$. Factoring, we get

$$f''(x) = 20(x^3 - 3x^2) = 20x^2(x-3) = 0.$$

So $x = 0$ or $x = 3$. Putting in values of x very close to zero, we see that $f''(x) = 20x^2(x-3)$ is negative for $x < 0$ and negative for $x > 0$, so $x = 0$ is not an inflection point for $f(x)$. Putting in values close to 3, we see that $f''(x) = 20x^2(x-3)$ is negative for $x < 3$ and positive for $x > 3$, so $x = 3$ is an inflection point for $f(x)$.

So $x = 0$ and $x = 4$ are critical points of $f(x)$, and $x = 3$ is an inflection point for $f(x)$.

To identify the nature of the critical points $x = 0$ and $x = 4$ that we have found, we can look at a graph of $f(x)$ for values of x near the critical points. Such a graph is shown in Figure 4.46. From the graph we see that $f(0)$ is a local maximum of f and $f(4)$ is a local minimum of f.

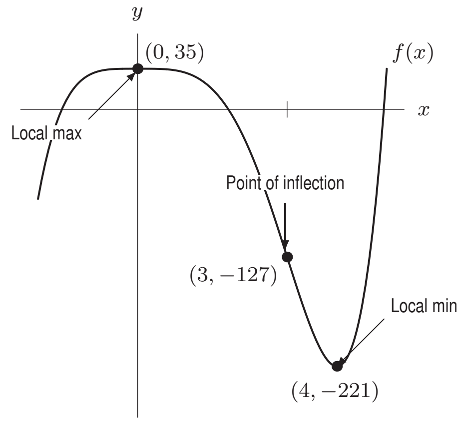


Figure 4.46

20. We see that $f'(x) = 15x^4 - 15x^2$ and $f''(x) = 60x^3 - 30x$. Since $f'(x) = 15x^2(x^2 - 1)$, the critical points are $x = 0, \pm 1$.

To find possible inflection points, we determine when $f''(x) = 0$. Since $f''(x) = 30x(2x^2 - 1)$, we have $f''(x) = 0$ when $x = 0$ or $x = \pm 1/\sqrt{2}$. Since $f''(x)$ changes sign at each of these points, all are inflection points.

We see from Figure 4.47 that the critical point $x = -1$ is a local maximum, $x = 1$ is a local minimum, and $x = 0$ is neither.

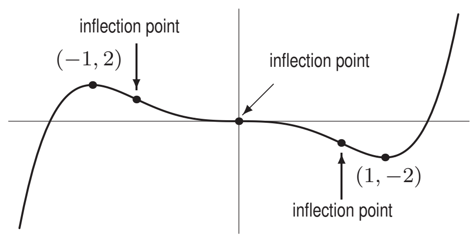
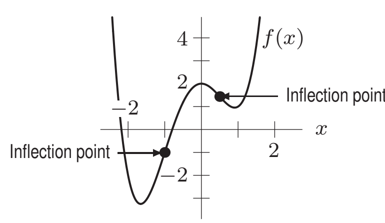


Figure 4.47

21. From the graph of $f(x)$ in the figure below, we see that the function must have two inflection points. We calculate $f'(x) = 4x^3 + 3x^2 - 6x$, and $f''(x) = 12x^2 + 6x - 6$. Solving $f''(x) = 0$ we find that:

$$x_1 = -1 \quad \text{and} \quad x_2 = \frac{1}{2}.$$

Since $f''(x) > 0$ for $x < x_1$, $f''(x) < 0$ for $x_1 < x < x_2$, and $f''(x) > 0$ for $x_2 < x$, it follows that both points are inflection points.



22. **(a)** We set the derivative equal to zero and solve for x to find critical points:

$$\begin{aligned} f'(x) &= 4x^3 - 4ax = 0 \\ 4x(x^2 - a) &= 0. \end{aligned}$$

We see that there are three critical points:

$$\text{Critical points:} \quad x = 0, \quad x = \sqrt{a}, \quad x = -\sqrt{a}.$$

To find possible inflection points, we set the second derivative equal to zero and solve for x:

$$f''(x) = 12x^2 - 4a = 0.$$

There are two possible inflection points:

$$\text{Possible inflection points:} \quad x = \sqrt{\frac{a}{3}}, \quad x = -\sqrt{\frac{a}{3}}.$$

To see if these are inflection points, we determine whether concavity changes by evaluating f'' at values on either side of each of the potential inflection points. We see that

$$f''(-2\sqrt{\frac{a}{3}}) = 12(4\frac{a}{3}) - 4a = 16a - 4a = 12a > 0,$$

so f is concave up to the left of $x = -\sqrt{a/3}$. Also,

$$f''(0) = -4a < 0,$$

so f is concave down between $x = -\sqrt{a/3}$ and $x = \sqrt{a/3}$. Finally, we see that

$$f''(2\sqrt{\frac{a}{3}}) = 12(4\frac{a}{3}) - 4a = 16a - 4a = 12a > 0,$$

so f is concave up to the right of $x = \sqrt{a/3}$. Since $f(x)$ changes concavity at $x = \sqrt{a/3}$ and $x = -\sqrt{a/3}$, both points are inflection points.

(b) The only positive critical point is at $x = \sqrt{a}$, so to have a critical point at $x = 2$, we substitute:

$$x = \sqrt{a}$$
$$2 = \sqrt{a}$$
$$a = 4.$$

Since the critical point is at the point $(2, 5)$, we have

$$\begin{aligned} f(2) &= 5 \\ 2^4 - 2(4)2^2 + b &= 5 \\ 16 - 32 + b &= 5 \\ b &= 21. \end{aligned}$$

The function is $f(x) = x^4 - 8x^2 + 21$.

(c) We have seen that $a = 4$, so the inflection points are at $x = \sqrt{4/3}$ and $x = -\sqrt{4/3}$.

23. See Figure 4.48.

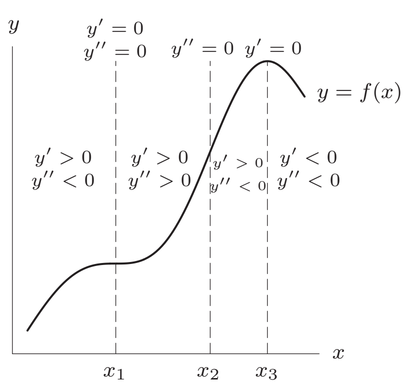


Figure 4.48

24. See Figure 4.49.

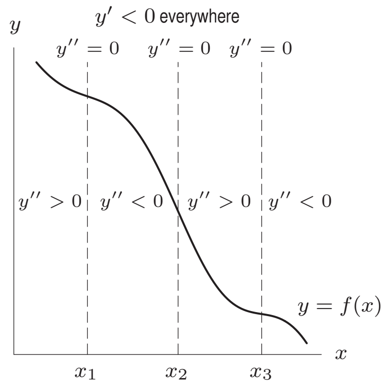


Figure 4.49

25. See Figure 4.50.

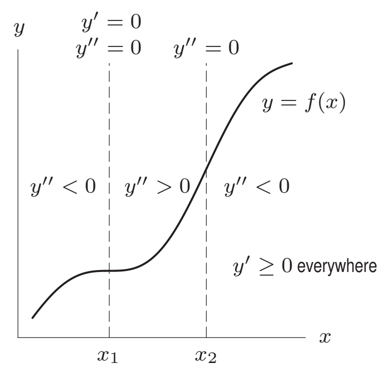


Figure 4.50

26. See Figure 4.51.

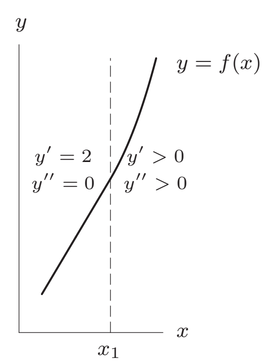
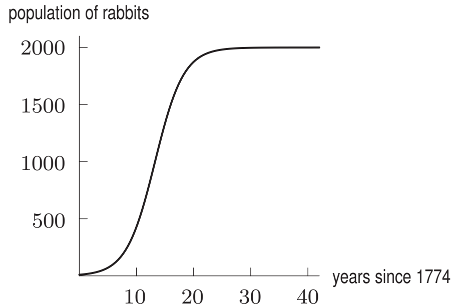


Figure 4.51

27. **(a)**

$$P(t) = \frac{2000}{1 + e^{(5.3-0.4t)}}$$



From the graph we see that the population levels off at about 2000 rabbits.

(b) The population appears to have been growing fastest when there were about 1000 rabbits, approximately 13 years after Captain Cook left the original rabbits on the island.

(c) The inflection point coincides with the point of most rapid increase of the rabbit population, that is, the inflection point occurs approximately 13 years after 1774 when the rabbit population is about 1000 rabbits.

(d) The rabbits reproduce quickly, so their population initially grew very rapidly. Limited food and space availability and perhaps predators on the island probably account for the population being unable to grow past 2000.

28. **(a)** Since the volume of water in the container is proportional to its depth, and the volume is increasing at a constant rate,

$$d(t) = \text{ Depth at time } t = Kt,$$

where K is some positive constant. So the graph is linear, as shown in Figure 4.52. Since initially no water is in the container, we have $d(0) = 0$, and the graph starts from the origin.

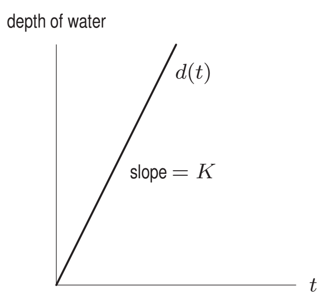


Figure 4.52

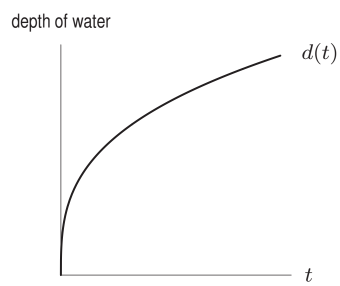


Figure 4.53

(b) As time increases, the additional volume needed to raise the water level by a fixed amount increases. Thus, although the depth, $d(t)$, of water in the cone at time t, continues to increase, it does so more and more slowly. This means $d'(t)$ is positive but decreasing, i.e., $d(t)$ is concave down. See Figure 4.53.

29. See Figure 4.54.

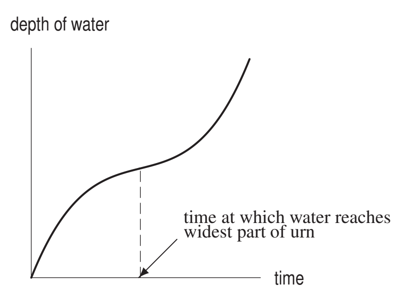


Figure 4.54

30. See Figure 4.55.

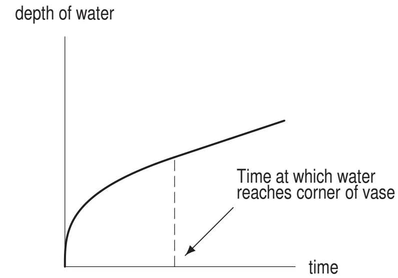


Figure 4.55

31. See Figure 4.56.

Suppose t_1 is the time to fill the left side to the top of the middle ridge. Since the container gets wider as you go up, the rate dH/dt decreases with time. Therefore, for $0 \le t \le t_1$, graph is concave down.

At $t = t_1$, water starts to spill over to right side and so depth of left side does not change. It takes as long for the right side to fill to the ridge as the left side, namely t_1. Thus the graph is horizontal for $t_1 \le t \le 2t_1$.

For $t \ge 2t_1$, water level is above the central ridge. The graph is climbing because the depth is increasing, but at a slower rate than for $t \le t_1$ because the container is wider. The graph is concave down because width is increasing with depth. Time t_3 represents the time when container is full.

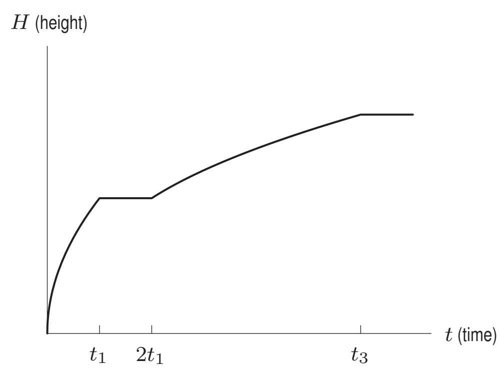


Figure 4.56

32. **(a)** The concavity changes at t_1 and t_3, as shown in Figure 4.57.

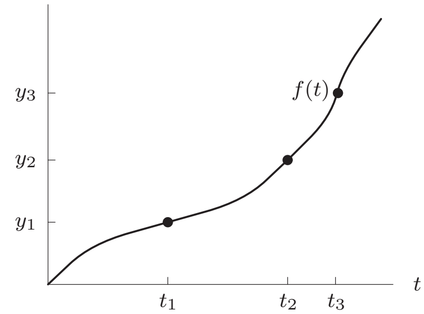


Figure 4.57

(b) $f(t)$ grows most quickly where the vase is skinniest (at y_3) and most slowly where the vase is widest (at y_1). The diameter of the widest part of the vase looks to be about 4 times as large as the diameter at the skinniest part. Since the area of a cross section is given by πr^2, where r is the radius, the ratio between areas of cross sections at these two places is about 4^2, so the growth rates are in a ratio of about 1 to 16 (the wide part being 16 times slower).

33. Since the x^3 term has coefficient 1, the polynomial is of the form

$$y = x^3 + ax^2 + bx + c.$$

Differentiating gives

$$\frac{dy}{dx} = 3x^2 + 2ax + b.$$

There is a critical point at $x = 2$, so $dy/dx = 0$ at $x = 2$. Thus

$$\left.\frac{dy}{dx}\right|_{x=2} = 3(2^2) + 2a(2) + b = 12 + 4a + b = 0, \text{ so } 4a + b = -12.$$

We take the second derivative to look for the inflection point. We find

$$\frac{d^2y}{dx^2} = 6x + 2a,$$

and for an inflection point at $x = 1$, we have $6 + 2a = 0$, so $a = -3$. We now use $a = -3$ and the relationship $4a + b = -12$, which gives $4(-3) + b = -12$, so $b = 0$.

We now have

$$y = x^3 - 3x^2 + c,$$

and using the point (1,4) gives

$$\begin{aligned} 1 - 3 + c &= 4, \\ c &= 6. \end{aligned}$$

Thus, $y = x^3 - 3x^2 + 6$.

34. Differentiating $y = bxe^{-ax}$ gives

$$\frac{dy}{dx} = be^{-ax} - abxe^{-ax} = be^{-ax}(1 - ax).$$

Since we have a critical point at $x = 3$, we know that $1 - 3a = 0$, so $a = 1/3$.

If $b > 0$, the first derivative goes from positive values to the left of $x = 3$ to negative values on the right of $x = 3$, so we know this critical point is a local maximum. Since the function value at this local maximum is 6, we have

$$6 = 3be^{-3/3} = \frac{3b}{e},$$

so $b = 2e$ and

$$y = 2xe^{1-x/3}.$$

35. (a) An inflection point occurs whenever the concavity of $f(x)$ changes. If the graph shown is that of $f(x)$, then an inflection point will occur whenever its concavity changes, or equivalently when the tangent line moves from above the curve to below or vice-versa. Such points are shown in Figure 4.58.

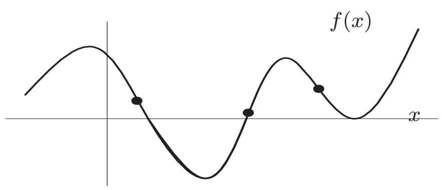


Figure 4.58

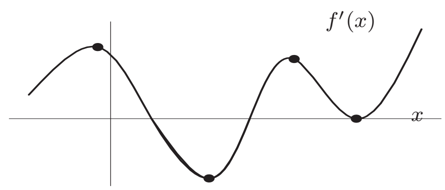


Figure 4.59

(b) To find inflection points of the function f we must find points where f'' changes sign. However, because f'' is the derivative of f', any point where f'' changes sign will be a local maximum or minimum on the graph of f'. Such points are shown in Figure 4.59.

(c) The inflection points of f are the points where f'' changes sign. If the graph shown is that of $f''(x)$, then we are looking for where the given graph passes from above the x-axis to below, or vice versa. Such points are shown in Figure 4.60:

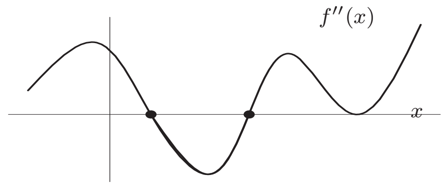
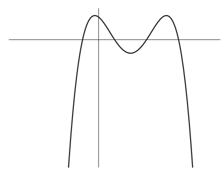


Figure 4.60

36. **(a)** This is one of many possible graphs.



(b) Since f must have a bump between each pair of zeros, f could have at most four zeros.
(c) f could well have no zeros at all. To see this, consider the graph of the above function shifted vertically downward.
(d) f must have at least two inflection points. Since f has 3 maxima or minima, it has 3 critical points. Consequently f' will have 3 corresponding zeros. Between each consecutive pair of these zeroes f' must have a local maximum or minimum. Thus f' will have one local maximum and one local minimum, which implies that f'' will have two zeros. These values, where the second derivative is zero, correspond to points of inflection on the graph of f.
(e) The 3 critical points are zeros of f', so degree$(f') \geq 3$. Thus degree$(f) \geq 4$.
(f) For example:

$$f(x) = -(x+1)(x-1)(x-3)(x-5)$$

will look something like the graph in part (a). Many other answers are possible.

Solutions for Section 4.3

1. See Figure 4.61.

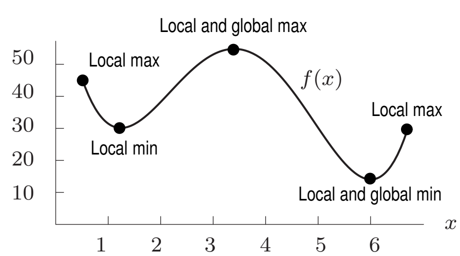


Figure 4.61

2. See Figure 4.62.

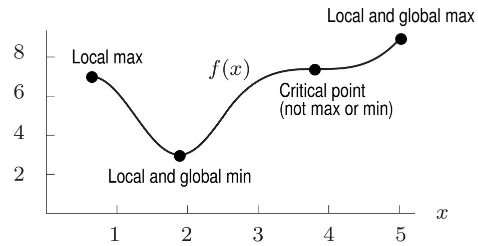


Figure 4.62

3. **(a)** (IV)
(b) (I)
(c) (III)
(d) (II)

4. See Figure 4.63.

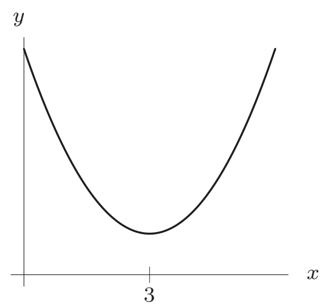


Figure 4.63

5. See Figure 4.64.

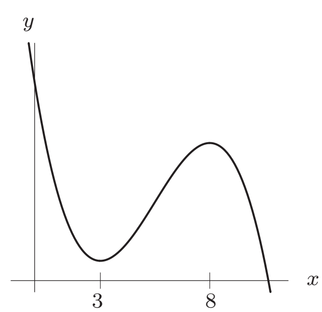


Figure 4.64

6. See Figure 4.65.

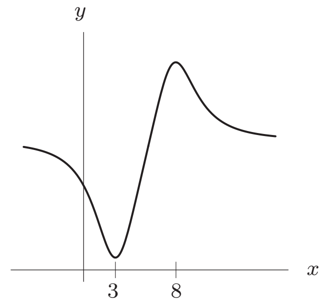


Figure 4.65

7. See Figure 4.66.

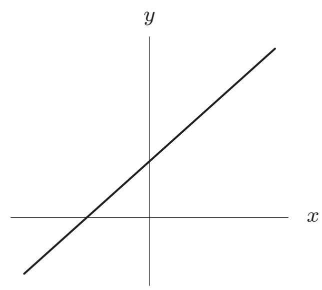


Figure 4.66

8. True. If the maximum is not at an endpoint, then it must be at critical point of f. But $x = 0$ is the only critical point of $f(x) = x^2$ and it gives a minimum, not a maximum.

9. See Figure 4.67.

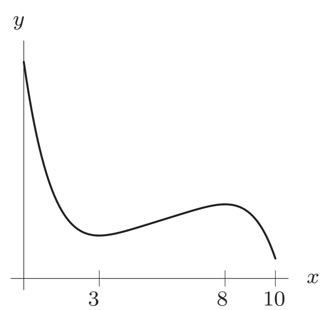


Figure 4.67

10. See Figure 4.68.

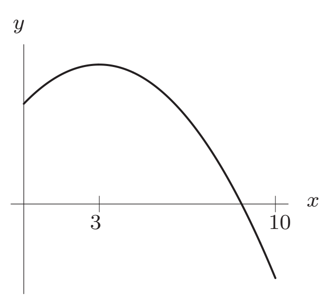


Figure 4.68

11. See Figure 4.69.

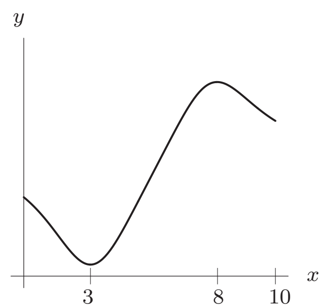


Figure 4.69

12. See Figure 4.70.

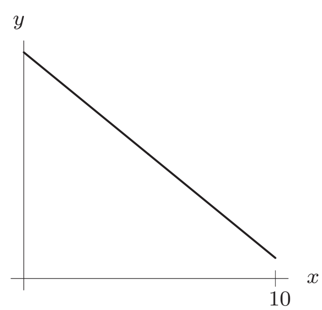


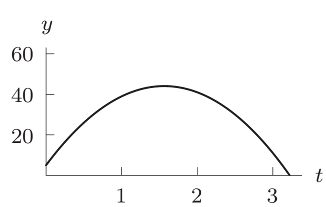
Figure 4.70

13. We want to maximize the height, y, of the grapefruit above the ground, as shown in the figure below. Using the derivative we can find exactly when the grapefruit is at the highest point. We can think of this in two ways. By common sense, at the peak of the grapefruit's flight, the velocity, dy/dt, must be zero. Alternately, we are looking for a global maximum of y, so we look for critical points where $dy/dt = 0$. We have

$$\frac{dy}{dt} = -32t + 50 = 0 \qquad \text{and so} \qquad t = \frac{-50}{-32} \approx 1.56 \text{ sec.}$$

Thus, we have the time at which the height is a maximum; the maximum value of y is then

$$y \approx -16(1.56)^2 + 50(1.56) + 5 = 44.1 \text{ feet.}$$



14. **(a)** We see in Figure 4.71 that the maximum occurs between $x = 1$ and $x = 2$ and the maximum value of y is about $y = 30$.

(b) The maximum value occurs at a critical point, so we find all critical points of y:

$$\frac{dy}{dx} = 18 - 10x = 0$$
$$x = 1.8.$$

Since y is a quadratic polynomial with negative leading coefficient, this critical point gives a local maximum, which is also a global maximum. We find the maximum value of y by substituting $x = 1.8$:

$$\text{Maximum value of } y = 12 + 18(1.8) - 5(1.8)^2 = 28.2.$$

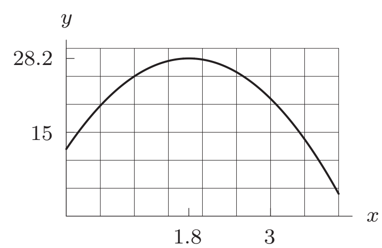


Figure 4.71

15. Using a computer to graph the function, $f(x) = x^3 - e^x$, and its derivative, $f'(x) = 3x^2 - e^x$, we find that the derivative crosses the x-axis three times in the interval $-1 \le x \le 4$ and twice in the interval $-3 \le x \le 2$. See Figures 4.72 and 4.73.

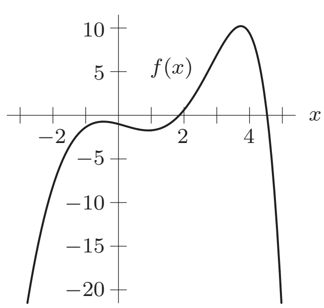


Figure 4.72: Function, f

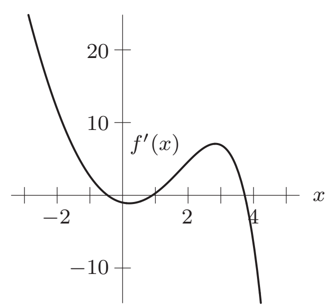


Figure 4.73: Derivative, f'

Through trial and error, we obtain approximations: local maximum at $x \approx 3.73$, local minimum at $x \approx 0.91$ and local maximum at $x \approx -0.46$. We can use the approximate values at these points, along with a picture as a guide, to find the global maximum and minimum on any interval.

(a) We find the global minimum and maximum on the interval $-1 \le x \le 4$ by examining the critical points above as well as the endpoints. Since $f(-1) = -1.3679$, $f(-0.46) = -0.7286$, $f(0.91) = -1.7308$, $f(3.73) = 10.2160$, $f(4) = 9.4018$, we see $x \approx 0.91$ gives a global minimum on the interval and $x \approx 3.73$ gives a global maximum.

(b) We find the global minimum and maximum on the interval $-3 \le x \le 2$ by examining the critical points above as well as the endpoints. Since $f(-3) = -27.0498$, $f(-0.46) = -0.7286$, $f(0.91) = -1.7308$, $f(2) = 0.6109$, we see $x = -3$ gives a global min and $x = 2$ a global max. (Even though $x \approx -0.46$ gives a local maximum, it does not give the greatest maximum on this interval; even though $x \approx 0.91$ gives a local minimum, it is not the smallest minimum on this interval.)

16. **(a)** The maximum photosynthesis rate occurs when $t \approx 50$ days.

(b) The leaf is always growing, since the photosynthesis rate is always positive. The leaf is growing fastest when the photosynthesis rate is greatest, that is $t \approx 50$ days.

17. **(a)** We have

$$T(D) = \left(\frac{C}{2} - \frac{D}{3}\right) D^2 = \frac{CD^2}{2} - \frac{D^3}{3},$$

and

$$\frac{dT}{dD} = CD - D^2 = D(C - D).$$

Since, by this formula, dT/dD is zero when $D = 0$ or $D = C$, negative when $D > C$, and positive when $D < C$, we have (by the first derivative test) that the temperature change is maximized when $D = C$.

(b) The sensitivity is $dT/dD = CD - D^2$; its derivative is $d^2T/dD^2 = C - 2D$, which is zero if $D = C/2$, negative if $D > C/2$, and positive if $D < C/2$. Thus by the first derivative test the sensitivity is maximized at $D = C/2$.

18. **(a)** Differentiating $f(x) = x^3 - 3x^2$ produces $f'(x) = 3x^2 - 6x$. A second differentiation produces $f''(x) = 6x - 6$.

(b) $f'(x)$ is defined for all x and $f'(x) = 0$ when $x = 0, 2$. Thus 0 and 2 are the critical points of f.

(c) $f''(x)$ is defined for all x and $f''(x) = 0$ when $x = 1$. When $x < 1$, $f''(x) < 0$ and when $x > 1$, $f''(x) > 0$. Thus the concavity of the graph of f changes at $x = 1$. Hence $x = 1$ is an inflection point.

(d) $f(-1) = -4$, $f(0) = 0$, $f(2) = -4$, $f(3) = 0$. So f has a local maximum at $x = 0$, a local minimum at $x = 2$, global maxima at $x = 0$ and $x = 3$, and global minima at $x = -1$ and $x = 2$.

(e) Plotting the function $f(x)$ for $-1 \le x \le 3$ gives the graph shown in Figure 4.74.

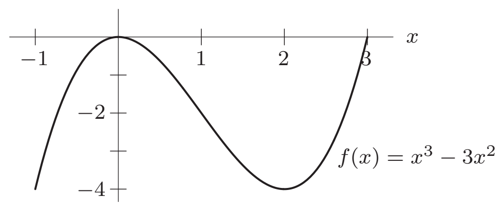


Figure 4.74

19. **(a)** Differentiating $f(x) = 2x^3 - 9x^2 + 12x + 1$ produces $f'(x) = 6x^2 - 18x + 12$. A second differentiation produces $f''(x) = 12x - 18$.

(b) $f'(x)$ is defined for all x and $f'(x) = 0$ when $x = 1, 2$. Thus $x = 1, 2$ are critical points.

(c) $f''(x)$ is defined for all x and $f''(x) = 0$ when $x = \frac{3}{2}$. Since the concavity of f changes at this point, it is an inflection point.

(d) $f(-0.5) = -7.5$, $f(3) = 10$, $f(1) = 6$, $f(2) = 5$. So f has a local maximum at $x = 1$ and a local minimum at $x = 2$, a global maximum at $x = 3$ and a global minimum at $x = -0.5$

(e) Plotting the function $f(x)$ for $-0.5 \le x \le 3$ gives the graph shown in Figure 4.75.

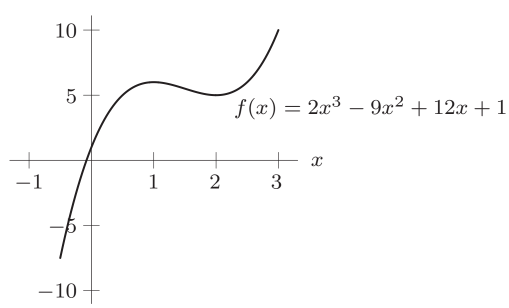


Figure 4.75

20. **(a)** Differentiating $f(x) = x^3 - 3x^2 - 9x + 15$ produces $f'(x) = 3x^2 - 6x - 9$. A second differentiation produces $f''(x) = 6x - 6$.

(b) $f'(x)$ is defined for all x and $f'(x) = 0$ when $x = -1, 3$. Thus $x = -1, 3$ are critical points.

(c) $f''(x)$ is defined for all x and $f''(x) = 0$ when $x = 1$. Since the concavity of f changes at this point, it is an inflection point.

(d) $f(-5) = -140$, $f(4) = -5$, $f(-1) = 2$, $f(3) = -12$. So f has a global maximum at $x = -1$ and a global minimum at $x = -5$, and a local minimum at $x = 3$

(e) Plotting the function $f(x)$ for $-5 \le x \le 4$ gives the graph shown in Figure 4.76:

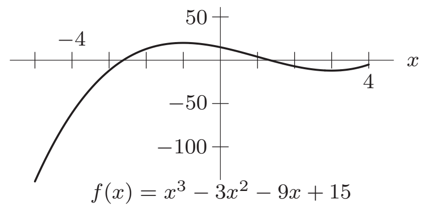


Figure 4.76

21. (a) Differentiating $f(x) = x + \sin x$ produces $f'(x) = 1 + \cos x$. A second differentiation produces $f''(x) = -\sin x$.

(b) $f'(x)$ is defined for all x and $f'(x) = 0$ when $x = \pi$. Thus π is the critical point of f.

(c) $f''(x)$ is defined for all x and $f''(x) = 0$ when $x = 0$, $x = \pi$ and $x = 2\pi$. Since the concavity of f changes at each of these points they are all inflection points.

(d) $f(0) = 0$, $f(2\pi) = 2\pi$, $f(\pi) = \pi$. So f has a global minimum at $x = 0$ and a global maximum at $x = 2\pi$.

(e) Plotting the function $f(x)$ for $0 \le x \le 2\pi$ gives the graph shown in Figure 4.77:

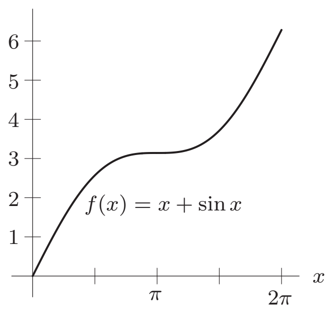


Figure 4.77

22. (a) Differentiating $f(x) = e^{-x}\sin x$ produces $f'(x) = -e^{-x}\sin x + e^{-x}\cos x$. A second differentiation produces $f''(x) = -2e^{-x}\cos x$.

(b) $f'(x)$ is defined for all x and $f'(x) = 0$ when $x = \pi/4$ and when $x = 5\pi/4$. Thus $\pi/4$ and $5\pi/4$ are the critical points of f.

(c) $f''(x)$ is defined for all x and $f''(x) = 0$ when $x = \pi/2, 3\pi/2$. Since the concavity of f changes at both of these points they are both inflection points.

(d) $f(0) = 0$, $f(2\pi) = 0$, $f(\pi/4) = e^{-\pi/4}\sin\pi/4$ and $f(5\pi/4) = e^{-5\pi/4}\sin(5\pi/4)$. So f has a global maximum at $x = \pi/4$ and a global minimum at $x = 5\pi/4$.

(e) Plotting the function $f(x)$ for $0 \le x \le 2\pi$ gives the graph shown in Figure 4.78.

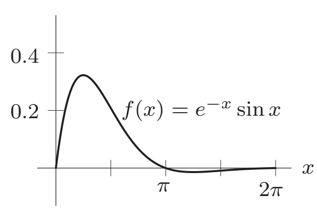


Figure 4.78

23. This is a parabola opening downward. We find the critical points by setting $g'(x) = 0$:

$$\begin{aligned} g'(x) = 4 - 2x &= 0 \\ x &= 2. \end{aligned}$$

Since $g'(x) > 0$ for $x < 2$ and $g'(x) < 0$ for $x > 2$, the critical point at $x = 2$ is a local maximum.

As $x \to \pm\infty$, the value of $g(x) \to -\infty$. Thus, the local maximum at $x = 2$ is a global maximum of $g(2) = 4 \cdot 2 - 2^2 - 5 = -1$. There is no global minimum. See Figure 4.79.

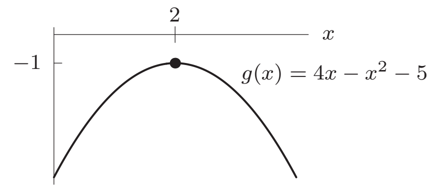


Figure 4.79

24. Differentiating gives

$$f'(x) = 1 - \frac{1}{x^2},$$

so the critical points satisfy

$$\begin{aligned} 1 - \frac{1}{x^2} &= 0 \\ x^2 &= 1 \\ x &= 1 \quad \text{(We want } x > 0\text{).} \end{aligned}$$

Since f' is negative for $0 < x < 1$ and f' is positive for $x > 1$, there is a local minimum at $x = 1$.

Since $f(x) \to \infty$ as $x \to 0^+$ and as $x \to \infty$, the local minimum at $x = 1$ is a global minimum; there is no global maximum. See Figure 4.80. The the global minimum is $f(1) = 2$.

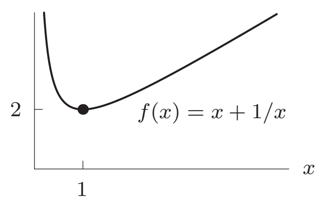


Figure 4.80

25. Differentiating using the product rule gives

$$g'(t) = 1 \cdot e^{-t} - te^{-t} = (1 - t)e^{-t},$$

so the critical point is $t = 1$.

Since $g'(t) > 0$ for $0 < t < 1$ and $g'(t) < 0$ for $t > 1$, the critical point is a local maximum.

As $t \to \infty$, the value of $g(t) \to 0$, and as $t \to 0^+$, the value of $g(t) \to 0$. Thus, the local maximum at $x = 1$ is a global maximum of $g(1) = 1e^{-1} = 1/e$. In addition, the value of $g(t)$ is positive for all $t > 0$; it tends to 0 but never reaches 0. Thus, there is no global minimum. See Figure 4.81.

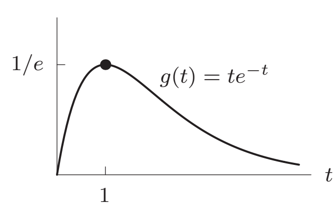


Figure 4.81

26. Differentiating gives

$$f'(x) = 1 - \frac{1}{x},$$

so the critical points satisfy

$$\begin{aligned} 1 - \frac{1}{x} &= 0 \\ \frac{1}{x} &= 1 \\ x &= 1. \end{aligned}$$

Since f' is negative for $0 < x < 1$ and f' is positive for $x > 1$, there is a local minimum at $x = 1$.

Since $f(x) \to \infty$ as $x \to 0^+$ and as $x \to \infty$, the local minimum at $x = 1$ is a global minimum; there is no global maximum. See Figure 4.82. Thus, the global minimum is $f(1) = 1$.

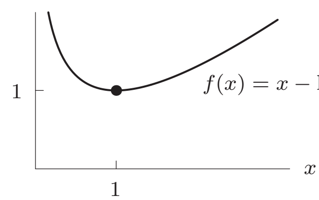


Figure 4.82

27. Differentiating using the quotient rule gives

$$f'(t) = \frac{1(1+t^2) - t(2t)}{(1+t^2)^2} = \frac{1-t^2}{(1+t^2)^2}.$$

The critical points are the solutions to

$$\begin{aligned} \frac{1-t^2}{(1+t^2)^2} &= 0 \\ t^2 &= 1 \\ t &= \pm 1. \end{aligned}$$

Since $f'(t) > 0$ for $-1 < t < 1$ and $f'(t) < 0$ otherwise, there is a local minimum at $t = -1$ and a local maximum at $t = 1$.

As $t \to \pm\infty$, we have $f(t) \to 0$. Thus, the local maximum at $t = 1$ is a global maximum of $f(1) = 1/2$, and the local minimum at $t = -1$ is a global minimum of $f(-1) = -1/2$. See Figure 4.83.

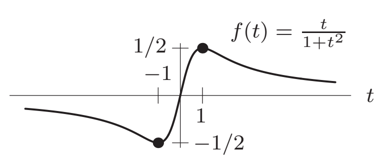


Figure 4.83

28. Differentiating using the product rule gives

$$\begin{aligned} f'(t) = 2\sin t \cos t \cdot \cos t - (\sin^2 t + 2)\sin t &= 0 \\ \sin t(2\cos^2 t - \sin^2 t - 2) &= 0 \\ \sin t(2(1 - \sin^2 t) - \sin^2 t - 2) &= 0 \\ \sin t(-3\sin^2 t) = -3\sin^3 t &= 0. \end{aligned}$$

Thus, the critical points are where $\sin t = 0$, so

$$t = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots.$$

Since $f'(t) = -3\sin^3 t$ is negative for $-\pi < t < 0$, positive for $0 < t < \pi$, negative for $\pi < t < 2\pi$, and so on, we find that $t = 0, \pm 2\pi, \ldots$ give local minima, while $t = \pm\pi, \pm 3\pi, \ldots$ give local maxima. Evaluating gives

$$f(0) = f(\pm 2\pi) = (0+2)1 = 2$$
$$f(\pm\pi) = f(\pm 3\pi) = (0+2)(-1) = -2.$$

Thus, the global maximum of $f(t)$ is 2, occurring at $t = 0, \pm 2\pi, \ldots$, and the global minimum of $f(t)$ is -2, occurring at $t = \pm\pi, \pm 3\pi, \ldots$. See Figure 4.84.

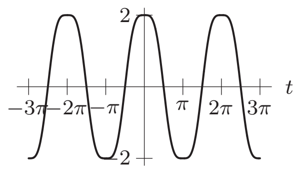


Figure 4.84

29. We find all critical points:

$$\frac{dy}{dx} = 2ax + b = 0$$
$$x = -\frac{b}{2a}.$$

Since y is a quadratic polynomial, its graph is a parabola which opens up if $a > 0$ and down if $a < 0$. The critical value is a maximum if $a < 0$ and a minimum if $a > 0$.

30. To find the value of w that minimizes S, we set dS/dw equal to zero and solve for w. To find dS/dw, we first solve for S:

$$S - 5pw = 3qw^2 - 6pq$$
$$S = 5pw + 3qw^2 - 6pq.$$

We now find the critical points:

$$\frac{dS}{dw} = 5p + 6qw = 0$$
$$w = -\frac{5p}{6q}.$$

There is one critical point. Since S is a quadratic function of w with a positive leading coefficient, the function has a minimum at this critical point.

31. One possible graph of g is in Figure 4.85.

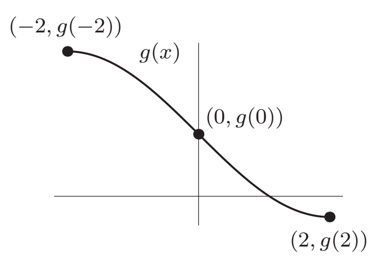


Figure 4.85

(a) From left to right, the graph of $g(x)$ starts "flat", decreases slowly at first then more rapidly, most rapidly at $x = 0$. The graph then continues to decrease but less and less rapidly until flat again at $x = 2$. The graph should exhibit symmetry about the point $(0, g(0))$.

(b) The graph has an inflection point at $(0, g(0))$ where the slope changes from negative and decreasing to negative and increasing.
(c) The function has a global maximum at $x = -2$ and a global minimum at $x = 2$.
(d) Since the function is decreasing over the interval $-2 \le x \le 2$

$$g(-2) = 5 > g(0) > g(2).$$

Since the function appears symmetric about $(0, g(0))$, we have

$$g(-2) - g(0) = g(0) - g(2).$$

32. To find the minimum value of E, we solve

$$\begin{aligned} \frac{dE}{dF} = 0.25 - \frac{2(1.7)}{F^3} &= 0 \\ F^3 &= \frac{2(1.7)}{0.25} \\ F &= \left(\frac{2(1.7)}{0.25}\right)^{1/3} = 2.4 \text{ hours.} \end{aligned}$$

Since dE/dF is negative for $F < 2.4$ and dE/dF is positive for $F > 2.4$, a foraging time of $F = 2.4$ hours gives a local minimum. This is the global minimum for $F > 0$. See Figure 4.86.

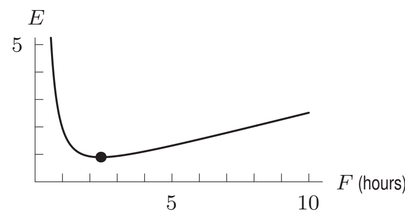
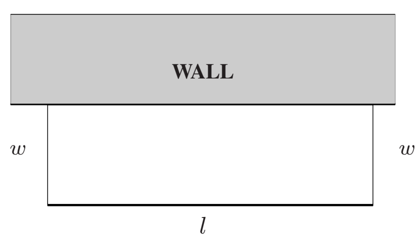


Figure 4.86

33. Let w and l be the width and length, respectively, of the rectangular area you wish to enclose. Then

$$\begin{aligned} w + w + l &= 100 \text{ feet} \\ l &= 100 - 2w \\ \text{Area} = w \cdot l &= w(100 - 2w) = 100w - 2w^2 \end{aligned}$$



To maximize area, we solve $A' = 0$ to find critical points. This gives $A' = 100 - 4w = 0$, so $w = 25$, $l = 50$. So the area is $25 \cdot 50 = 1250$ square feet. This is a local maximum by the second derivative test because $A'' = -4 < 0$. Since the graph of A is a parabola, the local maximum is in fact a global maximum.

34. (a) Suppose the height of the box is h. The box has six sides, four with area xh and two, the top and bottom, with area x^2. Thus,

$$4xh + 2x^2 = A.$$

So

$$h = \frac{A - 2x^2}{4x}.$$

Then, the volume, V, is given by

$$V = x^2 h = x^2 \left(\frac{A - 2x^2}{4x}\right) = \frac{x}{4}\left(A - 2x^2\right)$$
$$= \frac{A}{4}x - \frac{1}{2}x^3.$$

(b) The graph is shown in Figure 4.87. We are assuming A is a positive constant. Also, we have drawn the whole graph, but we should only consider $V > 0$, $x > 0$ as V and x are lengths.

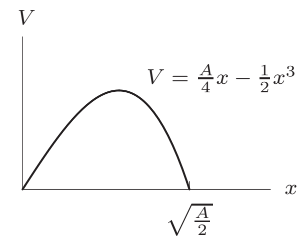


Figure 4.87

(c) To find the maximum, we differentiate, regarding A as a constant:

$$\frac{dV}{dx} = \frac{A}{4} - \frac{3}{2}x^2.$$

So $dV/dx = 0$ if

$$\frac{A}{4} - \frac{3}{2}x^2 = 0$$
$$x = \pm\sqrt{\frac{A}{6}}.$$

For a real box, we must use $x = \sqrt{A/6}$. Figure 4.87 makes it clear that this value of x gives the maximum. Evaluating at $x = \sqrt{A/6}$, we get

$$V = \frac{A}{4}\sqrt{\frac{A}{6}} - \frac{1}{2}\left(\sqrt{\frac{A}{6}}\right)^3 = \frac{A}{4}\sqrt{\frac{A}{6}} - \frac{1}{2}\cdot\frac{A}{6}\sqrt{\frac{A}{6}} = \left(\frac{A}{6}\right)^{3/2}.$$

35. (a) Since the crow makes $n(x)$ trips to a height of x meters each, the total vertical distance traveled upward is

$$h = x \cdot n(x) = x + \frac{27}{x} \text{ meters.}$$

(b) At the minimum,

$$\frac{dh}{dx} = 1 - \frac{27}{x^2} = 0$$
$$x^2 = 27$$
$$x = 5.2 \text{ meters.}$$

Since dh/dx is negative for $x < 5.2$ and dh/dx is positive for $x > 5.2$, there is a local minimum at $x = 5.2$. This is the global minimum for $x > 0$. See Figure 4.88.

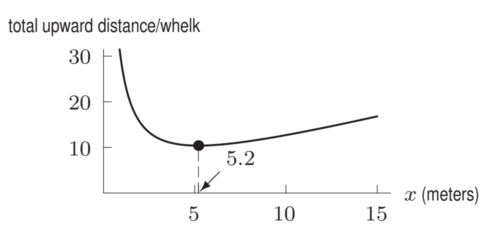


Figure 4.88

The dropping height actually observed by biologists is about 5.2 meters.

36. Rewriting the expression for I using the properties of logs gives

$$I = 192(\ln S - \ln 762) - S + 763.$$

Differentiating with respect to S gives

$$\frac{dI}{dS} = \frac{192}{S} - 1.$$

At a critical point

$$\frac{192}{S} - 1 = 0$$
$$S = 192.$$

Since

$$\frac{d^2I}{dS^2} = -\frac{192}{S^2},$$

we see that if $S = 192$, we have $d^2I/dS^2 < 0$, so $S = 192$ is a local maximum. From Figure 4.89, we see that it is a global maximum. The maximum possible number of infected children is therefore

$$I = 192 \ln\left(\frac{192}{762}\right) - 192 + 763 = 306 \text{ children.}$$

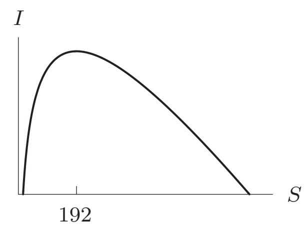


Figure 4.89

37. (a) Suppose the farmer plants x trees per km^2. Then, for $x \leq 200$, the yield per tree is 400 kg, so

$$\text{Total yield, } y = 400x \text{ kg.}$$

For $x > 200$, the yield per tree is reduced by 1 kg for each tree over 200, so the yield per tree is $400 - (x - 200)$ kg $= (600 - x)$ kg. Thus,

$$\text{Total yield, } y = (600 - x)x = 600x - x^2 \text{ kg.}$$

The graph of total yield is in Figure 4.90. It is a straight line for $x \leq 200$ and a parabola for $x > 200$.

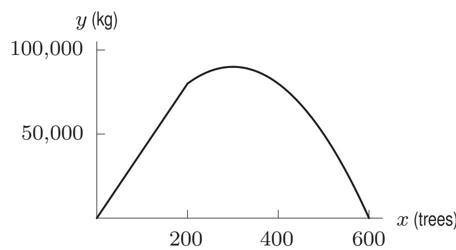


Figure 4.90

(b) The maximum occurs where the derivative of the quadratic is zero, or

$$\frac{dy}{dx} = 600 - 2x = 0$$
$$x = 300 \text{ trees/km}^2.$$

38. (a) Differentiating twice using the product rule yields $dy/dx = (1-bx)ae^{-bx}$ and $d^2y/dx^2 = -(2-bx)abe^{-bx}$. Setting $dy/dx = 0$, we find that there is a critical point where

$$1 - bx = 0$$
$$x = \frac{1}{b}.$$

At $x = \frac{1}{b}$, we have $y = \frac{a}{b}e^{-1}$.

This is a local maximum since at this point $d^2y/dx^2 = -abe^{-1} < 0$.

(b) Since we are concerned only with positive values of x we consider $x = 0$ and $\lim_{x\to\infty} y$. When $x = 0$, $y = 0$, and $\lim_{x\to\infty} y = 0$ also. Since $\frac{a}{b}e^{-1}$ is positive, $(\frac{1}{b}, \frac{a}{b}e^{-1})$ is in fact a global maximum. This implies that a large population may actually produce *fewer* offspring than a smaller one.

39. (a) The variables are S and H. At the maximum value of S, using the product rule, we have

$$\begin{aligned}\frac{dS}{dH} = ae^{-bH} + aH(-b)e^{-bH} &= 0\\ ae^{-bH}(1-bH) &= 0\\ bH &= 1\\ H &= \frac{1}{b}.\end{aligned}$$

Thus, the maximum value of S occurs when $H = 1/b$. To find the maximum value of S, we substitute $H = 1/b$, giving

$$S = a\frac{1}{b}e^{-b(1/b)} = \frac{a}{b}e^{-1}.$$

(b) Increasing a increases the maximum value of S. Increasing b decreases the maximum value of S.

40. (a) When $t = 0$, we have $q(0) = 20(e^{-0} - e^{-2\cdot 0}) = 20(1-1) = 0$. This is as we would expect: initially there is none of the drug in the bloodstream.

(b) The maximum value of $q(t)$ occurs where

$$\begin{aligned}q'(t) = 20(-e^{-t} + 2e^{-2t}) &= 0\\ -e^{-t} &= -2e^{-2t}\\ \frac{e^{-t}}{e^{-2t}} &= 2\\ e^{t} &= 2\\ t &= \ln 2 = 0.69 \text{ hours.}\end{aligned}$$

When $t = 0.69$ hours, we have $q(0.69) = 5$ mg.

(c) To see what happens in the long run, we let $t \to \infty$. Since $e^{-t} \to 0$ and $e^{-2t} \to 0$, we see that $q(t) \to 0$ as $t \to \infty$. This is as we would expect: in the long run, the drug will be metabolized or excreted and leave the body.

41. (a) To maximize benefit (surviving young), we pick 10, because that's the highest point of the benefit graph.

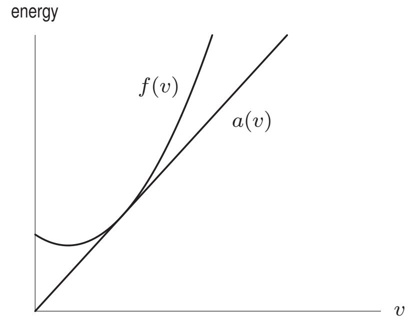
(b) To optimize the vertical distance between the curves, we can either do it by inspection or note that the slopes of the two curves will be the same where the difference is maximized. Either way, one gets approximately 9.

42. (a) At higher speeds, more energy is used so the graph rises to the right. The initial drop is explained by the fact that the energy it takes a bird to fly at very low speeds is greater than that needed to fly at a slightly higher speed. When it flies slightly faster, the amount of energy consumed decreases. But when it flies at very high speeds, the bird consumes a lot more energy (this is analogous to our swimming in a pool).

(b) $f(v)$ measures energy per second; $a(v)$ measures energy per meter. A bird traveling at rate v will in 1 second travel v meters, and thus will consume $v \cdot a(v)$ joules of energy in that 1 second period. Thus $v \cdot a(v)$ represents the energy consumption per second, and so $f(v) = v \cdot a(v)$.

(c) Since $v \cdot a(v) = f(v)$, $a(v) = f(v)/v$. But this ratio has the same value as the slope of a line passing from the origin through the point $(v, f(v))$ on the curve (see figure). Thus $a(v)$ is minimal when the slope of this line is minimal. To find the value of v minimizing $a(v)$, we solve $a'(v) = 0$. By the quotient rule,

$$a'(v) = \frac{vf'(v) - f(v)}{v^2}.$$



Thus $a'(v) = 0$ when $vf'(v) = f(v)$, or when $f'(v) = f(v)/v = a(v)$. Since $a(v)$ is represented by the slope of a line through the origin and a point on the curve, $a(v)$ is minimized when this line is tangent to $f(v)$, so that the slope $a(v)$ equals $f'(v)$.

(d) The bird should minimize $a(v)$ assuming it wants to go from one particular point to another, i.e. where the distance is set. Then minimizing $a(v)$ minimizes the total energy used for the flight.

43. **(a)** We want the maximum value of I. Using the properties of logarithms, we rewrite the expression for I as

$$I = k(\ln S - \ln S_0) - S + S_0 + I_0.$$

Since k and S_0 are constant, differentiating with respect to S gives

$$\frac{dI}{dS} = \frac{k}{S} - 1.$$

Thus, the critical point is at $S = k$. Since dI/dS is positive for $S < k$ and dI/dS is negative for $S > k$, we see that $S = k$ is a local maximum.

We only consider positive values of S. Since $S = k$ is the only critical point, it gives the global maximum value for I, which is

$$I = k(\ln k - \ln S_0) - k + S_0 + I_0.$$

(b) Since both k and S_0 are in the expression for the maximum value of I, both the particular disease and how it starts influence the maximum.

44. The triangle in Figure 4.91 has area, A, given by

$$A = \frac{1}{2}x \cdot y = \frac{1}{2}x^3 e^{-3x}.$$

If the area has a maximum, it occurs where

$$\begin{aligned}\frac{dA}{dx} = \frac{3}{2}x^2 e^{-3x} - \frac{3}{2}x^3 e^{-3x} &= 0\\ \frac{3}{2}x^2\left(1-x\right)e^{-3x} &= 0\\ x &= 0, 1.\end{aligned}$$

The value $x = 0$ gives the minimum area, $A = 0$, for $x \geq 0$. Since

$$\frac{dA}{dx} = \frac{3}{2}x^2(1-x)e^{-3x},$$

we see that

$$\frac{dA}{dx} > 0 \text{ for } 0 < x < 1 \qquad \text{and} \qquad \frac{dA}{dx} < 0 \text{ for } x > 1.$$

Thus, $x = 1$ gives the local and global maximum of

$$A = \frac{1}{2}1^3 e^{-3\cdot 1} = \frac{1}{2e^3}.$$

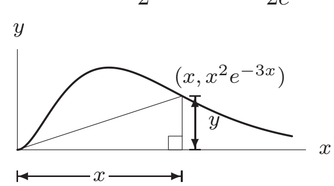


Figure 4.91

45. **(a)** The maximum and minimum values of p can be found without taking derivatives, since the function $20\sin(2.5\pi t)$ has maximum and minimum values of 20 and -20, respectively. Thus, the maximum value of p is 120 mm Hg and the minimum value is 80 mm Hg.

(b) The time between successive maxima is the period, which is $2\pi/(2.5\pi) = 0.8$ seconds.

(c) See Figure 4.92.

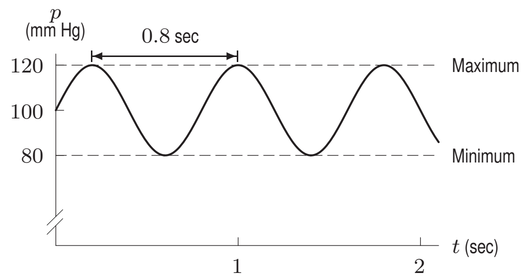


Figure 4.92

46. **(a)** If we expect the rate to be nonnegative, then we must have $0 \le y \le a$. See Figure 4.93.

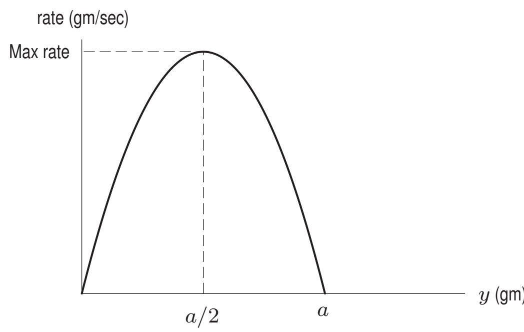


Figure 4.93

(b) The maximum value of the rate occurs at $y = a/2$, as can be seen from Figure 4.93, or by setting

$$\frac{d}{dy}(\text{rate}) = 0$$
$$\frac{d}{dy}(\text{rate}) = \frac{d}{dy}(kay - ky^2) = ka - 2ky = 0$$
$$y = \frac{a}{2}.$$

From the graph, we see that $y = a/2$ gives the global maximum.

47. **(a)** If we expect the rate to be nonnegative, we must have $0 \le y \le a$ and $0 \le y \le b$. Since we assume $a < b$, we restrict y to $0 \le y \le a$.

In fact, the expression for the rate is nonnegative for y greater than b, but these values of y are not meaningful for the reaction. See Figure 4.94.

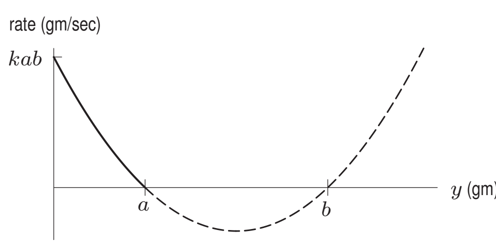


Figure 4.94

(b) From the graph, we see that the maximum rate occurs when $y = 0$; that is, at the start of the reaction.

Solutions for Section 4.4

1. The profit function is positive when $R(q) > C(q)$, and negative when $C(q) > R(q)$. It's positive for $5.5 < q < 12.5$, and negative for $0 < q < 5.5$ and $12.5 < q$. Profit is maximized when $R(q) > C(q)$ and $R'(q) = C'(q)$ which occurs at about $q = 9.5$. See Figure 4.95.

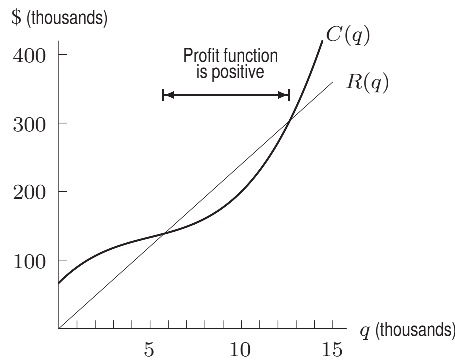
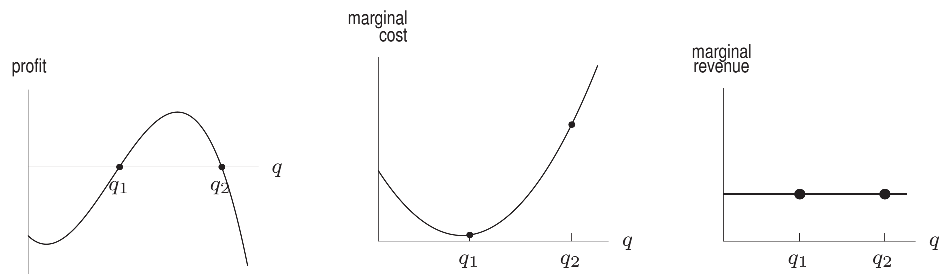


Figure 4.95

2.



3. **(a)** Profit is maximized when $R(q)-C(q)$ is as large as possible. This occurs at $q = 2500$, where profit $= 7500-5500 =$ \$2000.

(b) We see that $R(q) = 3q$ and so the price is $p = 3$, or \$3 per unit.

(c) Since $C(0) = 3000$, the fixed costs are \$3000.

4. **(a)** The total revenue is given by $R = pq$ so $R = 400q - 2q^2$.

(b) The marginal revenue, MR, is given by

$$\frac{dR}{dq} = 400 - 4q.$$

When $q = 10$ the marginal revenue is 360 \$/unit.

(c) The total revenue when $q = 10$ is \$3800, and for $q = 11$ it is \$4158 so the increase in revenue is \$358 compared with the marginal revenue of 360 \$/unit calculated in part (b).

5. **(a)** The profit earned by the 51^{st} is the revenue earned by the 51^{st} item minus the cost of producing the 51^{st} item. This can be approximated by

$$\pi'(50) = R'(50) - C'(50) = 84 - 75 = \$9.$$

Thus the profit earned from the 51^{st} item will be approximately \$9.

(b) The profit earned by the 91^{st} item will be the revenue earned by the 91^{st} item minus the cost of producing the 91^{st} item. This can be approximated by

$$\pi'(90) = R'(90) - C'(90) = 68 - 71 = -\$3.$$

Thus, approximately three dollars are lost in the production of the 91^{st} item.

(c) If $R'(78) > C'(78)$, production of a 79^{th} item would increase profit. If $R'(78) < C'(78)$, production of one less item would increase profit. Since profit is maximized at $q = 78$, we must have

$$C'(78) = R'(78).$$

6. **(a)** See Figure 4.96. We find q_1 and q_2 by checking to see where the slope of the tangent line to $C(q)$ is equal to the slope of $R(q)$. Because the slopes of $C(q)$ and $R(q)$ represent marginal cost and marginal revenue, respectively, at q's where the slopes are equal, the cost of producing an additional unit of q exactly equals the revenue gained from selling an additional unit. In other words, q_1 and q_2 are levels of production at which the additional or marginal profit from producing an additional unit of q is zero.

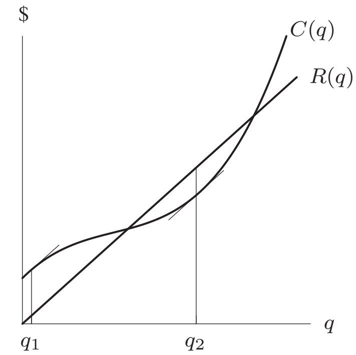


Figure 4.96

(b) We can think of the vertical distance between the cost and revenue curves as representing the firm's total profits. The quantity q_1 is the production level at which total profits are at a minimum. We see this by noticing that for points slightly to the left of q_1, the slope of $C(q)$ is slightly greater than the slope of $R(q)$. This means that the cost of producing an additional unit of q is greater than the revenue earned from selling it. So for points to the left of q_1, additional production decreases profits. For points slightly to the right of q_1, the slope of $C(q)$ is less than the slope of $R(q)$. Thus additional production results in an increase in profits. The production level q_1 is the level at which the firm ceases to take a loss on each additional item and begins to make a profit on each additional item. Note that the total profit is still negative, and remains so until the graphs cross, i.e, where total cost equals total revenue.

Similar reasoning applies for q_2, except it is the level of production at which profits are maximized. For points slightly to the left of q_2, the slope of $C(q)$ is less than the slope of $R(q)$. Thus the cost of producing an additional unit is less than the revenue gained from selling it. So by selling an additional unit, the firm can increase profits. For points to the right of q_2, the slope of $C(q)$ is greater than the slope of $R(q)$. This means that the profit from producing and selling an additional unit of q will be negative, decreasing total profits. The point q_2 is the level of production at which the firm stops making a profit on each additional item sold and begins to take a loss.

At both points q_1 and q_2, note that the vertical distance between $C(q)$ and $R(q)$ is at a local maximum. This represents the fact that q_1 and q_2 are local profit minimum and local profit maximum points.

7. **(a)** At $q = 5000$, $MR > MC$, so the marginal revenue to produce the next item is greater than the marginal cost. This means that the company will make money by producing additional units, and production should be increased.

(b) Profit is maximized where $MR = MC$, and where the profit function is going from increasing ($MR > MC$) to decreasing ($MR < MC$). This occurs at $q = 8000$.

8. We know that the maximum (or minimum) profit can occur when

$$\text{Marginal cost} = \text{Marginal revenue} \quad \text{or} \quad MC = MR.$$

From the table it appears that $MC = MR$ at $q \approx 2500$ and $q \approx 4500$. To decide which one corresponds to the maximum profit, look at the marginal profit at these points. Since

$$\text{Marginal profit} = \text{Marginal revenue} - \text{Marginal cost}$$

(or $M\pi = MR - MC$), we compute marginal profit at the different values of q in Table 4.1:

Table 4.1

q	1000	2000	3000	4000	5000	6000
$M\pi = MR - MC$	-22	-4	4	7	-5	-22

From the table, at $q \approx 2500$, we see that profit changes from decreasing to increasing, so $q \approx 2500$ gives a local minimum. At $q \approx 4500$, profit changes from increasing to decreasing, so $q \approx 4500$ is a local maximum. See Figure 4.97. Therefore, the global maximum occurs at $q = 4500$ or at the endpoint $q = 1000$.

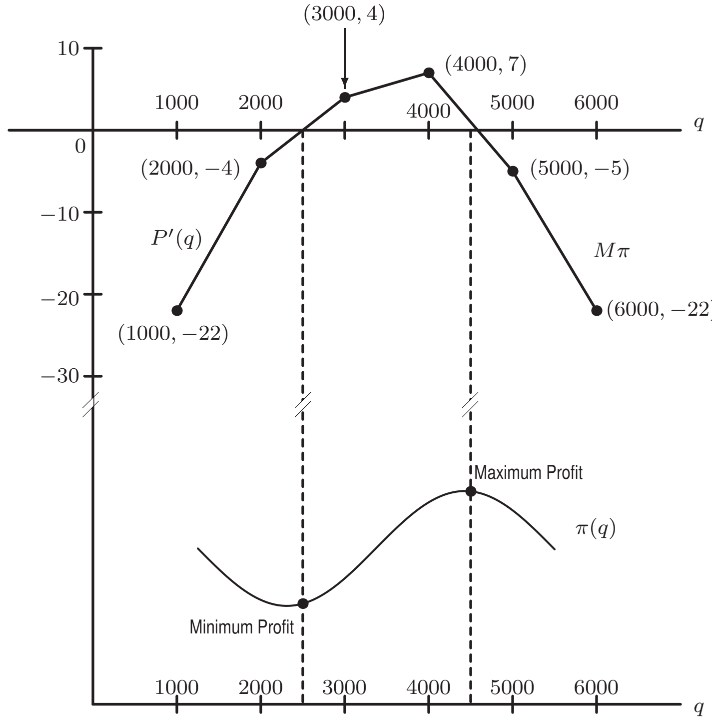


Figure 4.97

9. The marginal revenue, MR, is given by differentiating the total revenue function, R. We use the chain rule so

$$MR = \frac{dR}{dq} = \frac{1}{1+1000q^2} \cdot \frac{d}{dq}\left(1000q^2\right) = \frac{1}{1+1000q^2} \cdot 2000q.$$

When $q = 10$,

$$\text{Marginal revenue } = \frac{2000 \cdot 10}{1+1000 \cdot 10^2} = \$0.20\text{/item}.$$

When 10 items are produced, each additional item produced gives approximately \$0.20 in additional revenue.

10. (a) Profit $= \pi = R - C$; profit is maximized when the slopes of the two graphs are equal, at around $q = 350$. See Figure 4.98.

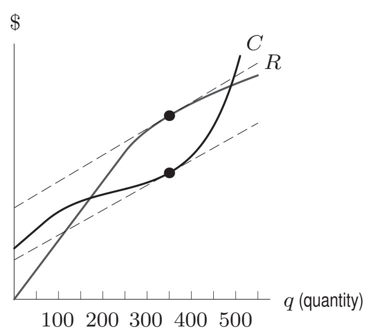


Figure 4.98

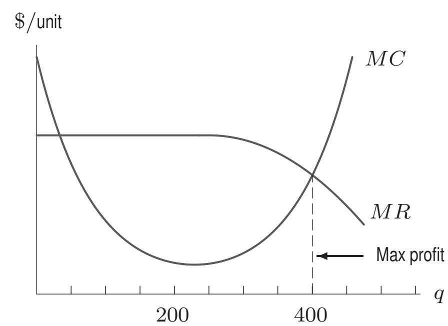


Figure 4.99

(b) The graphs of MR and MC are the derivatives of the graphs of R and C. Both R and C are increasing everywhere, so MR and MC are everywhere positive. The cost function is concave down and then concave up, so MC is decreasing and then increasing. The revenue function is linear and then concave down, so MR is constant and then decreasing. See Figure 4.99.

11. From the given graph, we can sketch the graph of marginal profit $= M\pi = MR - MC$: see Figure 4.100.

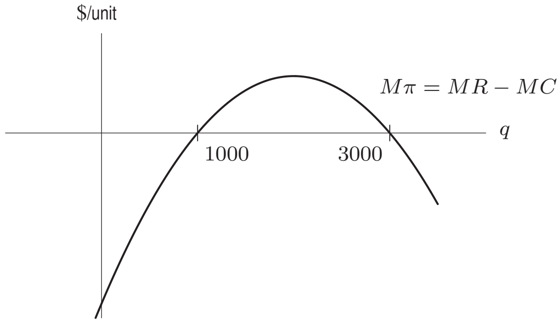


Figure 4.100

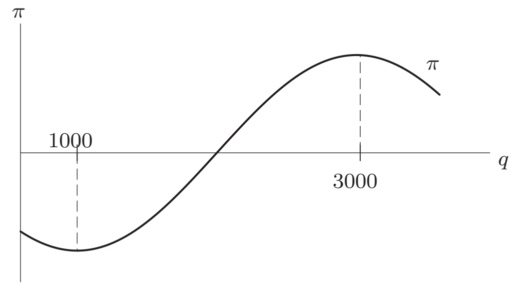


Figure 4.101

And, from the graph of marginal profit, we can make a sketch showing the shape of the shape of the graph of the profit function, since the marginal profit curve is the graph of the derivative of the profit function. See Figure 4.101. Because we don't know the value of the profit when $q = 0$, the graph in Figure 4.101 may be shifted vertically.

Since $M\pi < 0$ for $q < 1000$ and for $q > 3000$, while $M\pi > 0$ for $1000 < q < 3000$ we see that profits appear to be at a minimum when $q = 1000$ and at a maximum for $q = 3000$. We see from Figure 4.101 that there is a local maximum at $q = 3000$. Therefore, the profit has a global maximum either at $q = 3{,}000$ or at the endpoint $q = 0$.

Because the profit function could be shifted vertically downward, notice that the maximum profit could be zero or negative.

12. The company should increase production if $MR > MC$, since increasing production then adds more to revenue than to cost—a net gain for the company.

(a) Since $MC(25) = 17.75$ and $MR(25) = 30$, the company should increase production.
(b) Since $MC(50) = 39$ and $MR(50) = 30$, the company should decrease production.
(c) Since $MC(80) = 114$ and $MR(80) = 30$, the company should decrease production.

13. We have $MR = 30$ for all quantities q. The profit is a maximum when $MC = MR$ and $MC < MR$ to the left of the point and $MC > MR$ to the right of the point. There appear to be two points with $MC = MR$, one between $q = 0$ and $q = 10$ and another between $q = 40$ and $q = 50$.

At $q = 0$ we have $MC = 34 > MR$, and at $q = 10$ we have $MC = 23 < MR$. Thus, the point between $q = 0$ and $q = 10$ does not give a maximum profit.

At $q = 40$ we have $MC = 26 < MR$, and at $q = 50$ we have $MC = 39 > MR$. Thus, the point between $q = 40$ and $q = 50$ gives a maximum profit.

14. The profit is maximized at the point where the difference between revenue and cost is greatest. Thus the profit is maximized at approximately $q = 4000$.

15. **(a)** We know that marginal cost $= C'(q)$ or the slope of the graph of $C(q)$ at q. Similarly, marginal revenue $= R'(q)$ or the slope of the graph of $R(q)$ at q. From the graph, the slope of $R(q)$ at $q = 3000$ is greater than the slope of $C(q)$ at $q = 3000$. Therefore, marginal revenue is greater than marginal cost at $q = 3000$. Production should be increased.
(b) The graph of $C(q)$ is steeper than the graph of $R(q)$ at $q = 5000$. Therefore, marginal cost is greater than marginal revenue at $q = 5000$. Production should be decreased.

16. Since marginal revenue is larger than marginal cost around $q = 2000$, as you produce more of the product your revenue increases faster than your costs, so profit goes up, and maximal profit will occur at a production level above 2000.

17. First find marginal revenue and marginal cost.

$$MR = R'(q) = 450$$
$$MC = C'(q) = 6q$$

Setting $MR = MC$ yields $6q = 450$, so marginal cost is equal to marginal revenue when

$$q = \frac{450}{6} = 75 \text{ units.}$$

Is profit maximized at $q = 75$? Profit $= R(q) - C(q)$;

$$\begin{aligned} R(75) - C(75) &= 450(75) - (10{,}000 + 3(75)^2) \\ &= 33{,}750 - 26{,}875 = \$6875. \end{aligned}$$

Testing $q = 74$ and $q = 76$:

$$\begin{aligned} R(74) - C(74) &= 450(74) - (10{,}000 + 3(74)^2) \\ &= 33{,}300 - 26{,}428 = \$6872. \end{aligned}$$

$$\begin{aligned} R(76) - C(76) &= 450(76) - (10{,}000 + 3(76)^2) \\ &= 34{,}200 - 27{,}328 = \$6872. \end{aligned}$$

Since profit at $q = 75$ is more than profit at $q = 74$ and $q = 76$, we conclude that profit is maximized locally at $q = 75$. The only endpoint we need to check is $q = 0$.

$$\begin{aligned} R(0) - C(0) &= 450(0) - (10{,}000 + 3(0)^2) \\ &= -\$10{,}000. \end{aligned}$$

This is clearly not a maximum, so we conclude that the profit is maximized globally at $q = 75$, and the total profit at this production level is $\$6{,}875$.

18. We first need to find an expression for $R(q)$, or revenue in terms of quantity sold. We know that $R(q) = pq$, where p is the price of one item. Here $p = 45 - 0.01q$, so we make the substitution

$$R(q) = (45 - .01q)q = 45q - 0.01q^2.$$

This is the function we want to maximize. Finding the derivative and setting it equal to 0 yields

$$\begin{aligned} R'(q) &= 0 \\ 45 - 0.02q &= 0 \\ 0.02q &= 45 \text{ so} \\ q &= 2250. \end{aligned}$$

Is this a maximum?

$$\begin{aligned} R'(q) &> 0 \quad \text{for } q < 2250 \text{ and} \\ R'(q) &< 0 \quad \text{for } q > 2250. \end{aligned}$$

So we conclude that $R(q)$ has a local maximum at $q = 2250$. Testing $q = 0$, the only endpoint, $R(0) = 0$, which is less than $R(2250) = \$50{,}625$. So we conclude that revenue is maximized at $q = 2250$. The price of each item at this production level is

$$p = 45 - .01(2250) = \$22.50$$

and total revenue is

$$pq = \$22.50(2250) = \$50{,}625,$$

which agrees with the above answer.

19. **(a)** The marginal cost at $q = 400$ is the slope of the tangent line to $C(q)$ at $q = 400$. Looking at the graph, we can estimate a slope of about 1. Thus, the marginal cost is about \$1.

(b) At $q = 500$, we can see that slope of the cost function is greater than the slope of the revenue function. Thus, the marginal cost is greater than the marginal revenue and thus the 500th item will incur a loss. So, the company should not produce the 500th item.

(c) The quantity which maximizes profit is at the point where marginal cost equals marginal revenue. This occurs when the slope of $R(q)$ equals $C(q)$, which occurs at approximately $q = 400$. Thus, the company should produce about 400 items.

20. **(a)** Since Profit = Revenue − Cost, we can calculate $\pi(q) = R(q) - C(q)$ for each of the q values given:

q	0	100	200	300	400	500
$R(q)$	0	500	1000	1500	2000	2500
$C(q)$	700	900	1000	1100	1300	1900
$\pi(q)$	−700	−400	0	400	700	600

We see that maximum profit is \$700 and it occurs when the production level q is 400. See Figure 4.102.

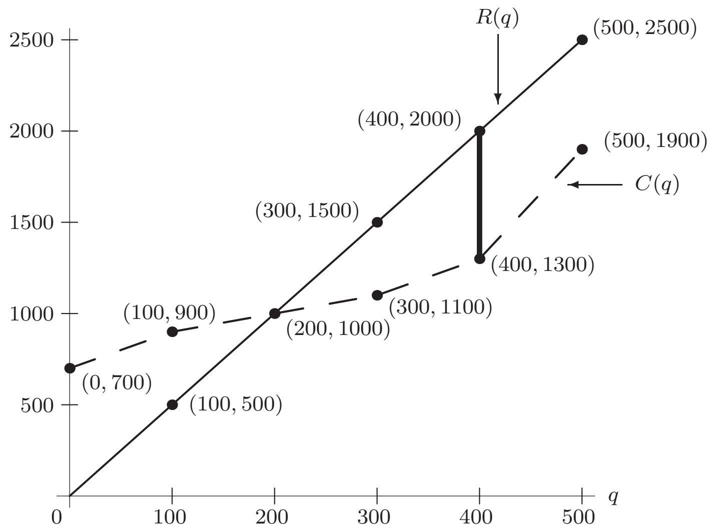


Figure 4.102

(b) Since revenue is \$500 when $q = 100$, the selling price is \$5 per unit.

(c) Since $C(0) = \$700$, the fixed costs are \$700.

21. (a) If $q = 3000$, the demand equation gives $p = 70 - 0.02 \cdot 3000 = 10$. That is, at a price of \$10, 3000 people attend. At this price,

$$\text{Revenue} = 3000 \text{ people} \cdot 10 \text{ dollars/person} = \$30{,}000.$$

To find total revenue at a price of \$20, first find the attendance at this price. Substituting $p = 20$ into the demand equation, $p = 70 - 0.02q$, gives

$$20 = 70 - 0.02q.$$

Solving for q, we get

$$\begin{aligned} -50 &= -0.02q \\ 2500 &= q. \end{aligned}$$

That is, at a price of \$20, attendance is 2500 people, and

$$\text{Revenue} = 2500 \cdot 20 = \$50,000.$$

Notice that, although demand is reduced, the revenue is higher at a price of \$20 than at \$10.

(b) Since Revenue $=$ Price $\times$ Quantity $= p \cdot q$ and $p = 70 - 0.002q$, we have

$$\begin{aligned} R(q) &= (70 - 0.02q)q \\ &= 70q - 0.02q^2. \end{aligned}$$

(c) To maximize revenue, find the critical points of the revenue function $R(q) = 70q - 0.02q^2$:

$$\begin{aligned} R'(q) &= 70 - 0.02 \cdot 2q \\ 0 &= 70 - 0.04q \\ 70 &= 0.04q \\ 1750 &= q. \end{aligned}$$

The graph of revenue is a parabola opening downward, so an attendance of 1750 gives the maximum revenue.

(d) Using the demand equation, we find the price corresponding to an attendance of 1750:

$$p = 70 - 0.02 \cdot 1750 = 70 - 35 = 35.$$

The optimal price for a ticket at the amusement park is \$35.

(e) When the optimal price of \$35 is charged, the attendance at the park is 1750 people. Thus, the maximum revenue is $R = pq = 35 \cdot 1750 = \$61{,}250$. The corresponding profit cannot be determined without knowing the costs.

22. To maximize revenue, we first must find an expression for revenue in terms of price. We know that $R(p) = pq$, where p=price and q=quantity sold. We now need to find an expression for q in terms of p. Using the information given, we find that

$$q = 4000 + \frac{(4.00 - p)}{0.25}(200)$$

Simplification of q yields

$$\begin{aligned} q &= 4000 + 800(4 - p) \\ &= 4000 + 3200 - 800p \\ &= 7200 - 800p \end{aligned}$$

We can now get an expression for revenue in terms of price.

$$\begin{aligned} R(p) = qp &= (7200 - 800p)p \\ &= 7200p - 800p^2 \end{aligned}$$

We want to maximize this function in terms of p. First find the critical points by finding the derivative.

$$R'(p) = 7200 - 1600p$$

Setting $R'(p) = 0$ and solving for p yields

$$\begin{aligned} 7200 - 1600p &= 0 \\ 1600p &= 7200 \\ p &= 4.5 \end{aligned}$$

Since $R'(p) > 0$ for $p < 4.5$ and $R'(p) < 0$ for $p > 4.5$, we conclude that revenue has a local maximum at $p = 4.5$. Since this is the only critical point, we conclude that it is the global maximum. So revenue is maximized at a price of \$4.50. The quantity sold at this amount is given by

$$q = 7200 - 800(4.50) = 3600$$

and the total revenue is

$$R(4.5) = 7200(4.5) - 800(4.5)^2 = \$16{,}200.$$

23. We first need to find an expression for revenue in terms of price. At a price of \$8, 1500 tickets are sold. For each \$1 above \$8, 75 fewer tickets are sold. This suggests the following formula for q, the quantity sold for any price p.

$$\begin{aligned} q &= 1500 - 75(p - 8) \\ &= 1500 - 75p + 600 \\ &= 2100 - 75p. \end{aligned}$$

We know that $R = pq$, so substitution yields

$$R(p) = p(2100 - 75p) = 2100p - 75p^2$$

To maximize revenue, we find the derivative of $R(p)$ and set it equal to 0.

$$\begin{aligned} &R'(p) = 2100 - 150p = 0 \\ &150p = 2100 \end{aligned}$$

so $p = \frac{2100}{150} = 14$. Does $R(p)$ have a maximum at $p = 14$? Using the first derivative test,

$$\begin{aligned} &R'(p) > 0 \ \text{ if } \ p < 14 \text{ and} \\ &R'(p) < 0 \ \text{ if } \ p > 14. \end{aligned}$$

So $R(p)$ has a local maximum at $p = 14$. Since this is the only critical point for $p \geq 0$, it must be a global maximum. So we conclude that revenue is maximized when the price is \$14.

24. **(a)** In dollars, the revenue earned is $R = pq = (-5q + 4000)q$. Thus

$$\begin{aligned}\text{Profit, } \pi = R - C &= (-5q + 4000)q - (6q + 5)\\ &= -5q^2 + 3994q - 5.\end{aligned}$$

(b) For maximum profit,

$$\frac{d\pi}{dq} = -10q + 3994 = 0,$$

so

$$q = 399.4.$$

(c) For $q = 399.4$, the profit is given in dollars by

$$\pi = -5(399.4)^2 + 3994(399.4) - 5 = 797{,}596.80.$$

25. **(a)** Cost $C =$ Fixed cost $+$ Total variable cost $= 10{,}000 + 2q$ dollars.

(b) We express the demand equation in the form

$$q = b + mp$$

where $q = 10{,}100$ when $p = 5$ and $q = 12{,}872$ when $p = 4.5$. Thus

$$\text{Slope} = m = \frac{12{,}872 - 10{,}100}{4.5 - 5} = \frac{2772}{-0.5} = -5544.$$

To find b, we substitute $q = 10{,}100$ and $p = 5$.

$$10{,}100 = b - 5544 \cdot 5.$$

Thus, $b = 37{,}820$, so $q = 37{,}820 - 5544p$.

(c) To find the profit as a function of q, we solve for p:

$$p = \frac{37{,}820}{5544} - \frac{1}{5544}q$$

so

$$p = 6.822 - 0.00018q.$$

Then we have

$$\begin{aligned}\text{Profit, } \pi &= \text{Revenue} - \text{Cost} = pq - C\\ &= (6.822 - 0.00018q)q - (10{,}000 + 2q)\\ &= -0.00018q^2 + 4.822q - 10{,}000.\end{aligned}$$

(d) At the maximum profit,

$$\frac{d\pi}{dq} = -0.00036q + 4.822 = 0,$$

so $q = 13{,}394.4$. Thus the company should produce 13,394 items. At that production level,

$$\pi = -0.00018(13{,}394)^2 + 4.822(13{,}394) - 10{,}000 = 22{,}294 \text{ dollars}.$$

26. Consider the rectangle of sides x and y shown in Figure 4.103.

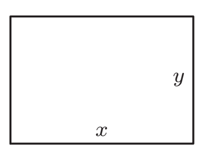


Figure 4.103

The total area is $xy = 3000$, so $y = 3000/x$. Suppose the left and right edges and the lower edge have the shrubs and the top edge has the fencing. The total cost is

$$\begin{aligned}C &= 45(x + 2y) + 20(x)\\ &= 65x + 90y.\end{aligned}$$

Since $y = 3000/x$, this reduces to

$$C(x) = 65x + 90(3000/x) = 65x + 270{,}000/x.$$

Therefore, $C'(x) = 65 - 270{,}000/x^2$. We set this to 0 to find the critical points:

$$\begin{aligned} 65 - \frac{270{,}000}{x^2} &= 0 \\ \frac{270{,}000}{x^2} &= 65 \\ x^2 &= 4153.85 \\ x &= 64.450 \text{ ft} \end{aligned}$$

so that

$$y = 3000/x = 46.548 \text{ ft}.$$

Since $C(x) \to \infty$ as $x \to 0^+$ and $x \to \infty$, we see that $x = 64.450$ is a minimum. The minimum total cost is then

$$C(64.450) \approx \$8378.54.$$

27. Let x equal the number of chairs ordered in excess of 300, so $0 \le x \le 100$.

$$\begin{aligned} \text{Revenue} = R &= (90 - 0.25x)(300 + x) \\ &= 27,000 - 75x + 90x - 0.25x^2 = 27,000 + 15x - 0.25x^2 \end{aligned}$$

At a critical point $dR/dx = 0$. Since $dR/dx = 15 - 0.5x$, we have $x = 30$, and the maximum revenue is $\$27,225$ since the graph of R is a parabola which opens downward. The minimum is $\$0$ (when no chairs are sold).

28. **(a)** Since a/q decreases with q, this term represents the ordering cost. Since bq increases with q, this term represents the storage cost.

(b) At the minimum,

$$\frac{dC}{dq} = \frac{-a}{q^2} + b = 0$$

giving

$$q^2 = \frac{a}{b} \quad \text{so} \quad q = \sqrt{\frac{a}{b}}.$$

Since

$$\frac{d^2C}{dq^2} = \frac{2a}{q^3} > 0 \quad \text{for} \quad q > 0,$$

we know that $q = \sqrt{a/b}$ gives a local minimum. Since $q = \sqrt{a/b}$ is the only critical point, this must be the global minimum.

29. **(a)** The business must reorder often enough to keep pace with sales. If reordering is done every t months, then,

$$\begin{aligned} \text{Quantity sold in } t \text{ months} &= \text{Quantity reordered in each batch} \\ rt &= q \\ t &= \frac{q}{r} \text{ months.} \end{aligned}$$

(b) The amount spent on each order is $a + bq$, which is spent every q/r months. To find the monthly expenditures, divide by q/r. Thus, on average,

$$\text{Amount spent on ordering per month} = \frac{a + bq}{q/r} = \frac{ra}{q} + rb \text{ dollars.}$$

(c) The monthly cost of storage is $kq/2$ dollars, so

$$\begin{aligned} C &= \text{Ordering costs} + \text{Storage costs} \\ C &= \frac{ra}{q} + rb + \frac{kq}{2} \text{ dollars.} \end{aligned}$$

(d) The optimal batch size minimizes C, so

$$\begin{aligned}\frac{dC}{dq} &= \frac{-ra}{q^2} + \frac{k}{2} = 0\\ \frac{ra}{q^2} &= \frac{k}{2}\\ q^2 &= \frac{2ra}{k}\end{aligned}$$

so

$$q = \sqrt{\frac{2ra}{k}} \text{ items per order.}$$

30. (a) Suppose n passengers sign up for the cruise. If $n \leq 100$, then the cruise's revenue is $R = 2000n$, so the maximum revenue is

$$R = 2000 \cdot 100 = 200{,}000.$$

If $n > 100$, then the price is

$$p = 2000 - 10(n - 100)$$

and hence the revenue is

$$R = n(2000 - 10(n - 100)) = 3000n - 10n^2.$$

To find the maximum revenue, we set $dR/dn = 0$, giving $20n = 3000$ or $n = 150$. Then the revenue is

$$R = (2000 - 10 \cdot 50) \cdot 150 = 225{,}000.$$

Since this is more than the maximum revenue when $n \leq 100$, the boat maximizes its revenue with 150 passengers, each paying \$1500.

(b) We approach this problem in a similar way to part (a), except now we are dealing with the profit function π. If $n \leq 100$, we have

$$\pi = 2000n - 80{,}000 - 400n,$$

so π is maximized with 100 passengers yielding a profit of

$$\pi = 1600 \cdot 100 - 80{,}000 = \$80{,}000.$$

If $n > 100$, we have

$$\pi = n(2000 - 10(n - 100)) - (80{,}000 + 400n).$$

We again set $d\pi/dn = 0$, giving $2600 = 20n$, so $n = 130$. The profit is then \$89,000. So the boat maximizes profit by boarding 130 passengers, each paying \$1700. This gives the boat \$89,000 in profit.

31. For each month,

$$\begin{aligned}\text{Profit} &= \text{Revenue} - \text{Cost}\\ \pi &= pq - wL = pcK^\alpha L^\beta - wL\end{aligned}$$

The variable on the right is L, so at the maximum

$$\frac{d\pi}{dL} = \beta pcK^\alpha L^{\beta-1} - w = 0$$

Now $\beta - 1$ is negative, since $0 < \beta < 1$, so $1 - \beta$ is positive and we can write

$$\frac{\beta pcK^\alpha}{L^{1-\beta}} = w$$

giving

$$L = \left(\frac{\beta pcK^\alpha}{w}\right)^{\frac{1}{1-\beta}}$$

Since $\beta - 1$ is negative, when L is just above 0, the quantity $L^{\beta-1}$ is huge and positive, so $d\pi/dL > 0$. When L is large, $L^{\beta-1}$ is small, so $d\pi/dL < 0$. Thus the value of L we have found gives a global maximum, since it is the only critical point.

32. (a) The units of $f(L)$ are tons/month. If $f(1000) = 400$, then with 1000 hours of labor in a month the company can produce 400 tons of product per month.

(b) The units of L are hours/month. For the derivative of f with respect to L we have

$$\text{Units of } f'(L) = \frac{\text{Units of } f}{\text{Units of } L} = \frac{\text{tons/month}}{\text{hours/month}} = \frac{\text{tons}}{\text{hour}}.$$

If $f'(1000) = 2$, then increasing labor from 1000 to 1001 hours in a month increases production by about 2 tons per month.

(c) At a price of p dollars per ton, Q tons of product costs pQ dollars. One hour's labor earns w dollars, enough to purchase Q tons where

$$w = pQ.$$

Hence

$$\text{Real wage} = Q = \frac{w}{p}.$$

(d) The company's monthly cost is the price of the labor it employs.

$$C = \text{Hourly wage} \cdot \text{Hours of labor} = wL.$$

The monthly revenue is

$$R = \text{Product price} \cdot \text{Quantity} = pf(L).$$

Monthly profit is

$$\pi(L) = \text{Revenue} - \text{Cost} = pf(L) - wL.$$

(e) At maximum profit, the marginal profit $\pi'(L)$ is zero. We have

$$\pi'(L) = pf'(L) - w = 0.$$

Hence $pf'(L) = w$ and

$$f'(L) = \frac{w}{p},$$

which means that the marginal product of labor equals the real wage.

The marginal product of labor and the real wage are also equal when the company operates at minimum profit.

Solutions for Section 4.5

1. (a) Since the graph is concave down, the average cost gets smaller as q increases. This is because the cost per item gets smaller as q increases. There is no value of q for which the average cost is minimized since for any q_0 larger than q the average cost at q_0 is less than the average cost at q. Graphically, the average cost at q is the slope of the line going through the origin and through the point $(q, C(q))$. Figure 4.104 shows how as q gets larger, the average cost decreases.

(b) The average cost will be minimized at some q for which the line through $(0, 0)$ and $(q, c(q))$ is tangent to the cost curve. This point is shown in Figure 4.105.

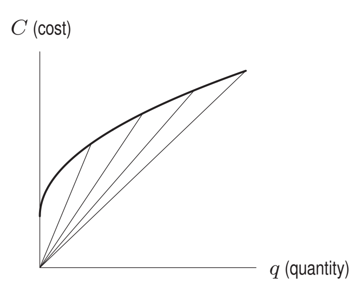


Figure 4.104

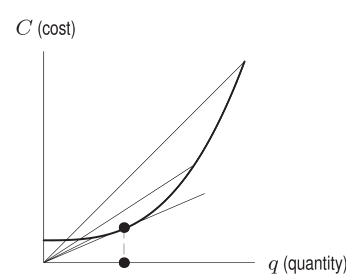


Figure 4.105

2. (a) (i) The average cost of quantity q is given by the formula $C(q)/q$. So average cost at $q = 25$ is given by $C(25)/25$. From the graph, we see that $C(25) \approx 200$, so $a(q) \approx 200/25 \approx \$8$ per unit. To interpret this graphically, note that $a(q) = C(q)/q = (C(q) - 0)/(q - 0)$. This is the formula for the slope of a line from the origin to a point $(q, C(q))$ on the curve. So $a(25)$ is the slope of a line connecting $(0, 0)$ to $(25, C(25))$. See Figure 4.106.

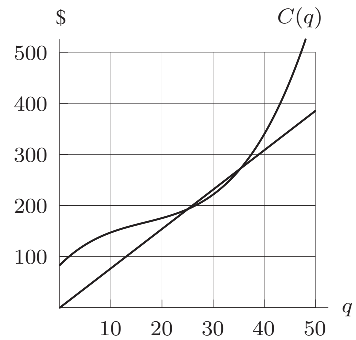


Figure 4.106

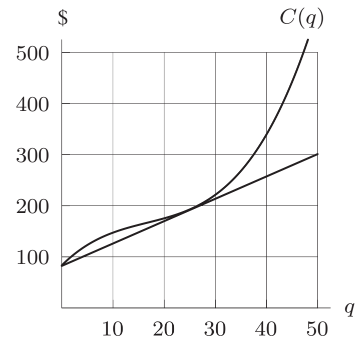


Figure 4.107

(ii) The marginal cost is $C'(q)$. This derivative is the slope of the tangent line to $C(q)$ at $q = 25$. To estimate this slope, we draw the tangent line, shown in Figure 4.107. From this plot, we see that the points $(50, 300)$ and $(0, 100)$ are approximately on this line, so its slope is approximately $(300 - 100)/(50 - 0) = 4$. Thus, $C'(25) \approx \$4$ per unit.

(b) We know that $a(q)$ is minimized where $a(q) = C'(q)$. Using the graphical interpretations from parts (i) and (ii), we see that $a(q)$ is minimized where the line passing from $(q, C(q))$ to the origin is also tangent to the curve. To find such points, a variety of lines passing through the origin and the curve are shown in Figure 4.108. The line which is also a tangent touches the curve at $q \approx 30$. So $q \approx 30$ units minimizes $a(q)$.

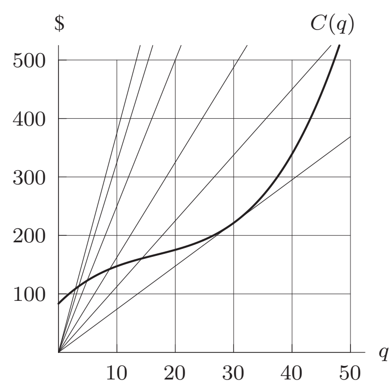


Figure 4.108

3. (a) The average cost of quantity q is given by the formula $C(q)/q$. So average cost at $q = 10{,}000$ is given by $C(10{,}000)/10{,}000$. From the graph, we see that $C(10{,}000) \approx 16{,}000$, so $a(q) \approx \frac{16{,}000}{10{,}000} \approx \$1.60$ per unit. The economic interpretation of this is that \$1.60 is each unit's share of the total cost of producing 10,000 units.

(b) To interpret this graphically, note that $a(q) = \frac{C(q)}{q} = \frac{C(q)-0}{q-0}$. This is exactly the formula for the slope of a line from the origin to a point $(q, C(q))$ on the curve. So $a(10{,}000)$ is the slope of a line connecting $(0, 0)$ to $(10{,}000, C(10{,}000))$. Such a line is shown below in Figure 4.109.

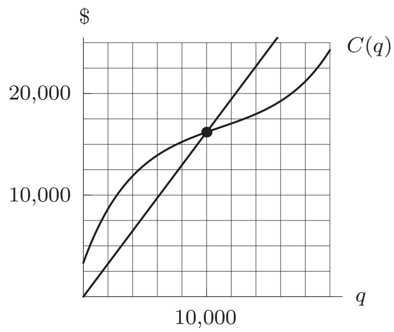


Figure 4.109

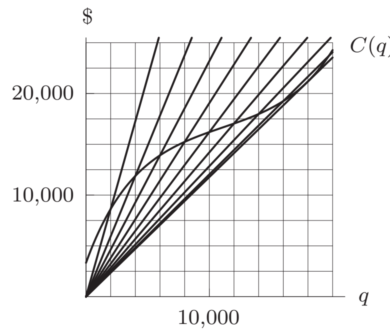


Figure 4.110

(c) We know that $a(q)$ is minimized where $a(q) = C'(q)$. Using the graphical interpretations from parts (b) and (c), this is equivalent to saying that the tangent line has the same slope as the line connecting the point on the curve to the origin. Since these two lines share a point, specifically the point $(q, C(q))$ on the curve, and have the same slope, they are in fact the same line. So $a(q)$ is minimized where the line passing from $(q, C(q))$ to the origin is also tangent to the curve. To find such points, a variety of lines passing through the origin and the curve are shown in Figure 4.110.

From this plot, we see that the line with the desired properties intersects the curve at $q \approx 18{,}000$. So $q \approx 18{,}000$ units minimizes $a(q)$.

4. **(a)** The marginal cost of producing the q_0^{th} item is simply $C'(q_0)$, or the slope of the graph at q_0. Since the slope of the cost function is always 12, the marginal cost of producing the 100^{th} item and the marginal cost of producing the 1000^{th} item is \$12.

(b) The average cost at $q = 100$ is given by

$$a(100) = \frac{C(100)}{100} = \frac{3700}{100} = \$37.$$

The average cost at $q = 1000$ is given by

$$a(1000) = \frac{C(1000)}{1000} = \frac{14{,}500}{1000} = \$14.50.$$

5. The cost function is $C(q) = 1000 + 20q$. The marginal cost function is the derivative $C'(q) = 20$, so the marginal cost to produce the 200^{th} unit is \$20 per unit. The average cost of producing 200 units is given by

$$a(200) = \frac{C(200)}{200} = \frac{5000}{200} = \$25/\text{unit}$$

6. The graph of the average cost function is shown in Figure 4.111.

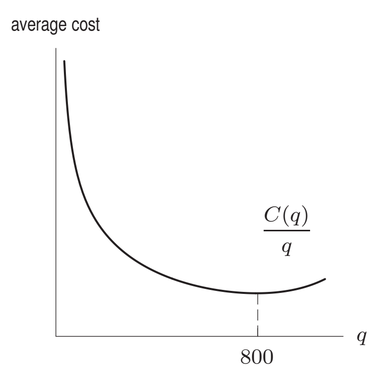


Figure 4.111

7. (a) The line connecting the origin and the graph of $C(q)$ in Figure 4.112 appears to have minimum slope at $q = 6$. Therefore we conclude that average cost is minimized at about $q = 6$.

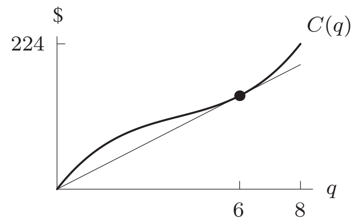


Figure 4.112

(b) The average cost of the first q items is given by

$$a(q) = \frac{C(q)}{q} = \frac{q^3 - 12q^2 + 60q}{q} = q^2 - 12q + 60$$

We want to minimize $a(q)$. Differentiating gives

$$a'(q) = 2q - 12.$$

Setting this equal to 0 and solving yields $q = 6$. Is this our minimum? We have $a'(q) < 0$ if $q < 6$ and $a'(q) > 0$ if $q > 6$, so $q = 6$ is a local minimum for $a(q)$. From Figure 4.112, we see that $q = 6$ is the global minimum for $0 \leq q \leq 8$.

8. (a) Profit equals revenue minus cost. Your monthly revenue is

$$1200 \text{ slippers} \times \$20/\text{slipper} = \$24000,$$

your monthly cost equals 1200 slippers $\times$ $\$2/\text{slipper} = \$2400$. Since you are earning a monthly profit of $\$24000 - \$2400 = \$21600$, you are making money.

(b) Since additional units produced cost about \$3 each, which is above the average cost, producing them increases average cost. Since additional pairs of slippers cost about \$3 to produce and can be sold for \$20, you can increase your profit by making and selling them. This is a case where marginal revenue, which is \$20 per slipper, is greater than marginal cost, which is \$3 per pair of slippers.

(c) You should recommend increase in production, since that increases profit. The fact that average cost of production increases is irrelevant to your decision.

9. (a) $a(q) = C(q)/q$, so $C(q) = 0.01q^3 - 0.6q^2 + 13q$.

(b) Taking the derivative of $C(q)$ gives an expression for the marginal cost:

$$C'(q) = MC(q) = 0.03q^2 - 1.2q + 13.$$

To find the smallest MC we take its derivative and find the value of q that makes it zero. So: $MC'(q) = 0.06q - 1.2 = 0$ when $q = 1.2/0.06 = 20$. This value of q must give a minimum because the graph of $MC(q)$ is a parabola opening upward. Therefore the minimum marginal cost is $MC(20) = 1$. So the marginal cost is at a minimum when the additional cost per item is \$1.

(c) $a'(q) = 0.02q - 0.6$

Setting $a'(q) = 0$ and solving for q gives $q = 30$ as the quantity at which the average is minimized, since the graph of a is a parabola which opens upward. The minimum average cost is $a(30) = 4$ dollars per item.

(d) The marginal cost at $q = 30$ is $MC(30) = 0.03(30)^2 - 1.2(30) + 13 = 4$. This is the same as the average cost at this quantity. Note that since $a(q) = C(q)/q$, we have $a'(q) = (qC'(q) - C(q))/q^2$. At a critical point, q_0, of $a(q)$, we have

$$0 = a'(q_0) = \frac{q_0 C'(q_0) - C(q_0)}{q_0^2},$$

so $C'(q_0) = C(q_0)/q_0 = a(q_0)$. Therefore $C'(30) = a(30) = 4$ dollars per item.

Another way to see why the marginal cost at $q = 30$ must equal the minimum average cost $a(30) = 4$ is to view $C'(30)$ as the approximate cost of producing the 30^{th} or 31^{st} good. If $C'(30) < a(30)$, then producing the 31^{st} good would lower the average cost, i.e. $a(31) < a(30)$. If $C'(30) > a(30)$, then producing the 30^{th} good would raise the average cost, i.e. $a(30) > a(29)$. Since $a(30)$ is the global minimum, we must have $C'(30) = a(30)$.

10. (a) The marginal cost tells us that additional units produced would cost about $\$10$ each, which is below the average cost, so producing them would reduce average cost.

(b) It is impossible to determine the effect on profit from the information given. Profit depends on both cost and revenue, $\pi = R - C$, but we have no information on revenue.

11. (a) $N = 100 + 20x$, graphed in Figure 4.113.

(b) $N'(x) = 20$ and its graph is just a horizontal line. This means that rate of increase of the number of bees with acres of clover is constant — each acre of clover brings 20 more bees.

On the other hand, $N(x)/x = 100/x + 20$ means that the average number of bees per acre of clover approaches 20 as more acres are put under clover. See Figure 4.114. As x increases, $100/x$ decreases to 0, so $N(x)/x$ approaches 20 (i.e. $N(x)/x \to 20$). Since the total number of bees is 20 per acre plus the original 100, the average number of bees per acre is 20 plus the 100 shared out over x acres. As x increases, the 100 are shared out over more acres, and so its contribution to the average becomes less. Thus the average number of bees per acre approaches 20 for large x.

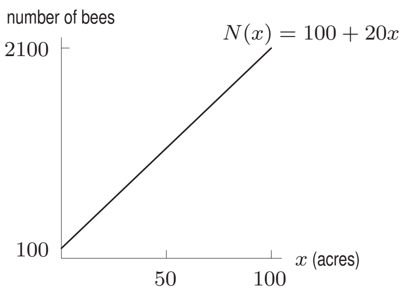


Figure 4.113

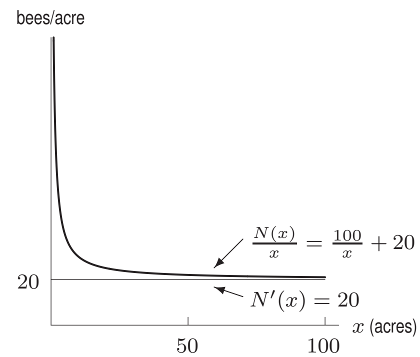


Figure 4.114

12. (a)

Table 4.2

i	Marginal Cost (of i^{th} filter)	Average Cost (for i^{th} filter)	Marginal Savings
0	$0	$0	$0
1	$5	$5	$64
2	$6	$5.50	$32
3	$7	$6	$16
4	$8	$6.50	$9
5	$9	$7	$3
6	$10	$7.50	$3
7	$11	$8	$0

(b) She should install four filters. Up to the fourth filter, the marginal cost is less than the marginal savings. This tells us that for each of the first four filters the developer buys, she will save more on the laundromat than she will have to pay for the filter. From the fifth filter onward, she pays more for each additional filter than she makes from the laundromat.

(c) If the rack costs $\$100$ she should not buy it. Instead she should let the laundromat protect itself. This is because, if she buys even one filter she has to spend $\$105$ on the filter but her savings as a result of the purchase only amount to $\$64$. If she buys two filters she has to spend $\$111$ on the filters but her total savings as a result of the purchase only amount to $\$96$. If she buys three filters she has to spend $\$118$ on the filters but her total savings as a result of the purchase only amount to $\$112$. If she buys four filters she has to spend $\$126$ on the filters but her total savings as a result of the purchase only amount to $\$121$ etc.

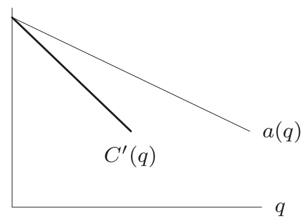
(d) If the rack costs $\$50$ she should buy it and install four filters since she covers the price of the rack with the first purchase of filters i.e., having bought one filter she will have spent $\$50 + \$5 = \$55$ but she will have saved $\$64$ on the laundromat business. Her marginal cost and marginal savings do not change after this point and so she still saves more on laundromat than she spends on filters if she buys the second, third and fourth filter.

13. It is interesting to note that to draw a graph of $C'(q)$ for this problem, you never have to know what $C(q)$ looks like, although you *could* draw a graph of $C(q)$ if you wanted to. By the definition of average cost, we know that $C(q) = q{\cdot}a(q)$. Using the product rule we get that $C'(q) = a(q) + q \cdot a'(q)$.

We are given a graph of $a(q)$ which is linear, so $a(q) = b + mq$, where $b = a(0)$ is the y-intercept and m is the slope. Therefore

$$\begin{aligned} C'(q) &= a(q) + q \cdot a'(q) = b + mq + q \cdot m \\ &= b + 2mq. \end{aligned}$$

In other words, $C'(q)$ is also linear, and it has twice the slope and the same y–intercept as $a(q)$.



14. Since $a(q) = C(q)/q$, we use the quotient rule to find

$$a'(q) = \frac{qC'(q) - C(q)}{q^2} = \frac{C'(q) - C(q)/q}{q} = \frac{C'(q) - a(q)}{q}.$$

Since marginal cost is C', if $C'(q) < a(q)$, then $C'(q) - a(q) < 0$, so $a'(q) < 0$.

15. Since $a(q) = C(q)/q$, we use the quotient rule to find

$$a'(q) = \frac{qC'(q) - C(q)}{q^2} = \frac{C'(q) - C(q)/q}{q} = \frac{C'(q) - a(q)}{q}.$$

Since marginal cost is C', if $C'(q) > a(q)$, then $C'(q) - a(q) > 0$, so $a'(q) > 0$.

16. **(a)** Differentiating $C(q)$ gives

$$C'(q) = \frac{K}{a}q^{(1/a)-1}, \quad C''(q) = \frac{K}{a}\left(\frac{1}{a} - 1\right)q^{(1/a)-2}.$$

If $a > 1$, then $C''(q) < 0$, so C is concave down.

(b) We have

$$\begin{aligned} a(q) &= \frac{C(q)}{q} = \frac{Kq^{1/a} + F}{q} \\ C'(q) &= \frac{K}{a}q^{(1/a)-1} \end{aligned}$$

so $a(q) = C'(q)$ means

$$\frac{Kq^{1/a} + F}{q} = \frac{K}{a}q^{(1/a)-1}.$$

Solving,

$$\begin{aligned} Kq^{1/a} + F &= \frac{K}{a}q^{1/a} \\ K\left(\frac{1}{a} - 1\right)q^{1/a} &= F \\ q &= \left[\frac{Fa}{K(1-a)}\right]^a. \end{aligned}$$

Solutions for Section 4.6

1. The effect on the quantity demanded is approximately E times the change in price. A price increase causes a decrease in quantity demanded and a price decrease causes an increase in quantity demanded.

(a) The quantity demanded decreases by about $0.5(3\%) = 1.5\%$.
(b) The quantity demanded increases by about $0.5(3\%) = 1.5\%$.

2. The effect on the quantity demanded is approximately E times the change in price. A price increase cause a decrease in quantity demanded and a price decrease cause an increase in quantity demanded.

(a) The quantity demanded decreases by about $2(3\%) = 6\%$.
(b) The quantity demanded increases by about $2(3\%) = 6\%$.

3. **(a)** We have $E = |p/q \cdot dq/dp| = |\text{dollars/tons} \cdot \text{tons/dollars}|$. All the units cancel, and so elasticity has no units.
(b) We have $E = |p/q \cdot dq/dp| = |\text{yen/liters} \cdot \text{liters/yen}|$. All the units cancel, and so elasticity has no units.
(c) Elasticity has no units. This is why it makes sense to compare elasticities of different products valued in different ways and measured in different units. Changing units of measurement will not change the value of elasticity.

4. If shoppers spend 1% more time in the store, then they spend about 1.3% more money.

5. Table 4.5 of Section 4.6 gives the elasticity of peaches as 1.49. Since $E > 1$, the demand is elastic, and a change in price will cause a larger percentage change in demand. So if the price of peaches goes up, fewer people will buy them: peaches are not essential, but a luxury item.

6. Table 4.5 of Section 4.6 gives the elasticity of potatoes as 0.27. Here $0 \leq E \leq 1$, so the demand is inelastic, meaning that changes in price will not change the demand so much. This is expected for potatoes; as a staple food item, people will continue to buy them even if the price goes up.

7. Elasticity will be high. As soon as the price of a brand is raised, many people will switch to another brand causing a drop in sales.

8. Demand for high-definition TV's will be elastic, since it is not a necessary item. If the prices are too high, people will not choose to buy them, so price changes will cause relatively large demand changes.

9. Elasticity will be low. Nearly everyone regards telephone service as a necessity. Since there is not another company they can turn to, they will keep their telephone service even if the company raises the price. The number of sales will change very little.

10. The elasticity of demand for a product, E, is given by

$$E = \left| \frac{p}{q} \cdot \frac{dq}{dp} \right| .$$

We first find $dq/dp = -4p$. At a price of \$5, the quantity demanded is $q = 200 - 50 = 150$ and $dq/dp = -20$, so

$$E = \left| \frac{5}{150} \cdot (-20) \right| = \frac{2}{3}.$$

Since $E < 1$ demand is inelastic.

11. **(a)** We use

$$E = \left| \frac{p}{q} \cdot \frac{dq}{dp} \right| .$$

So we approximate dq/dp at $p = 1.00$.

$$\frac{dq}{dp} \approx \frac{2440 - 2765}{1.25 - 1.00} = \frac{-325}{0.25} = -1300$$

$$\text{so} \qquad E = \left| \frac{p}{q} \cdot \frac{dq}{dp} \right| \approx \left| \frac{1.00}{2765} \cdot (-1300) \right| = 0.470$$

Since $E = 0.470 < 1$, demand for the candy is inelastic at $p = 1.00$.

(b) At $p = 1.25$,

$$\frac{dq}{dp} \approx \frac{1980 - 2440}{1.50 - 1.25} = \frac{-460}{0.25} = -1840$$

$$\text{so} \qquad E = \left|\frac{p}{q}\cdot\frac{dq}{dp}\right| \approx \left|\frac{1.25}{2440}\cdot(-1840)\right| = 0.943$$

At $p = 1.5$,

$$\frac{dq}{dp} \approx \frac{1660-1980}{1.75-1.50} = \frac{-320}{0.25} = -1280$$

$$\text{so} \qquad E = \left|\frac{p}{q}\cdot\frac{dq}{dp}\right| \approx \left|\frac{1.50}{1980}\cdot(-1280)\right| = 0.970$$

At $p = 1.75$,

$$\frac{dq}{dp} \approx \frac{1175-1660}{2.00-1.75} = \frac{-485}{0.25} = -1940$$

$$\text{so} \qquad E = \left|\frac{p}{q}\cdot\frac{dq}{dp}\right| \approx \left|\frac{1.75}{1660}\cdot(-1940)\right| = 2.05$$

At $p = 2.00$,

$$\frac{dq}{dp} \approx \frac{800-1175}{2.25-2.00} = \frac{-375}{0.25} = -1500$$

$$\text{so} \qquad E = \left|\frac{p}{q}\cdot\frac{dq}{dp}\right| \approx \left|\frac{2.00}{1175}\cdot(-1500)\right| = 2.55$$

At $p = 2.25$,

$$\frac{dq}{dp} \approx \frac{430-800}{2.50-2.25} = \frac{-370}{0.25} = -1480$$

$$\text{so} \qquad E = \left|\frac{p}{q}\cdot\frac{dq}{dp}\right| \approx \left|\frac{2.25}{800}\cdot(-1480)\right| = 4.16$$

Examination of the elasticities for each of the prices suggests that elasticity gets larger as price increases. In other words, at higher prices, an increase in price will cause a larger drop in demand than the same size price increase at a lower price level. This can be explained by the fact that people will not pay too much for candy, as it is somewhat of a "luxury" item.

(c) Elasticity is approximately equal to 1 at $p = \$1.25$ and $p = \$1.50$.

(d)

Table 4.3

$p(\$)$	q	Revenue $= p\cdot q$ ($)
1.00	2765	2765
1.25	2440	3050
1.50	1980	2970
1.75	1660	2905
2.00	1175	2350
2.25	800	1800
2.50	430	1075

We can see that revenue is maximized at $p = \$1.25$, with $p = \$1.50$ a close second, which agrees with part (c).

12. The elasticity of demand is given by

$$E = \left|\frac{p}{q}\cdot\frac{dq}{dp}\right|.$$

To evaluate dq/dp we solve $p = 90 - 10q$ for q, so $q = (90-p)/10$ and hence

$$\frac{dq}{dp} = -\frac{1}{10}.$$

When $p = 50$, we find $q = 4$, $dq/dp = -1/10$ so

$$E = \left|\frac{50}{4}\cdot\left(-\frac{1}{10}\right)\right| = 1.25.$$

Since the elasticity is

$$E = \frac{\text{Percent change in demand}}{\text{Percent change in price}}$$

when the price increases by 2% the percent change in demand is given by

$$\begin{aligned}\text{Percent change in demand} &= E \cdot \text{Percent change in price}\\ &= 2 \cdot (1.25) = 2.5.\end{aligned}$$

Therefore, the percentage change in demand is 2.5%. Since $dq/dp < 0$ this corresponds to a 2.5% fall in demand.

13. **(a)** If the price of yams is \$2/pound, the quantity sold will be

$$q = 5000 - 10(2)^2 = 5000 - 40 = 4960$$

so 4960 pounds will be sold.

(b) Elasticity of demand is given by

$$E = \left|\frac{p}{q} \cdot \frac{dq}{dp}\right| = \left|\frac{p}{q} \cdot \frac{d}{dp}(5000 - 10p^2)\right| = \left|\frac{p}{q} \cdot (-20p)\right| = \frac{20p^2}{q}$$

Substituting $p = 2$ and $q = 4960$ yields

$$E = \frac{20(2)^2}{4960} = \frac{80}{4960} = 0.016.$$

Since $E < 1$ the demand is inelastic, so it would be more accurate to say "People want yams and will buy them no matter what the price."

14. **(a)** At a price of \$2/pound, the quantity sold is

$$q = 5000 - 10(2)^2 = 5000 - 40 = 4960$$

so the total revenue is

$$R = pq = 2 \cdot 4960 = \$9{,}920$$

(b) We know that $R = pq$, and that $q = 5000 - 10p^2$, so we can substitute for q to find $R(p)$

$$R(p) = p(5000 - 10p^2) = 5000p - 10p^3$$

To find the price that maximizes revenue we take the derivative and set it equal to 0.

$$\begin{aligned}R'(p) &= 0\\ 5000 - 30p^2 &= 0\\ 30p^2 &= 5000\\ p^2 &= 166.67\\ p &= \pm 12.91\end{aligned}$$

We disregard the negative answer, so $p = 12.91$ is the only critical point. Is it the maximum? We use the first derivative test.

$$\begin{aligned}&R'(p) > 0 \text{ if } p < 12.91 \text{ and}\\ &R'(p) < 0 \text{ if } p > 12.91\end{aligned}$$

So $R(p)$ has a local maximum at $p = 12.91$. We also test the function at $p = 0$, which is the only endpoint.

$$R(0) = 5000(0) - 10(0)^3 = 0$$

$$R(12.91) = 5000(12.91) - 10(12.91)^3 = 64{,}550 - 21{,}516.85 = \$43{,}033.15$$

So we conclude that revenue is maximized at price of \$12.91/pound.

(c) At a price of \$12.91/pound the quantity sold is

$$q = 5000 - 10(12.91)^2 = 5000 - 1666.68 = 3333.32$$

so the total revenue is

$$R = pq = (3333.32)(12.91) = \$43{,}033.16$$

which agrees with part (b).

(d)

$$E = \left|\frac{p}{q}\cdot\frac{dq}{dp}\right| = \left|\frac{p}{q}\cdot\frac{d}{dp}(5000-10p^2)\right| = \left|\frac{p}{q}\cdot(-20p)\right| = \frac{20p^2}{q}$$

Substituting $p = 12.91$ and $q = 3333.32$ yields

$$E = \frac{20(12.91)^2}{3333.32} = \frac{3333.36}{3333.32} \approx 1$$

which agrees with the result that maximum revenue occurs when $E = 1$.

15. **(a)** There were good substitutes for slaves in city occupations - including free blacks. There were no good substitutes in the countryside.
(b) They were from the countryside, where there was no satisfactory substitute for slaves.

16. The revenue is maximized by finding the critical point of the revenue function:

$$R = pq = p(1000-2p^2) = 1000p - 2p^3.$$

Differentiate to find the critical points:

$$\begin{aligned}\frac{dR}{dp} = 1000 - 6p^2 &= 0\\ p^2 &= \frac{1000}{6}\\ p &\approx 12.91\end{aligned}$$

To maximize revenues, the price of the product should be $\$12.91$.

17. Demand is elastic at all prices. No matter what the price is, you can increase revenue by lowering the price. In the end, you would lower your prices all the way to zero. This is not a realistic example, but it is mathematically possible. It would correspond, for instance, to the demand equation $q = 1/p^2$, which gives revenue $R = pq = 1/p$ which is decreasing for all prices $p > 0$.

18. Demand is inelastic at all prices. No matter what the price is, you can increase revenue by raising the price, so there is no actual price for which your revenue is maximized. This is not a realistic example, but it is mathematically possible. It would correspond, for instance, to the demand equation $q = 1/\sqrt{p}$, which gives revenue $R = pq = \sqrt{p}$ which is increasing for all prices $p > 0$.

19. **(a)** Since $q = k/p$, we have $dq/dp = -k/p^2$ and $p/q = p/(k/p) = p^2/k$. Therefore

$$E = \left|\frac{p}{q}\cdot\frac{dq}{dp}\right| = \left|\frac{p^2}{k}\cdot\frac{-k}{p^2}\right| = 1$$

Elasticity is a constant equal to 1, independent of the price.
(b) Elasticity equal to 1 corresponds to critical points of the revenue function R. Since $R = pq = k$ is constant, $dR/dp = 0$, and so all prices are critical points of the revenue function.

20.

$$E = |p/q \cdot dq/dp| = |p^{r+1}/k \cdot -kr/p^{r+1}| = r$$

21. **(a)** We see that $d_1 = p_0$. In addition, we have

$$\frac{dp}{dq} = \text{Slope of demand curve} = \frac{\text{Rise}}{\text{Run}} = \frac{-d_2}{q_0}$$

so $d_2 = -q_0 \cdot dp/dq$. Therefore

$$\frac{d_1}{d_2} = \frac{p_0}{-q_0\cdot\frac{dp}{dq}} = -\frac{p_0}{q_0}\cdot\frac{dq}{dp} = E$$

(b) For prices near the maximum possible, we have $d_1 > d_2$ and so elasticity $E = d_1/d_2$ is greater than 1 for small quantities. For prices p near 0 (that is, near the q-axis), we have $d_1 < d_2$ and so $E = d_1/d_2$ is less than 1 for large quantities. Elasticity equals 1 where $d_1 = d_2$ at exactly half the maximum possible quantity. See Figure 4.115.

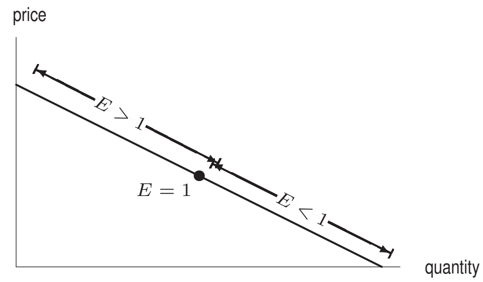


Figure 4.115

22. Since marginal revenue equals dR/dq and $R = pq$, we have, using the product rule,

$$\frac{dR}{dq} = \frac{d(pq)}{dq} = p \cdot 1 + \frac{dp}{dq} \cdot q = p\left(1 + \frac{q}{p} \cdot \frac{dp}{dq}\right) = p\left(1 - \frac{1}{-\frac{p}{q} \cdot \frac{dq}{dp}}\right) = p\left(1 - \frac{1}{E}\right).$$

23. Marginal cost equals $C'(q) = k$. On the other hand, by the preceding exercise, marginal revenue equals $p(1 - 1/E)$. Maximum profit will occur when the two are equal

$$k = p(1 - 1/E)$$

Thus

$$k/p = 1 - 1/E$$
$$1/E = 1 - k/p$$

On the other hand,

$$\frac{\text{Profit}}{\text{Revenue}} = \frac{\text{Revenue} - \text{Cost}}{\text{Revenue}} = 1 - \frac{\text{Cost}}{\text{Revenue}} = 1 - \frac{kq}{pq} = 1 - \frac{k}{p} = \frac{1}{E}$$

24. **(a)** The approximation $E_{C,q} \approx \frac{\Delta C/C}{\Delta q/q}$ shows that $E_{C,q}$ measures the ratio of the fractional change in the cost of production to the fractional change in the quantity produced. Thus, for example, a 1% increase in production will result in an $E_{C,q}\%$ increase in the cost of production.

(b) Since average cost equals C/q and marginal cost equals dC/dq, we have

$$\frac{\text{Marginal cost}}{\text{Average cost}} = \frac{dC/dq}{C/q} = \frac{q}{C} \cdot \frac{dC}{dq} = E_{C,q}$$

25. The approximation $E_{\text{cross}} \approx |\frac{\Delta q/q}{\Delta p/p}|$ shows that the cross-price elasticity measures the ratio of the fractional change in quantity of chicken demanded to the fractional change in the price of beef. Thus, for example, a 1% increase in the price of beef will stimulate a $E_{\text{cross}}\%$ increase in the demand for chicken, presumably because consumers will react to the price rise in beef by switching to chicken. The cross-price elasticity measures the strength of this switch.

26. The approximation $E_{\text{income}} \approx \left|\frac{\Delta q/q}{\Delta I/I}\right|$ shows that the income elasticity measures the ratio of the fractional change in quantity of the product demanded to the fractional change in the income of the consumer. Thus, for example, a 1% increase in income will translate into an $E_{\text{income}}\%$ increase in the quantity purchased. After an increase in income, the consumer will tend to buy more. The income elasticity measures the strength of this tendency.

Solutions for Section 4.7

1. **(a)** As t gets very very large, $e^{-0.08t} \to 0$ and the function becomes $P \approx 40/1$. Thus, this model implies that when t is very large, the population is 40 billion.

(b) A graph of P against t is shown in Figure 4.116.

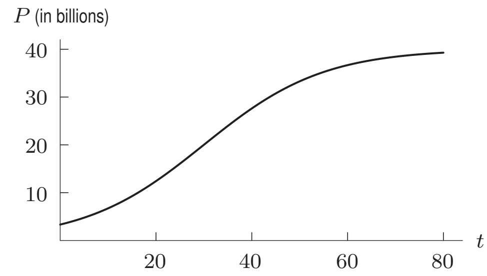


Figure 4.116

(c) We are asked to find the time t such that $P(t) = 20$. Solving we get

$$\begin{aligned}
20 = P(t) &= \frac{40}{1+11e^{-0.08t}} \\
1+11e^{-0.08t} &= \frac{40}{20} = 2 \\
11e^{-0.08t} &= 1 \\
e^{-0.08t} &= \frac{1}{11} \\
\ln e^{-0.08t} &= \ln \frac{1}{11} \\
-0.08t &\approx -2.4 \\
t &\approx \frac{-2.4}{-0.08} = 30
\end{aligned}$$

Thus 30 years from 1990 (the year 2020) the population of the world should be 20 billion.

We are asked to find the time t such that $P(t) = 39.9$. Solving we get

$$\begin{aligned}
39.9 = P(t) &= \frac{40}{1+11e^{-0.08t}} \\
1+11e^{-0.08t} &= \frac{40}{39.9} = 1.00251 \\
11e^{-0.08t} &= 0.00251 \\
e^{-0.08t} &= \frac{0.00251}{11} \\
\ln e^{-0.08t} &= \ln \frac{0.00251}{11} \\
-0.08t &\approx -8.39 \\
t &\approx \frac{-8.39}{-0.08} \approx 105
\end{aligned}$$

Thus 105 years from 1990 (the year 2095) the population of the world should be 39.9 billion.

2. **(a)** Figure 4.117 shows the rate of sales as a function of time. The point of diminishing returns is reached when the rate of sales is at a maximum; this happens during the 4th month.

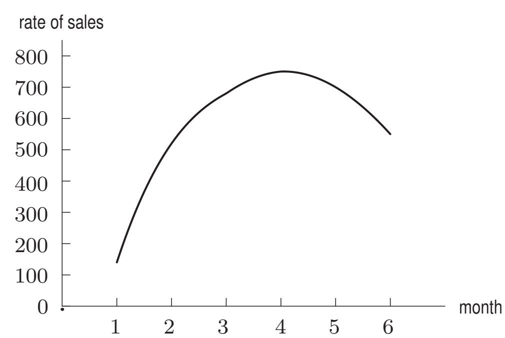


Figure 4.117

(b) Total sales for the first four months are

$$140 + 520 + 680 + 750 = 2090 \text{ sales.}$$

(c) Assuming logistic growth, the limiting value should be twice the value at the point of diminishing returns, so total sales should be $2(2090) = 4180$ sales.

3. (a) The data is plotted in Figure 4.118. On the intervals less than 10, the average rate of change increases, and greater than 10, this rate decreases, so the point of diminishing returns is about 10.

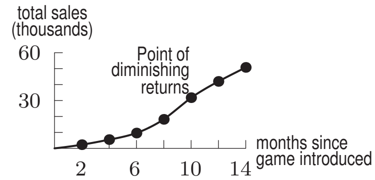


Figure 4.118

(b) The point of diminishing returns happens when total sales reach roughly 30,000. This predicts total sales of about $2(30) = 60{,}000$.

4. (a) Substituting the value $t = 0$ we get

$$\begin{aligned} N(0) &= \frac{400}{1 + 399e^{-0.4(0)}} \\ &= \frac{400}{1 + 399(1)} \\ &= 1 \end{aligned}$$

The fact that $N(0) = 1$ tells us that at the moment the rumor begins spreading, there is only one person who knows the content of the rumor.

(b) Substituting $t = 2$ we get

$$\begin{aligned} N(2) &= \frac{400}{1 + 399e^{-0.4(2)}} \\ &\approx \frac{400}{1 + 399(0.449)} \\ &\approx 2 \end{aligned}$$

Substituting in $t = 10$ we get

$$\begin{aligned} N(10) &= \frac{400}{1 + 399e^{-0.4(10)}} \\ &= \frac{400}{1 + 7.308} \\ &\approx 48 \end{aligned}$$

(c) The graph of $N(t)$ is shown in Figure 4.119.

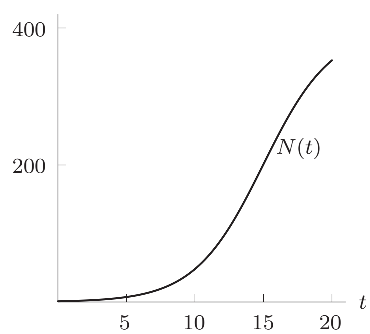


Figure 4.119

(d) We are asked to find the time t at which 200 people will have heard the rumor. In other words, we are asked to solve

$$200 = \frac{400}{1+399e^{-0.4t}}.$$

Solving we get

$$\begin{aligned}
1+399e^{-0.4t} &= \frac{400}{200} = 2 \\
399e^{-0.4t} &= 1 \\
e^{-0.4t} &= \frac{1}{399} \\
\ln e^{-0.4t} &= \ln \frac{1}{399} \\
-0.4t &\approx -5.989 \\
t &= \frac{-5.989}{-0.4} \approx 15.
\end{aligned}$$

Thus after 15 hours half the people will have heard the rumor.

We are asked to solve for the time at which virtually everyone will have heard the rumor. Since the value of $N(t)$ only makes sense after rounding to the nearest integer (i.e., it does not make sense to say that 2.5 people heard the rumor,) asking for the time at which all 400 people will have heard the rumor is equivalent to asking for the time t such that

$$N(t) = 399.5.$$

Solving this we get

$$\begin{aligned}
399.5 = N(t) &= \frac{400}{1+399e^{-0.4t}} \\
1+399e^{-0.4t} &= \frac{400}{399.5} \approx 1.00125 \\
399e^{-0.4t} &\approx 0.00125 \\
e^{-0.4t} &\approx \frac{0.00125}{399} \\
\ln e^{-0.4t} &\approx \ln \frac{0.00125}{399} \\
-0.4t &\approx -12.7 \\
t &\approx \frac{-12.7}{-0.4} = 31.75
\end{aligned}$$

Thus after approximately 32 hours virtually everyone will have heard the rumor.

(e) The rumor is spreading fastest at $L/2 = 400/2 = 200$ or when 200 people have already heard the rumor, so after about 15 hours.

5.

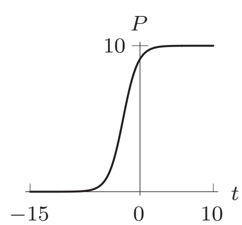


Figure 4.120: $C = 0.1$

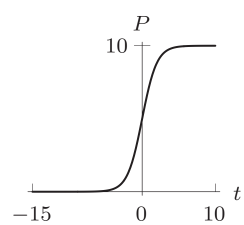


Figure 4.121: $C = 1$

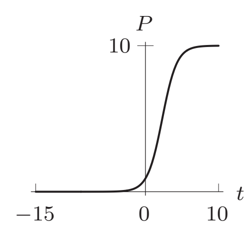


Figure 4.122: $C = 10$

See Figures 4.120–4.122. The value of C appears to affect where the curve cross the vertical axis. When $C = 1$, the graph crosses the vertical axis at $t = 0$ or at the point of diminishing returns. If $C > 1$, the graph crosses the the vertical axis before the point of diminishing returns; the higher C is, the sooner the graph crosses the vertical axis. If $C < 1$, the curve crosses the vertical axis after it changes concavity; the smaller C is, the closer to the limiting value the graph is before crossing the vertical axis.

6. Sales of a new product could very well follow a logistic curve. At first, sales will grow exponentially as more and more people hear of the product and decide to buy it. Eventually, though, everyone will know about the product and while people may still buy it, not as many will (sales will slow down). Eventually, it's possible that everyone who would buy the product already has, at which point sales will stop. It behooves the seller to notice the point of diminishing returns so they don't make more of their product than people will want to buy.

7. (a) The limiting value of $f(t)$ is about 36, so the total number of infected computers is about 36,000.
(b) The curve has an inflection point at about $t \approx 16$ hours, and then $n = f(16) \approx 18$.
(c) The virus was spreading fastest at about when $t = 16$, that is at 4 pm on July 19, 2001. At that time, about 18,000 computers were infected.
(d) At the inflection point, the number of computers infected is about half the number infected in the long run.

8. The slope of the curve, dy/dt, is given by

$$\frac{dy}{dt} = -50\left(1+6e^{-2t}\right)^{-2}\left(-12e^{-2t}\right) = 600e^{-2t}\left(1+6e^{-2t}\right)^{-2}.$$

If the slope has a maximum, it occurs where its derivative is zero, that is where

$$\begin{aligned}
\frac{d^2y}{dt^2} = 600\left(-2e^{-2t}\right)\left(1+6e^{-2t}\right)^{-2} - 1200e^{-2t}\left(1+6e^{-2t}\right)^{-3}\left(-12e^{-2t}\right) &= 0\\
\frac{1200e^{-2t}}{(1+6e^{-2t})^3}\left(-(1+6e^{-2t}) + 12e^{-2t}\right) &= 0\\
6e^{-2t} - 1 &= 0\\
e^{-2t} &= \frac{1}{6}\\
-2t &= \ln\left(\frac{1}{6}\right)\\
t &= -\frac{1}{2}\ln\left(\frac{1}{6}\right) = \frac{1}{2}\ln 6.
\end{aligned}$$

For this value of t,

$$y = \frac{50}{1+6e^{-2\left(\frac{1}{2}\ln 6\right)}} = \frac{50}{1+6e^{-\ln 6}} = \frac{50}{1+6\left(\frac{1}{6}\right)} = 25.$$

Since we see in Figure 4.123 that the slope increases as t increases from 0 and tends to 0 as $t \to \infty$, the only critical point $t = \frac{1}{2}\ln 6$ is a local and global maximum for the slope.

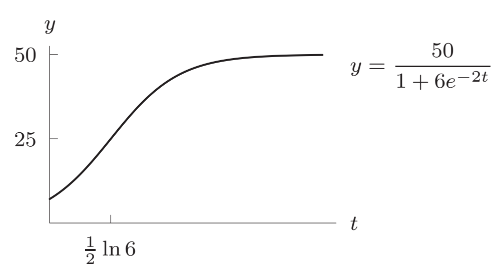


Figure 4.123

9. (a), (b) The graph of $P(t)$ with carrying capacity L and point of diminishing returns t_0 is in Figure 4.124. The derivative $P'(t)$ is also shown.

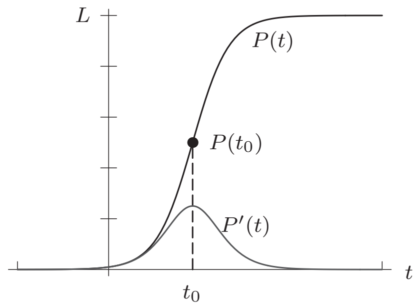


Figure 4.124

(c) Keeping track of rate of sales is the same as keeping track of the derivative $P'(t)$. The point of diminishing returns happens when the concavity of $P(t)$ changes, which is the when the derivative, $P'(t)$, switches from increasing to decreasing. This happens when $P'(t)$ reaches its maximum at $t = t_0$

10. **(a)** At $t = 0$, which corresponds to 1935, we have

$$P = \frac{1}{1 + 3e^{-0.0275(0)}} = 0.25$$

showing that 25% of the land was in use in 1935.

(b) This model predicts that as t gets very large, P approaches 1. That is, the model predicts that in the long run, all the land will be used for farming.

(c) To solve this graphically, enter the function into a graphing calculator and trace the resulting curve until it reaches a height of 0.5, which occurs when $t = 39.9 \approx 40$. Since $t = 0$ corresponds to 1935, $t = 40$ corresponds to $1935 + 40 = 1975$. According to this model, the Tojolobal were using half their land in 1975. See Figure 4.125.

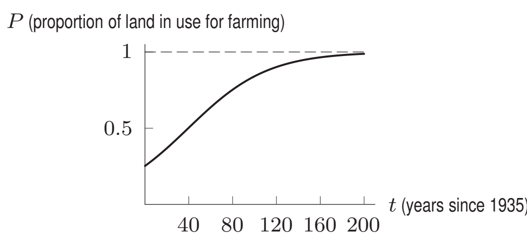


Figure 4.125

(d) The point of diminishing returns occurs when $P = L/2$ or at one-half the carrying capacity. In this case, $P = 1/2$ in 1975, as shown in part (c).

11. **(a)** Between day $t = 0$ and day $t = 5$,

$$\begin{array}{c}\text{Average}\\ \text{rate of change}\end{array} = \frac{\Delta P}{\Delta t} = \frac{222 - 95}{5 - 0} = 25.4 \text{ new cases per day.}$$

Similarly, we find the values in Table 4.4.

Table 4.4

Days	0–5	5–12	12–19	19–26	26–33	33–40	40–47
New cases per day	25.4	35.4	47.1	44.0	35.7	24.1	13.4
Days	47–54	54–61	61–68	68–75	75–81	81–87	
New cases per day	7.6	5.1	2.0	2.1	1.8	0.8	

(b) The first computation showing a reduced rate of change was made with the data for $t = 26$, which corresponds to April 12, 2003. On this date, epidemiologists could report that the rate of new cases had begun to slow.

(c) The rate of change increases from $t = 0$ to $t = 19$, so the graph is concave up there. After $t = 19$, the rate of change decreases and the graph is concave down. Exponential functions do not change concavity.

(d) The inflection point appears to occur near $t = 19$, when the average number of new cases per day stops increasing and begins decreasing. At this point, we have $P = 800$, so we estimate the limiting value of P to be about $2 \cdot 800 = 1600$ cases.

(e) The predicted limiting value is 1760, which is close to the actual value of 1755.

12. Substituting $t = 0, 10, 20, \ldots, 70$ into the function $P = 3.9(1.03)^t$ gives the values in Table 4.5. Notice that the agreement is very close, reflecting the fact that an exponential function models the growth well over the period 1790–1860.

Table 4.5 *Predicted versus actual US population 1790–1860, in millions. (exponential model)*

Year	Actual	Predicted	Year	Actual	Predicted
1790	3.9	3.9	1830	12.9	12.7
1800	5.3	5.2	1840	17.1	17.1
1810	7.2	7.0	1850	23.2	23.0
1820	9.6	9.5	1860	31.4	30.9

13. Substituting $t = 0, 10, 20, 30, \ldots$ into the function

$$P = \frac{187}{1 + 47e^{-0.0318t}}$$

gives the values in Table 4.6. Notice that the agreement is close between the predicted and actual values.

Table 4.6

Year	Actual	Predicted	Year	Actual	Predicted	Year	Actual	Predicted
1790	3.9	3.9	1840	17.1	17.7	1890	62.9	63.3
1800	5.3	5.3	1850	23.1	23.5	1900	76.0	77.2
1810	7.2	7.2	1860	31.4	30.8	1910	92.0	91.9
1820	9.6	9.8	1870	38.6	39.9	1920	105.7	106.7
1830	12.9	13.2	1880	50.2	50.7	1930	122.8	120.8
						1940	131.7	133.7

14. **(a)** We use $k = 1.78$ as a rough approximation. We let $L = 5000$ since the problem tells us that 5000 people eventually get the virus. This means the limiting value is 5000.

(b) We know that

$$P(t) = \frac{5000}{1 + Ce^{-1.78t}} \quad \text{and} \quad P(0) = 10$$

so

$$\begin{aligned} 10 &= \frac{5000}{1 + Ce^0} = \frac{5000}{1 + C} \\ 10(1 + C) &= 5000 \\ 1 + C &= 500 \\ C &= 499. \end{aligned}$$

(c) We have $P(t) = \dfrac{5000}{1 + 499e^{-1.78t}}$. This function is graphed in Figure 4.126.

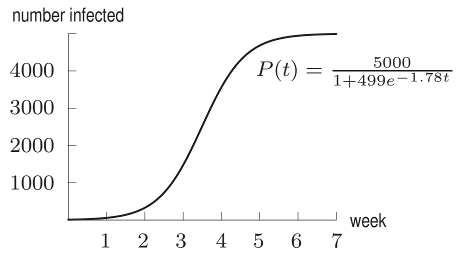


Figure 4.126

(d) The point of diminishing returns appears to be at the point $(3.5, 2500)$; that is, after 3 and a half weeks and when 2500 people are infected.

15. (a) The dose-response curve for this drug is shown if Figure 4.127.

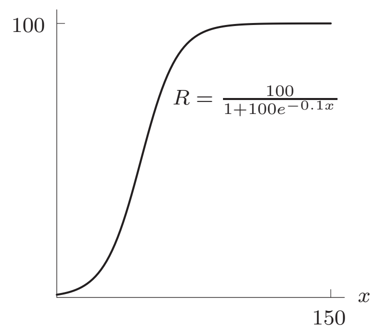


Figure 4.127

(b) First solve for x:

$$\begin{aligned}
R &= \frac{100}{1+100e^{-0.1x}} \\
R + 100Re^{-0.1x} &= 100 \\
100Re^{-0.1x} &= 100 - R \\
e^{-0.1x} &= \frac{100-R}{100R} \\
-0.1x &= \ln\left(\frac{100-R}{100R}\right) \\
x &= -10\ln\left(\frac{100-R}{100R}\right).
\end{aligned}$$

Substituting the value of $R = 50$ we get

$$x = -10\ln\left(\frac{100-50}{100\cdot 50}\right) \approx 46.0517019 \approx 46.05$$

(c) Evaluate the formula for x obtained in part (b) for $R = 20$ and $R = 70$.
For $R = 20$,

$$x = -10\ln\left(\frac{100-20}{100\cdot 20}\right) \approx 32.18875825 \approx 32.12$$

For $R = 70$,

$$x = -10\ln\left(\frac{100-70}{100\cdot 70}\right) \approx 54.5246805 \approx 54.52$$

So the range of doses that is both safe and effective is between 32.12 mg and 54.52 mg.

16. **(a)** The dose-response curve for product C crosses the minimum desired response line last, so it requires the largest dose to achieve the desired response. The dose-response curve for product B crosses the minimum desired response line first, so it requires the smallest dose to achieve the desired response.

(b) The dose-response curve for product A levels off at the highest point, so it has the largest maximum response. The dose-response curve for product B levels off at the lowest point, so it has the smallest maximum response.

(c) Product C is the safest to administer because its slope in the safe and effective region is the least, so there is a broad range of dosages for which the drug is both safe and effective.

17. **(a)** The fact that $f'(15) = 11$ means that the slope of the curve at the inflection point, $(15, 50)$ is 11. In terms of dose and response, this large slope tells us that the range of doses for which this drug is both safe and effective is small.

(b) As we can see from Figure 4.77 in the text, a dose-response curve starts out concave up (slope increasing) and switches to concave down (slope decreasing) at an inflection point. Since the slope at the inflection point $(15, 50)$ is 11, and the slope is increasing before the inflection point, $f'(10)$ is less than 11. Since the slope is decreasing after the inflection point, $f'(20)$ is also less than 11.

18. If the derivative of the dose-response curve is smaller, the slope is not as steep. Since the slope is not as steep, the response increases less at the same dosage. Therefore, there is a wider range of dosages that are both safe and effective, and consequently the dosage given to the patient does not have to be as exact.

19. The range of safe and effective doses begins at 10 mg where the drug is effective for 100 percent of patients. It ends at 18 mg where the percent lethal curve begins to rise above zero.

20. When 50 mg of the drug is administered, it is effective for 85 percent of the patients and lethal for 6 percent.

21. **(a)** We have

$$\begin{aligned} L - P &= L - \frac{L}{1 + Ce^{-kt}} \\ &= \frac{L(1 + Ce^{-kt}) - L}{1 + Ce^{-kt}} \\ &= \frac{CLe^{-kt}}{1 + Ce^{-kt}} \\ &= CPe^{-kt}. \end{aligned}$$

Dividing by P gives

$$\frac{L - P}{P} = Ce^{-kt}.$$

(b) The existing population is P. Since the carrying capacity is L, the additional population the environment can support is $L - P$.

22. **(a)** We have

$$\begin{aligned} f(V) &= \frac{1}{1 + e^{-(V+25)/2}} \\ &= \frac{1}{1 + e^{-25/2}e^{-\frac{1}{2}V}}. \end{aligned}$$

Comparison with

$$f(V) = \frac{L}{1 + Ce^{-kV}}$$

shows that $L = 1$, $k = 1/2 = 0.5$, and $C = e^{-25/2} = 3.7 \cdot 10^{-6}$.

(b) Ten percent of the channels are open at potential V where

$$\begin{aligned} f(V) &= 0.1 \\ \frac{1}{1 + e^{-(V+25)/2}} &= 0.1 \\ V &= -29.4 \text{ mV}. \end{aligned}$$

Half the channels are open when $f(V) = 0.5$, or $V = -25$ mV.
Ninety per cent of the channels are open when $f(V) = 0.9$, or $V = -20.6$ mV.

Solutions for Section 4.8

1. **(a)** See Figure 4.128.

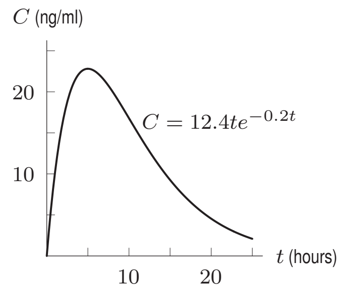


Figure 4.128

(b) The surge function $y = ate^{bt}$ changes from increasing to decreasing at $t = \frac{1}{b}$. For this function $b = 0.2$ so the peak is at $\frac{1}{0.2} = 5$ hours. We can now substitute this into the formula to compute the peak concentration:

$$C = 12.4(5)e^{-0.2(5)} = 22.8085254 \text{ ng/ml} \approx 22.8 \text{ ng/ml}.$$

(c) Tracing along the graph of $C = 12.4te^{-0.2t}$, we see it crosses the line $C = 10$ at $t \approx 1$ hour and at $t \approx 14.4$ hours. Thus, the drug is effective for $1 \le t \le 14.4$ hours.

(d) The drug drops below $C = 4$ for $t > 20.8$ hours. Thus, it is safe to take the other drug after 20.8 hours.

2.

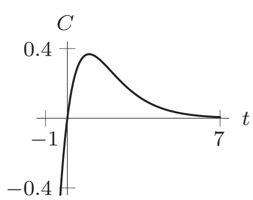


Figure 4.129: $a = 1$

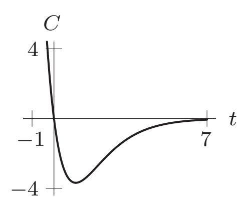


Figure 4.130: $a = -10$

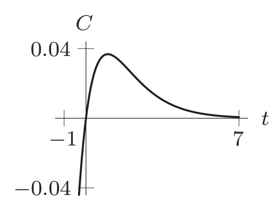


Figure 4.131: $a = 0.1$

The parameter a apparently affects the height and direction of $C = ate^{-bt}$. If a is positive, the "hump" is above the t-axis. If a is negative, it's below the t-axis. If $a > 0$, the larger the value of a, the larger the maximum value of C. If $a < 0$, the more negative the value of a, the smaller the minimum value of C.

3. Newborn babies are generally not as effective as adults in eliminating drugs from the body. See the figure in the text. The level of the drug in the body begins to decline after two hours in an adult, but not until six hours have elapsed in a newborn. In addition, the concentration in the blood reaches a higher level in the newborn than it does in the adult. Graphs such as this explain why physicians modify the doses for drugs given to pregnant women and women who are breastfeeding.

4.

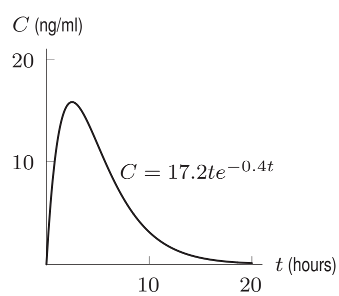


Figure 4.132

(a) See Figure 4.132. The first dose becomes ineffective when

$$10 = 17.2te^{-0.4t}.$$

To find this value of t, trace along the curve $C = 17.2te^{-0.4t}$, giving $t \approx 6$ hours. Thus, the second dose should be given after about 6 hours.

(b) By tracing, we find that the second dose becomes effective about 1 hour after it is given. We want it to become effective when $t \approx 6$, so the second dose should be given at $t \approx 5$ hours.

5. Large quantities of water dramatically increase the value of the peak concentration, but do not change the amount of time it takes to reach the peak concentration. The effect of the volume of water taken with the drug wears off after approximately 6 hours.

6. Figure 4.133 has its maximum at $t = 1.3$ hours, $C = 23.6$ ng/ml.

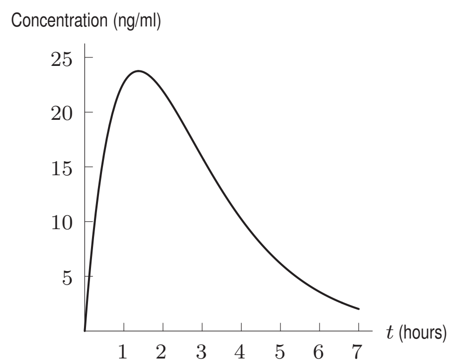


Figure 4.133

7. For cigarettes the peak concentration is approximately 17 ng/ml, the time until peak concentration is about 10 minutes, and the nicotine is at first eliminated quickly, but after about 45 minutes it is eliminated more slowly.

For chewing tobacco, the peak concentration is approximately 14 ng/ml and the time until peak concentration is about 30 minutes (when chewing stops). The nicotine is eliminated at a slow, somewhat erratic rate.

For nicotine gum the peak concentration is about 10 ng/ml and the time until peak concentration is approximately 45 minutes. The nicotine is eliminated at a very slow but steady rate.

8. **(a)** Differentiating using the product rule gives

$$\begin{aligned} C'(t) &= 20e^{-0.03t} + 20t(-0.03)e^{-0.03t} \\ &= 20(1 - 0.03t)e^{-0.03t}. \end{aligned}$$

At the peak concentration, $C'(t) = 0$, so

$$\begin{aligned} 20(1 - 0.03t)e^{-0.03t} &= 0 \\ t = \frac{1}{0.03} &= 33.3 \text{ minutes.} \end{aligned}$$

When $t = 33.3$, the concentration is

$$C = 20(33.3)e^{-0.03(33.3)} \approx 245 \text{ ng/ml.}$$

See Figure 4.134. The curve peaks after 33.3 minutes with a concentration of 244.9 ng/ml.

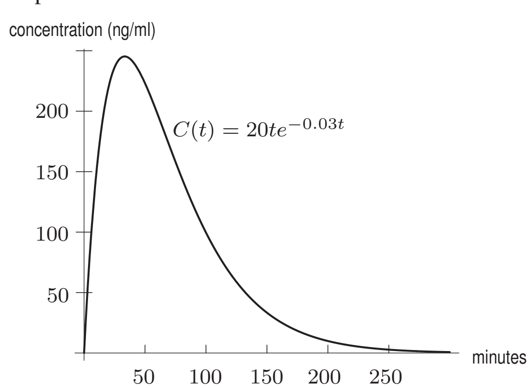


Figure 4.134

(b) After 15 minutes, the drug concentration will be $C(15) = 20(15)e^{(-0.03)(15)} \approx 191$ ng/ml. After an hour, the concentration will be $C(60) = 20(60)e^{-0.03(60)} \approx 198$ ng/ml.

(c) We want to know where $C(t) = 10$. We estimate from the graph. It looks like $C(t) = 10$ after 190 minutes or a little over 3 hours.

9. We find values of the parameters in the function $C = ate^{-bt}$ to create a local maximum at the point $(1.3, 23.6)$. We first set the derivative equal to zero and solve for t to find critical points. Using the product rule, we have:

$$\begin{aligned}\frac{dC}{dt} = at(e^{-bt}(-b)) + a(e^{-bt}) &= 0\\ ae^{-bt}(-bt+1) &= 0\\ t &= \frac{1}{b}.\end{aligned}$$

The only critical point is at $t = 1/b$. Since we want a critical point at $t = 1.3$, we substitute and solve for b:

$$\begin{aligned}1.3 &= \frac{1}{b}\\ b &= \frac{1}{1.3} = 0.769.\end{aligned}$$

To find the value of a, we use the fact that $C = 23.6$ when $t = 1.3$. We have:

$$\begin{aligned}a(1.3)e^{-0.769(1.3)} &= 23.6\\ a \cdot 1.3e^{-1} &= 23.6\\ a &= \frac{23.6}{1.3e^{-1}} = 49.3.\end{aligned}$$

We have $a = 49.3$ and $b = 0.769$.

10. **(a)** The IV method reaches peak concentration the fastest, it in fact begins at its peak.
The P-IM method reaches peak concentration the slowest.

(b) The IV method has the largest peak concentration.
The PO method has the smallest peak concentration.

(c) The IV method wears off the fastest.
The P-IM method wears off the slowest.

(d) The P-IM method has the longest effective duration.
The IV method has the shortest effective duration.

(e) It is effective for approximately 5 hours.

11. Food dramatically increases the value of the peak concentration but does not affect the time it takes to reach the peak concentration. The effect of food is stronger during the first 8 hours.

12. **(a)** The value $f(t) = e^{-t} - e^{-2t}$ is the vertical distance between the two curves in Figure 4.135. That distance is 0 at $t = 0$. The distance first increases as t increases from 0. As t gets larger and larger, the distance eventually begins to decrease toward 0 because both e^{-t} and e^{-2t} approach the same horizontal asymptote, the t-axis. Thus the graph of $f(t)$ shows a rapid increase from zero followed by a slow decrease back toward zero; this is the shape of a surge.

(b) At a critical point we have

$$f'(t) = -e^{-t} + 2e^{-2t} = 0.$$

Hence

$$\begin{aligned}e^{-t} &= 2e^{-2t}\\ e^t &= 2\\ t &= \ln 2 = 0.693.\end{aligned}$$

At the critical point, $f(\ln 2) = 1/4 = 0.250$. The critical point is $(\ln 2, 1/4) = (0.693, 0.250)$.

At an inflection point we have

$$f''(t) = e^{-t} - 4e^{-2t} = 0.$$

Hence

$$\begin{aligned}e^{-t} &= 4e^{-2t}\\ e^t &= 4\\ t &= \ln 4 = 1.386\\ f(\ln 4) &= \frac{3}{16} = 0.1875.\end{aligned}$$

The inflection point is $(2\ln 2, 3/16) = (1.386, 0.187)$. See Figure 4.136

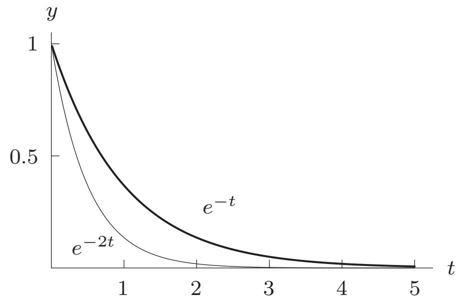


Figure 4.135

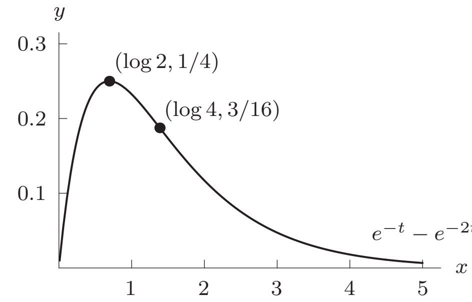


Figure 4.136

13. **(a)** Products A and B have much higher peak concentrations than products C and D. Product A reaches its peak concentration slightly before products B, C, and D, which all take about the same time to reach peak concentration.

(b) If the minimum effective concentration were low, perhaps 0.2, and the maximum safe concentration were also low, perhaps 1.0, then products C and D would be the preferred drugs since they do not enter the unsafe range while being well within the effective range.

(c) If the minimum effective concentration were high, perhaps 1.2, and the maximum safe concentration were also high, perhaps 2.0, then product A would be the preferred drug since it does not enter the unsafe range and it is the only drug that is in the effective range for a substantial amount of time.

14. In general, a faster dissolution rate corresponds to a larger peak concentration. Dissolution rate does not affect time to reach peak concentration, as the four products have different dissolution rates but virtually the same time to reach peak concentration.

Solutions for Chapter 4 Review

1. See Figure 4.137.

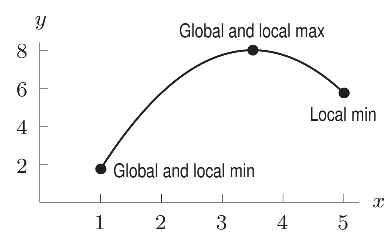


Figure 4.137

2. The global maximum is achieved at the two local maxima, which are at the same height. See Figure 4.138.

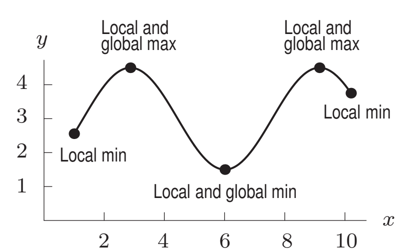


Figure 4.138

3. **(a)** We have $f'(x) = 10x^9 - 10 = 10(x^9 - 1)$. This is zero when $x = 1$, so $x = 1$ is a critical point of f. For values of x less than 1, x^9 is less than 1, and thus $f'(x)$ is negative when $x < 1$. Similarly, $f'(x)$ is positive for $x > 1$. Thus $f(1) = -9$ is a local minimum.

We also consider the endpoints $f(0) = 0$ and $f(2) = 1004$. Since $f'(0) < 0$ and $f'(2) > 0$, we see $x = 0$ and $x = 2$ are local maxima.

(b) Comparing values of f shows that the global minimum is at $x = 1$, and the global maximum is at $x = 2$.

4. **(a)** $f'(x) = 1 - 1/x$. This is zero only when $x = 1$. Now $f'(x)$ is positive when $1 < x \leq 2$, and negative when $0.1 < x < 1$. Thus $f(1) = 1$ is a local minimum. The endpoints $f(0.1) \approx 2.4026$ and $f(2) \approx 1.3069$ are local maxima.

(b) Comparing values of f shows that $x = 0.1$ gives the global maximum and $x = 1$ gives the global minimum.

5. We rewrite $h(z)$ as $h(z) = z^{-1} + 4z^2$.

Differentiating gives

$$h'(z) = -z^{-2} + 8z,$$

so the critical points satisfy

$$\begin{aligned} -z^{-2} + 8z &= 0 \\ z^{-2} &= 8z \\ 8z^3 &= 1 \\ z^3 &= \frac{1}{8} \\ z &= \frac{1}{2}. \end{aligned}$$

Since h' is negative for $0 < z < 1/2$ and h' is positive for $z > 1/2$, there is a local minimum at $z = 1/2$.

Since $h(z) \to \infty$ as $z \to 0^+$ and as $z \to \infty$, the local minimum at $z = 1/2$ is a global minimum; there is no global maximum. See Figure 4.139. Thus, the global minimum is $h(1/2) = 3$.

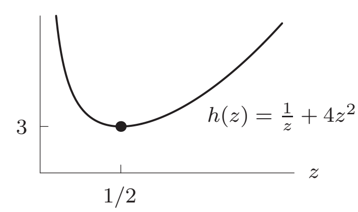


Figure 4.139

6. Since $g(t)$ is always decreasing for $t \geq 0$, we expect it to a global maximum at $t = 0$ but no global minimum. At $t = 0$, we have $g(0) = 1$, and as $t \to \infty$, we have $g(t) \to 0$.

Alternatively, rewriting as $g(t) = (t^3 + 1)^{-1}$ and differentiating using the chain rule gives

$$g'(t) = -(t^3 + 1)^{-2} \cdot 3t^2.$$

Since $3t^2 = 0$ when $t = 0$, there is a critical point at $t = 0$, and g decreases for all $t > 0$. See Figure 4.140.

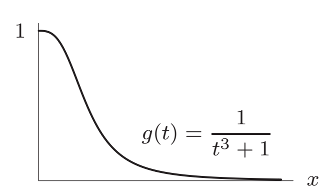


Figure 4.140

7. We begin by rewriting $f(x)$:

$$f(x) = \frac{1}{(x-1)^2+2} = ((x-1)^2+2)^{-1} = (x^2-2x+3)^{-1}.$$

Differentiating using the chain rule gives

$$f'(x) = -(x^2-2x+3)^{-2}(2x-2) = \frac{2-2x}{(x^2-2x+3)^2},$$

so the critical points satisfy

$$\begin{aligned}\frac{2-2x}{(x^2-2x+3)^2} &= 0\\ 2-2x &= 0\\ 2x &= 2\\ x &= 1.\end{aligned}$$

Since f' is positive for $x<1$ and f' is negative for $x>1$, there is a local maximum at $x=1$.

Since $f(x)\to 0$ as $x\to\infty$ and as $x\to-\infty$, the local maximum at $x=1$ is a global maximum; there is no global minimum. See Figure 4.141. Thus, the global maximum is $f(1)=1/2$.

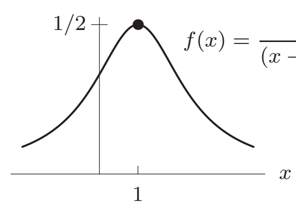


Figure 4.141

8. There are several possibilities. The price could have been increasing during the last few days of June, reaching a high point on July 1, then going back down during the first few days of July. In this case there was a local maximum in the price on July 1.

The price could have been decreasing during the last few days of June, reaching a low point on July 1, then going back up during the first few days of July. In this case there was a local minimum in the price on July 1.

It is also possible that there was neither a local maximum nor a local minimum in the price on July 1. This could have happened two ways. On the one hand, the price could have been rising in late June, then held steady with no change around July 1, after which the price increased some more. On the other hand, the price could have been falling in late June, then held steady with no change around July 1, after which the price fell some more. The key feature in these critical point scenarios is that there was no appreciable change in the price of the stock around July 1.

9. **(a)** Varying the a parameter changes the peak concentration proportionally, but does not change the time to reach peak concentration .
(b) The value of the peak concentration decreases as b increases and the time until peak concentration decreases as b increases.
(c) As a gets larger, the time until peak concentration decreases because it is only affected by the constant in the exponent. The value of the peak concentration is not affected as a changes because the peak always occurs at $t=1/b$ and if $a=b$, the value of the peak concentration is $c = a\cdot\frac{1}{a}e^{-a\cdot 1/a} = e^{-1}$, independent of a.

10. **(a)** Increasing for $x>0$, decreasing for $x<0$.
(b) $f(0)$ is a local and global minimum, and f has no global maximum.

11. **(a)** Increasing for all x.
(b) No maxima or minima.

12. **(a)** Decreasing for $x<0$, increasing for $0<x<4$, and decreasing for $x>4$.
(b) $f(0)$ is a local minimum, and $f(4)$ is a local maximum.

13. **(a)** Decreasing for $x<-1$, increasing for $-1<x<0$, decreasing for $0<x<1$, and increasing for $x>1$.
(b) $f(-1)$ and $f(1)$ are local minima, $f(0)$ is a local maximum.

14. (a) We know that Profit = Revenue − Cost, so differentiating with respect to q gives:

$$\text{Marginal Profit} = \text{Marginal Revenue} - \text{Marginal Cost}.$$

We see from the figure in the problem that just to the left of $q = a$, marginal revenue is less than marginal cost, so marginal profit is negative there. To the right of $q = a$ marginal revenue is greater than marginal cost, so marginal profit is positive there. At $q = a$ marginal profit changes from negative to positive. This means that profit is decreasing to the left of a and increasing to the right. The point $q = a$ corresponds to a local minimum of profit, and does not maximize profit. It would be a terrible idea for the company to set its production level at $q = a$.

(b) We see from the figure in the problem that just to the left of $q = b$ marginal revenue is greater than marginal cost, so marginal profit is positive there. Just to the right of $q = b$ marginal revenue is less than marginal cost, so marginal profit is negative there. At $q = b$ marginal profit changes from positive to negative. This means that profit is increasing to the left of b and decreasing to the right. The point $q = b$ corresponds to a local maximum of profit. In fact, since the area between the MC and MR curves in the figure in the text between $q = a$ and $q = b$ is bigger than the area between $q = 0$ and $q = a$, $q = b$ is in fact a global maximum.

15. (a) Centimeters per week.
(b) At week 24 the fetus is growing at a rate of 1.6 cm/week.

16. (a) The tangent line to the graph is steeper at 20 weeks then at 36 weeks, so $f'(20)$ is greater than $f'(36)$.
(b) The fetus increases its length more rapidly at week 20 than at week 36.

17. (a) The inflection point occurs at about week 14 where the graph changes from concave up to concave down.
(b) The fetus increases its length faster at week 14 than at any other time during its gestation.

18. We estimate the derivatives at 20 and 36 weeks by drawing tangent lines to the length graph, shown in Figure 4.142, and calculating their slopes.

(a) Two points on the tangent line at 20 weeks are $(6, 0)$ and $(32, 49)$. Thus,

$$f'(20) = \frac{49 - 0}{32 - 6} = 1.9 \text{ cm/week}.$$

(b) Two points on the tangent line at 36 weeks are $(0, 20)$ and $(40, 50)$. Thus

$$f'(36) = \frac{50 - 20}{40 - 0} = 0.75 \text{ cm/week}.$$

(c) The average rate of growth is the slope of the secant line from $(0, 0)$ to $(40, 50)$. Thus

$$\text{Average rate of change } = \frac{50 - 0}{40 - 0} = 1.25 \text{ cm/week}.$$

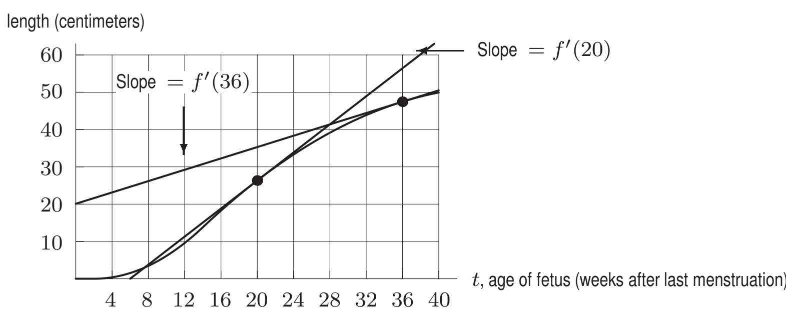


Figure 4.142

19. Using the product rule on the function $f(x) = axe^{bx}$, we have $f'(x) = ae^{bx} + abxe^{bx} = ae^{bx}(1 + bx)$. We want $f(\frac{1}{3}) = 1$, and since this is to be a maximum, we require $f'(\frac{1}{3}) = 0$. These conditions give

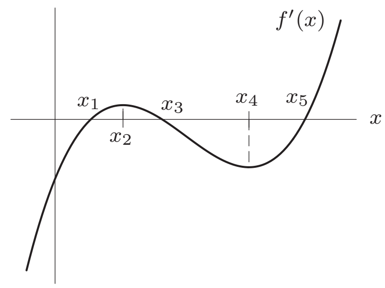
$$f(1/3) = a(1/3)e^{b/3} = 1,$$
$$f'(1/3) = ae^{b/3}(1 + b/3) = 0.$$

Since $ae^{(1/3)b}$ is non-zero, we can divide both sides of the second equation by $ae^{(1/3)b}$ to obtain $0 = 1 + \frac{b}{3}$. This implies $b = -3$. Plugging $b = -3$ into the first equation gives us $a(\frac{1}{3})e^{-1} = 1$, or $a = 3e$. How do we know we have a maximum at $x = \frac{1}{3}$ and not a minimum? Since $f'(x) = ae^{bx}(1 + bx) = (3e)e^{-3x}(1 - 3x)$, and $(3e)e^{-3x}$ is always positive, it follows that $f'(x) > 0$ when $x < \frac{1}{3}$ and $f'(x) < 0$ when $x > \frac{1}{3}$. Since f' is positive to the left of $x = \frac{1}{3}$ and negative to the right of $x = \frac{1}{3}$, $f(\frac{1}{3})$ is a local maximum.

20. Local maximum for some θ, with $1.1 < \theta < 1.2$, since $f'(1.1) > 0$ and $f'(1.2) < 0$.
Local minimum for some θ, with $1.5 < \theta < 1.6$, since $f'(1.5) < 0$ and $f'(1.6) > 0$.
Local maximum for some θ, with $2.0 < \theta < 2.1$, since $f'(2.0) > 0$ and $f'(2.1) < 0$.

21. **(a)** The derivative of $f(x) = x^5 + x + 7$ is $f'(x) = 5x^4 + 1$. The derivative is always positive.
(b) Since $f'(x) \neq 0$ for any value of x, there are no critical points for the function. Since $f'(x)$ is positive for all x, the function is increasing for all x, it crosses the x-axis at most once. Since $f(x) \to +\infty$ as $x \to +\infty$ and $f(x) \to -\infty$ as $x \to -\infty$, the graph of f crosses the x-axis once. So we conclude that $f(x)$ has one real root.

22. **(a)**



(b) $f'(x)$ changes sign at x_1, x_3, and x_5.
(c) $f'(x)$ has local extrema at x_2 and x_4.

23. The local maxima and minima of f correspond to places where f' is zero and changes sign or, possibly, to the endpoints of intervals in the domain of f. The points at which f changes concavity correspond to local maxima and minima of f'. The change of sign of f', from positive to negative corresponds to a maximum of f and change of sign of f' from negative to positive corresponds to a minimum of f.

24. Since the function is positive, the graph lies above the x-axis. If there is a global maximum at $x = 3$, $t'(x)$ must be positive, then negative. Since $t'(x)$ and $t''(x)$ have the same sign for $x < 3$, they must both be positive, and thus the graph must be increasing and concave up. Since $t'(x)$ and $t''(x)$ have opposite signs for $x > 3$ and $t'(x)$ is negative, $t''(x)$ must again be positive and the graph must be decreasing and concave up. A possible sketch of $y = t(x)$ is shown in Figure 4.143.

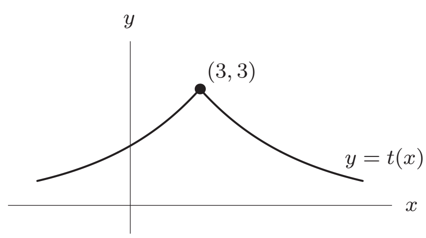


Figure 4.143

25. **(a)** Quadratic polynomial (degree 2) with negative leading coefficient.
(b) Exponential.
(c) Logistic.
(d) Logarithmic.
(e) This is a quadratic polynomial (degree 2) and positive leading coefficient.
(f) Exponential.
(g) Surge

26. First find the marginal revenue and marginal cost. Note that each product sells for \$788, so revenue is given by $R(q) = 788q$.

$$MR = R'(q) = 788,$$
$$MC = C'(q) = 3q^2 - 120q + 1400.$$

Setting $MR = MC$ yields

$$3q^2 - 120q + 1400 = 788$$
$$3q^2 - 120q + 612 = 0.$$

This factors to

$$3(q-34)(q-6) = 0,$$

so $MR = MC$ at $q = 34$ and $q = 6$. We now find the profit at these points:

$$R(6) - C(6) = 788 \cdot 6 - \left[6^3 - 60 \cdot 6^2 + 1400 \cdot 6 + 1000\right] = -\$2728.$$
$$R(34) - C(34) = 788 \cdot 34 - \left[34^3 - 60 \cdot 34^2 + 1400 \cdot 34 + 1000\right] = \$8248$$

We must also try the endpoints

$$R(0) - C(0) = 788 \cdot 0 - \left[0^3 - 60 \cdot 0^2 + 1400 \cdot 0 + 1000\right] = -\$1000$$
$$R(50) - C(50) = 788 \cdot 50 - \left[50^3 - 60 \cdot 50^2 + 1400 \cdot 50 + 1000\right] = -\$6600.$$

From this we see that profit is maximized at $q = 34$ units. The total cost at $q = 34$ is $C(34) = \$18{,}544$. The total revenue at $q = 34$ is $R(34) = \$26{,}792$ and

$$\text{Total profit } = 26{,}792 - 18{,}544 = \$8248.$$

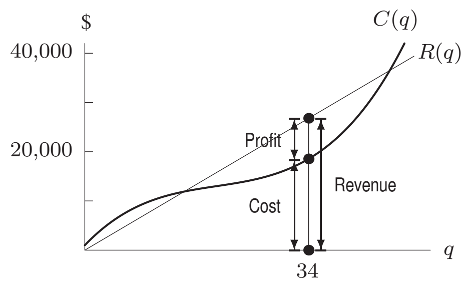


Figure 4.144

27. **(a)** The fixed cost is 0 because $C(0) = 0$.

(b) Profit, $\pi(q)$, is equal to money from sales, $7q$, minus total cost to produce those items, $C(q)$.

$$\pi = 7q - 0.01q^3 + 0.6q^2 - 13q$$
$$\pi' = -0.03q^2 + 1.2q - 6$$

$$\pi' = 0 \quad \text{if} \quad q = \frac{-1.2 \pm \sqrt{(1.2)^2 - 4(0.03)(6)}}{-0.06} \approx 5.9 \quad \text{or} \quad 34.1.$$

Now $\pi'' = -0.06q + 1.2$, so $\pi''(5.9) > 0$ and $\pi''(34.1) < 0$. This means $q = 5.9$ is a local min and $q = 34.1$ a local max. We now evaluate the endpoint, $\pi(0) = 0$, and the points nearest $q = 34.1$ with integer q-values:

$$\pi(35) = 7(35) - 0.01(35)^3 + 0.6(35)^2 - 13(35) = 245 - 148.75 = 96.25,$$

$$\pi(34) = 7(34) - 0.01(34)^3 + 0.6(34)^2 - 13(34) = 238 - 141.44 = 96.56.$$

So the (global) maximum profit is $\pi(34) = 96.56$. The money from sales is \$238, the cost to produce the items is \$141.44, resulting in a profit of \$96.56.

(c) The money from sales is equal to price$\times$quantity sold. If the price is raised from \$7 by $\$x$ to $\$(7 + x)$, the result is a reduction in sales from 34 items to $(34 - 2x)$ items. So the result of raising the price by $\$x$ is to change the money from sales from $(7)(34)$ to $(7 + x)(34 - 2x)$ dollars. If the production level is fixed at 34, then the production costs are fixed at \$141.44, as found in part (b), and the profit is given by:

$$\pi(x) = (7 + x)(34 - 2x) - 141.44$$

This expression gives the profit as a function of change in price x, rather than as a function of quantity as in part (b). We set the derivative of π with respect to x equal to zero to find the change in price that maximizes the profit:

$$\frac{d\pi}{dx} = (1)(34 - 2x) + (7 + x)(-2) = 20 - 4x = 0$$

So $x = 5$, and this must give a maximum for $\pi(x)$ since the graph of π is a parabola which opens downward. The profit when the price is $\$12\,(= 7 + x = 7 + 5)$ is thus $\pi(5) = (7 + 5)(34 - 2(5)) - 141.44 = \$146.56$. This is indeed higher than the profit when the price is \$7, so the smart thing to do is to raise the price by \$5.

28. Note that profit $= \pi(q) = R(q) - C(q)$, so to find profit, we must find an expression for $R(q)$. If p is the price of a single item,

$$R(q) = p \cdot q.$$

Substituting $p = b_1 - a_1 q$ gives

$$\begin{aligned} R(q) &= (b_1 - a_1 q) \cdot q \\ &= b_1 q - a_1 q^2 \quad \text{and} \\ \pi(q) &= R(q) - C(q) \\ &= b_1 q - a_1 q^2 - b_2 - a_2 q. \end{aligned}$$

Finding the derivative and setting it equal to 0 yields

$$\begin{aligned} \pi'(q) &= b_1 - 2a_1 q - a_2 = 0 \\ \text{so } q &= \frac{-a_2 + b_1}{2a_1} \end{aligned}$$

is a critical point for $\pi(q)$. Using the second derivative test, we see π is concave down,

$$\pi''(q) = -2a_1$$

Since a_1 is positive, $-2a_1$ will always be negative, so, since $\pi''(q) < 0$ for all q, $q = (-a_2 + b_1)/(2a_1)$ is a local maximum. This q is the only critical point, so it is the global maximum for $\pi(q)$.

29. **(a)** $\pi(q)$ is maximized when $R(q) > C(q)$ and they are as far apart as possible. See Figure 4.145.
(b) $\pi'(q_0) = R'(q_0) - C'(q_0) = 0$ implies that $C'(q_0) = R'(q_0) = p$.

Graphically, the slopes of the two curves at q_0 are equal. This is plausible because if $C'(q_0)$ were greater than p or less than p, the maximum of $\pi(q)$ would be to the left or right of q_0, respectively. In economic terms, if the cost were rising more quickly than revenues, the profit would be maximized at a lower quantity (and if the cost were rising more slowly, at a higher quantity).

(c) See Figure 4.146.

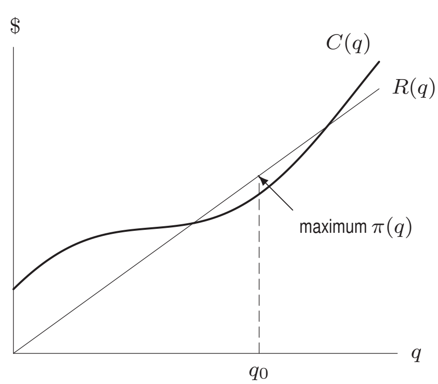


Figure 4.145

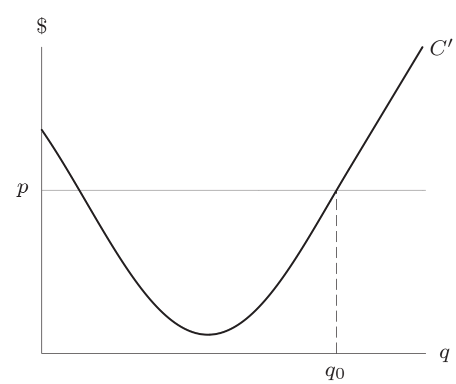


Figure 4.146

30. **(a)** We know that the average cost is given by

$$a(q) = \frac{C(q)}{q}.$$

Thus the average cost is

$$a(q) = 0.04q^2 - 3q + 75 + \frac{96}{q}.$$

(b) Average cost is graphed in Figure 4.147.

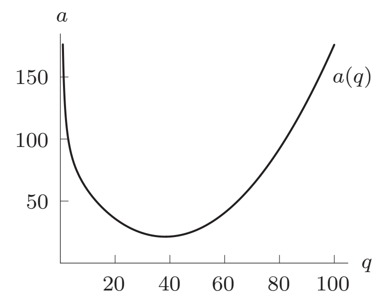


Figure 4.147

(c) Looking at the graph of the average value we see that it is decreasing for $q < 38$. Likewise we see that $a(q)$ is increasing for $q > 38$.

(d) Thus the average cost hits its minimum at about $q = 38$ with the value of about \$21.

31. If the minimum average cost occurs at a production level of 15,000 units, the line from the origin to the curve is tangent to the curve at that point. The slope of this line is 25, so the cost of producing 15,000 units is $25(15{,}000) = 375{,}000$. See Figure 4.148.

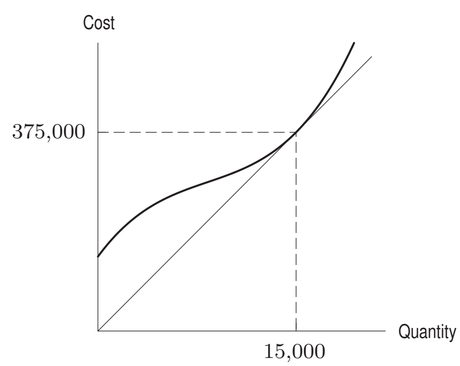


Figure 4.148

32. The point F representing the fixed costs is the vertical intercept of the cost function, $C(q)$.

The break-even level of production is where $R(q) = C(q)$, so the point B is where the two graphs intersect. There are two such points in Figure 4.149. We have marked the one at which production first becomes profitable.

The marginal cost is $C'(q)$, so the point M is where the slope of $C(q)$ is minimum.

The average cost is minimized when $a(q) = C'(q)$. Since $a(q)$ equals the slope of a line joining the origin to a point $(q, C(q))$ on the graph of $C(q)$, the point A is where such a line is tangent to the graph of $C(q)$.

The profit is $P(q) = R(q) - C(q)$, so the point P occurs where the two graphs are furthest apart and $R(q) > C(q)$. See Figure 4.149.

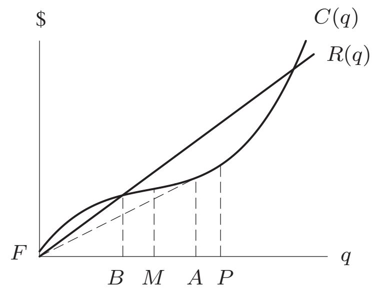


Figure 4.149

33. $C'(q)$ decreases with q because $C(q)$ is concave down. Therefore, $C'(2)$ is larger then $C'(3)$.

34. Note that $(C(5) - C(3))/(5-3)$ is the slope of the secant line between $q = 3$ and $q = 5$. The value of $(C(5) - C(3))/(5-3)$ is larger because the slope of the secant line is larger than the slope of the tangent line at its right endpoint.

35. Both quantities are slopes of secant lines with left endpoint at $q = 50$. The value of $(C(75) - C(50))/25$ is larger as the slope of the secant line decreases as the right endpoint moves to the right.

36. Both quantities are slopes of secant lines. Since the secant line between $q = 25$ and $q = 75$ is to the left of the secant line between $q = 50$ and $q = 100$, and $C(q)$ is concave down, $(C(75) - C(25))/50$ is larger.

37. Since $C'(3)$ is the slope at $q = 3$, and $C(3)/3$ is the slope of the line from the origin to the point $q = 3$ on the curve $C(q)$, the value of $C(3)/3$ is larger.

38. Since $C(q)/q$ is the slope of the line from the origin to the point q on the curve $C(q)$, the concavity of $C(q)$ tells us that the value of $C(10)/10$ is larger.

39. $E = \left|\frac{p}{q}\frac{dq}{dp}\right| = \left|\frac{p}{q} \cdot \frac{d}{dp}(2000 - 5p)\right| = \left|\frac{p}{q} \cdot (-5)\right|$, so $E = \frac{5p}{q}$. At a price of \$20.00, the number of items produced is $q = 2000 - 5(20) = 1900$, so at $p = 20$, we have

$$E = \frac{5(20)}{(1900)} \approx 0.05.$$

Since $0 \leq E < 1$, this product has inelastic demand–a 1% change in price will only decrease demand by 0.05%.

40. Since $R = pq$, we have $dR/dp = p(dq/dp) + q$. We are assuming that

$$E = \left|\frac{p}{q} \cdot \frac{dq}{dp}\right| > 1$$

so, removing the absolute values

$$-E = \frac{p}{q}\frac{dq}{dp} < -1.$$

Multiplication by q gives

$$p\frac{dq}{dp} < -q$$

and hence

$$\frac{dR}{dp} = p\frac{dq}{dp} + q < 0$$

41. Since $R = pq$, we have $dR/dp = p(dq/dp) + q$. We are assuming that

$$0 \leq E = \left|\frac{p}{q} \cdot \frac{dq}{dp}\right| < 1$$

so, removing the absolute values

$$0 \geq -E = \frac{p}{q}\frac{dq}{dp} > -1$$

Multiplication by q gives

$$p\frac{dq}{dp} > -q$$

and hence

$$\frac{dR}{dp} = p\frac{dq}{dp} + q > 0$$

42. **(a)** If we graph the data, we see that it looks like logistic growth. (See Figure 4.150.) But logistic growth also makes sense from a common-sense viewpoint. As cable television "catches on," the percentage of households which have it will at first grow exponentially but then slow down after more and more people have it. Eventually, nearly everyone who will subscribe already has and the percentage levels off.

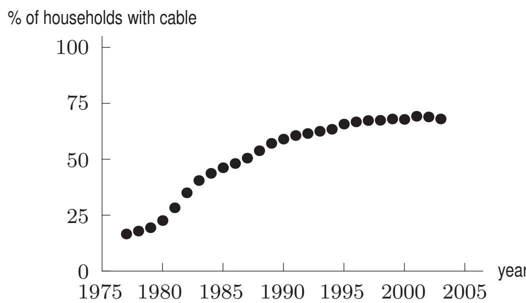


Figure 4.150

(b) The point of diminishing returns happens around 35%. This predicts a carrying capacity of 70% which is pretty close to the 68.9% we see in 2002 and 68.0% we see in 2003.

(c) 68.8%

(d) The limiting value predicts the percentage of households that will eventually have cable. This model predicts that there will never be a time when more than 68.8% of households have cable.

43. Let l and w be the length and width of the rectangle. Since the perimeter is 200 meters, we have

$$\begin{aligned} 2w + 2l &= 200 \\ l &= 100 - w. \end{aligned}$$

The area is

$$A(w) = w \cdot l = w(100 - w) = 100w - w^2.$$

At the critical points

$$A'(w) = 100 - 2w = 0,$$

so

$$w = 50, \quad l = 50.$$

The rectangle of perimeter 200 meters with maximum area is the 50 meter by 50 meter square. Its area is $w \cdot l = 50 \cdot 50 =$ 2500 meters^2.

44. The volume is given by $V = x^2 y$. The surface area is given by

$$\begin{aligned} S &= 2x^2 + 4xy \\ &= 2x^2 + 4xV/x^2 = 2x^2 + 4V/x. \end{aligned}$$

To find the dimensions which minimize the area, find x such that $dS/dx = 0$:

$$\begin{aligned} \frac{dS}{dx} = 4x - \frac{4V}{x^2} &= 0 \\ x^3 &= V. \end{aligned}$$

Solving for x gives $x = \sqrt[3]{V} = y$. To see that this gives a minimum, note that for small x, $S \approx 4V/x$ is decreasing. For large x, $S \approx 2x^2$ is increasing. Since there is only one critical point, it must give a global minimum. Therefore, when the width equals the height, the surface area is minimized.

45. **(a)** Setting the derivative of $p(1-p)^4$ equal to 0

$$\begin{aligned}\frac{d}{dp}(p(1-p)^4) = (1-p)^4 - 4p(1-p)^3 &= 0\\ (1-p)^3(1-p-4p) &= 0\\ (1-p)^3(1-5p) &= 0\\ p &= 1/5, 1.\end{aligned}$$

Thus, the critical points are $p = 1/5$ and $p = 1$.

(b) Since

$$\begin{aligned}\frac{d^2}{dp^2}(p(1-p)^4) &= \frac{d}{dp}((1-p)^4 - 4p(1-p)^3)\\ &= -4(1-p)^3 - 4(1-p)^3 + 12p(1-p)^2\\ &= 4(1-p)^2(-2(1-p)+3p)\\ &= 4(1-p)^2(-2+5p),\end{aligned}$$

substituting $p = 1/5$ and $p = 1$, we have

$$\left.\frac{d^2}{dp^2}(p(1-p)^4)\right|_{p=1/5} = 4\left(\frac{4}{5}\right)^2(-1) < 0 \quad \text{and} \quad \left.\frac{d^2}{dp^2}(p(1-p)^4)\right|_{p=1} = 0$$

Thus $p = 1/5$ is a local maximum. The second derivative test does not enable us to classify $p = 1$. However, $p(1-p)^4$ is positive everywhere except at $p = 0$ and $p = 1$, where it is 0. Thus, $p = 1$ is a local minimum.

(c) The global maximum occurs at the local maximum, at $p = 1/5$, so

$$\text{Maximum } = \frac{1}{5}\left(1-\frac{1}{5}\right)^4 = \frac{4^4}{5^5} = \frac{256}{3125}.$$

The global minimum occurs at the end points, so

$$\text{Minimum } = 0(1-0)^4 = 1(1-1)^4 = 0.$$

46. The triangle in Figure 4.151 has area, A, given by

$$A = \frac{1}{2}xy = \frac{x}{2}e^{-x/3}.$$

At a critical point,

$$\begin{aligned}\frac{dA}{dx} = \frac{1}{2}e^{-x/3} - \frac{x}{6}e^{-x/3} &= 0\\ \frac{1}{6}e^{-x/3}(3-x) &= 0\\ x &= 3.\end{aligned}$$

Substituting the critical point and the endpoints into the formula for the area gives:

For $x = 1$, we have $A = \frac{1}{2}e^{-1/3} = 0.358$
For $x = 3$, we have $A = \frac{3}{2}e^{-1} = 0.552$
For $x = 5$, we have $A = \frac{5}{2}e^{-5/3} = 0.472$

Thus, the maximum area is 0.552 and the minimum area is 0.358.

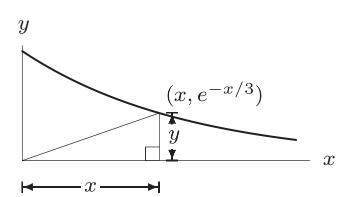


Figure 4.151

47. **(a)** The distance the pigeon flies over water is

$$\overline{BP} = \frac{\overline{AB}}{\sin\theta} = \frac{500}{\sin\theta},$$

and over land is

$$\overline{PL} = \overline{AL} - \overline{AP} = 2000 - \frac{500}{\tan\theta} = 2000 - \frac{500\cos\theta}{\sin\theta}.$$

Therefore the energy required is

$$\begin{aligned} E &= 2e\left(\frac{500}{\sin\theta}\right) + e\left(2000 - \frac{500\cos\theta}{\sin\theta}\right) \\ &= 500e\left(\frac{2-\cos\theta}{\sin\theta}\right) + 2000e, \quad \text{for} \quad \arctan\left(\frac{500}{2000}\right) \le \theta \le \frac{\pi}{2}. \end{aligned}$$

(b) Notice that E and the function $f(\theta) = \dfrac{2-\cos\theta}{\sin\theta}$ must have the same critical points since the graph of E is just a stretch and a vertical shift of the graph of f. The graph of $\dfrac{2-\cos\theta}{\sin\theta}$ for $\arctan(\frac{500}{2000}) \le \theta \le \frac{\pi}{2}$ in Figure 4.152 shows that E has precisely one critical point, and that a minimum for E occurs at this point.

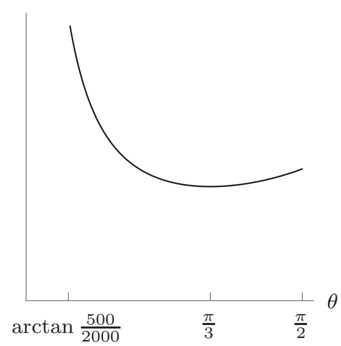


Figure 4.152: Graph of $f(\theta) = \frac{2-\cos\theta}{\sin\theta}$ for $\arctan(\frac{500}{2000}) \le \theta \le \frac{\pi}{2}$

To find the critical point θ, we solve $f'(\theta) = 0$ or

$$\begin{aligned} E' = 0 &= 500e\left(\frac{\sin\theta\cdot\sin\theta - (2-\cos\theta)\cdot\cos\theta}{\sin^2\theta}\right) \\ &= 500e\left(\frac{1-2\cos\theta}{\sin^2\theta}\right). \end{aligned}$$

Therefore $1 - 2\cos\theta = 0$ and so $\theta = \pi/3$.

(c) Letting $a = \overline{AB}$ and $b = \overline{AL}$, our formula for E becomes

$$\begin{aligned} E &= 2e\left(\frac{a}{\sin\theta}\right) + e\left(b - \frac{a\cos\theta}{\sin\theta}\right) \\ &= ea\left(\frac{2-\cos\theta}{\sin\theta}\right) + eb, \quad \text{for} \quad \arctan\left(\frac{a}{b}\right) \le \theta \le \frac{\pi}{2}. \end{aligned}$$

Again, the graph of E is just a stretch and a vertical shift of the graph of $\dfrac{2-\cos\theta}{\sin\theta}$. Thus, the critical point $\theta = \pi/3$ is independent of e, a, and b. But the maximum of E *on the domain* $\arctan(a/b) \le \theta \le \frac{\pi}{2}$ is dependent on the ratio $a/b = \dfrac{\overline{AB}}{\overline{AL}}$. In other words, the optimal angle is $\theta = \pi/3$ provided $\arctan(a/b) \le \frac{\pi}{3}$; otherwise, the optimal angle is $\arctan(a/b)$, which means the pigeon should fly over the lake for the entire trip—this occurs when $a/b > 1.733$.

48. (a) Differentiating using the chain rule gives

$$p'(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)} \cdot \left(-2\frac{(x-\mu)}{2\sigma^2}\right) = -\frac{(x-\mu)}{\sigma^3\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}.$$

So $p'(x) = 0$ where $x - \mu = 0$, so $x = \mu$.

Differentiating again using the product rule gives

$$\begin{aligned} p''(x) &= \frac{-1}{\sigma^3\sqrt{2\pi}} \left(1 \cdot e^{-(x-\mu)^2/(2\sigma^2)} + (x-\mu)e^{-(x-\mu)^2/(2\sigma^2)} \cdot \left(\frac{-2(x-\mu)}{2\sigma^2}\right)\right) \\ &= -\frac{e^{-(x-\mu)^2/(2\sigma^2)}}{\sigma^5\sqrt{2\pi}}(\sigma^2 - (x-\mu)^2). \end{aligned}$$

Substituting $x = \mu$ gives

$$p''(\mu) = -\frac{e^0}{\sigma^5\sqrt{2\pi}}\sigma^2 = -\frac{1}{\sigma^3\sqrt{2\pi}}.$$

Since $p''(\mu) < 0$, there is a local maximum at $x = \mu$. Since this local maximum is the only critical point, it is a global maximum. See Figure 4.153.

(b) Using the formula for $p''(x)$, we see that $p''(x) = 0$ where

$$\begin{aligned} \sigma^2 - (x-\mu)^2 &= 0 \\ x - \mu &= \pm\sigma \\ x &= \mu \pm \sigma. \end{aligned}$$

Since $p''(x)$ changes sign at $x = \mu + \sigma$ and $x = \mu - \sigma$, these are both points of inflection. See Figure 4.153.

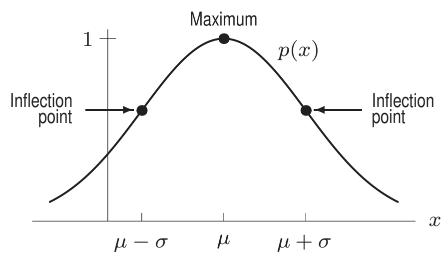


Figure 4.153

49. (a) Before differentiating, we multiply out, giving

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right) = kP - \frac{kP^2}{L}.$$

Since k and L are constant, the chain rule gives

$$\begin{aligned} \frac{d^2P}{dt^2} = \frac{d}{dt}\left(\frac{dP}{dt}\right) &= \frac{d}{dt}\left(kP - \frac{kP^2}{L}\right) \\ &= k\frac{dP}{dt} - \frac{2kP}{L}\frac{dP}{dt}. \end{aligned}$$

(b) Setting $d^2P/dt^2 = 0$ gives

$$0 = k\frac{dP}{dt} - \frac{2kP}{L}\frac{dP}{dt} = \frac{kdP}{dt}\left(1 - \frac{2P}{L}\right).$$

So

$$1 - \frac{2P}{L} = 0,$$

that is, when

$$P = \frac{L}{2},$$

we have $d^2P/dt^2 = 0$; that is, we are at the point of diminishing returns.

50. When bemetizide is taken with triamterene, rather than by itself, the peak concentration is lower, the time it takes to reach peak concentration is about the same, the time until onset of effectiveness is slightly less, and the duration of effectiveness is less. If bemetizide became unsafe in a concentration greater than 70 ng/ml, then it would be wise to take it with triamterene.

51. To find the intercepts of $f(x)$, we first find the y-intercept, which occurs at $f(0) = \sin(0^2) = 0$. To find the x-intercepts on the given interval, we use a calculator to graph $f(x)$ as shown in Figure 4.154:

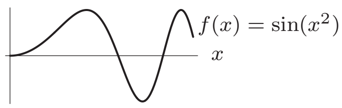


Figure 4.154

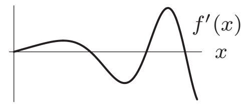


Figure 4.155

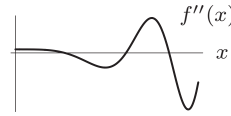


Figure 4.156

We can use a calculator's root-finding capability to find where $f(x) = 0$. We get:

$$x = 0, \qquad \text{and} \qquad x = 1.77, \qquad \text{and} \qquad x = 2.51,$$

so our intercepts are

$$(0, 0), \qquad (1.77, 0), \qquad (2.51, 0).$$

To find critical points, we look for where $f'(x) = 0$ or f' is undefined. Using a calculator's differentiation and graphing features provides a graph of $f'(x)$ shown in Figure 4.155.

Since in Figure 4.155 all the critical points $(x, f(x))$ have $f'(x) = 0$, we use the calculator's root-finding capability to find the critical points where $f'(x) = 0$:

$$x = 0, \qquad x = 1.25, \qquad x = 2.17, \qquad \text{and} \qquad x = 2.80.$$

Writing these out as coordinates, the critical points are at

$$(0, 0), \qquad (1.25, 1), \qquad (2.17, -1), \qquad (2.80, 1).$$

Similarly, inflection points occur where $f''(x)$ changes from negative to positive or vice versa. We can look for such points on a graph of $f''(x)$, shown in Figure 4.156.

We can use the calculator's root-finding capability on the $f''(x)$ to get these inflection points:

$$x = 0.81, \qquad \text{and} \qquad x = 1.81, \qquad \text{and} \qquad x = 2.52$$

The inflection points have coordinates $(0.81, f(0.81))$, etcetera and so they are

$$(0.81, 0.61), \qquad (1.81, -0.13), \qquad (2.52, 0.07).$$

Notice that the intercepts can also be computed algebraically, since $\sin(x^2) = 0$ when $x^2 = 0, \pi, 2\pi$. The solutions are $x = 0$, $x = \sqrt{\pi} = 1.77$, $x = \sqrt{2\pi} = 2.51$. Similarly, if we set $f'(x) = 2x\cos(x^2) = 0$, we get $x = 0$ or $x^2 = \pi/2, 3\pi/2, 5\pi/2$. Thus, the critical points are $x = 0$, $x = \sqrt{\pi/2} = 1.25$, $x = \sqrt{3\pi/2} = 2.17$, and $x = \sqrt{5\pi/2} = 2.80$.

52. **(a)** See Figure 4.157.

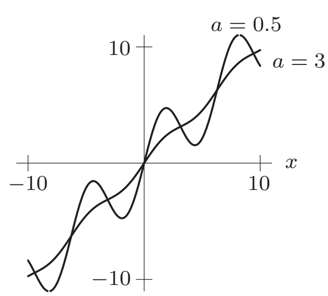


Figure 4.157

(b) The function $f(x) = x + a\sin x$ is increasing for all x if $f'(x) > 0$ for all x. We have $f'(x) = 1 + a\cos x$. Because $\cos x$ varies between -1 and 1, we have $1 + a\cos x > 0$ for all x if $-1 < a < 1$ but not otherwise. When $a = 1$, the function $f(x) = x + \sin x$ is increasing for all x, as is $f(x) = x - \sin x$, obtained when $a = -1$. Thus $f(x)$ is increasing for all x if $-1 \le a \le 1$.

53. (a) See Figure 4.158.

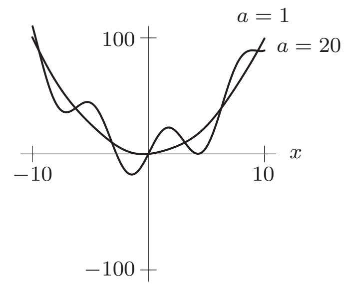


Figure 4.158

(b) The function $f(x) = x^2 + a\sin x$ is concave up for all x if $f''(x) > 0$ for all x. We have $f''(x) = 2 - a\sin x$. Because $\sin x$ varies between -1 and 1, we have $2 - a\sin x > 0$ for all x if $-2 < a < 2$ but not otherwise. Thus $f(x)$ is concave up for all x if $-2 < a < 2$.

54. (a) For a point (t, s), the line from the origin has rise $= s$ and run $= t$; See Figure 4.159. Thus, the slope of the line OP is s/t.

(b) Sketching several lines from the origin to points on the curve (see Figure 4.160), we see that the maximum slope occurs at the point P, where the line to the origin is tangent to the graph. Reading from the graph, we see $t \approx 2$ hours at this point.

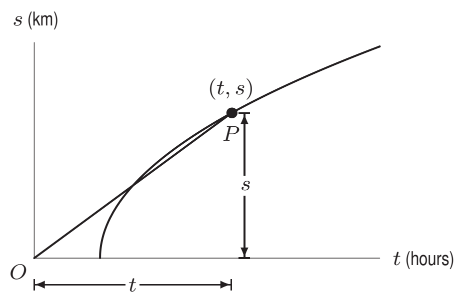


Figure 4.159

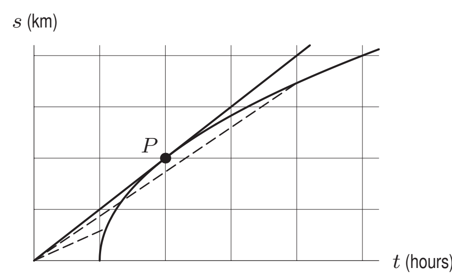


Figure 4.160

(c) The instantaneous speed of the cyclist at any time is given by the slope of the corresponding point on the curve. At the point P, the line from the origin is tangent to the curve, so the quantity s/t equals the cyclist's speed at the point P.

55. (a) The line in Figure 4.161 has slope equal to the rate worms arrive. To understand why, see line (1) in Figure 4.162. (This is the same line.) For any point Q on the loading curve, the line PQ has slope

$$\frac{QT}{PT} = \frac{QT}{PO + OT} = \frac{\text{load}}{\text{traveling time} + \text{searching time}}.$$

(b) The slope of the line PQ is maximized when the line is tangent to the loading curve, which happens with line (2). The load is then approximately 7 worms.

(c) If the traveling time is increased, the point P moves to the left, to point P', say. If line (3) is tangent to the curve, it will be tangent to the curve further to the right than line (2), so the optimal load is larger. This makes sense: if the bird has to fly further, you'd expect it to bring back more worms each time.

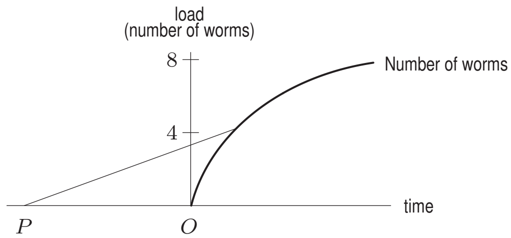


Figure 4.161

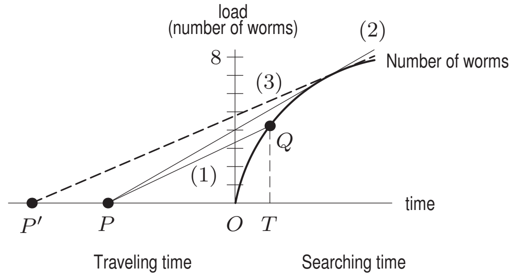


Figure 4.162

56. **(a)** See Figure 4.163. The capacity appears to be 200 cars. The parking lot is full just before 8:30 am.

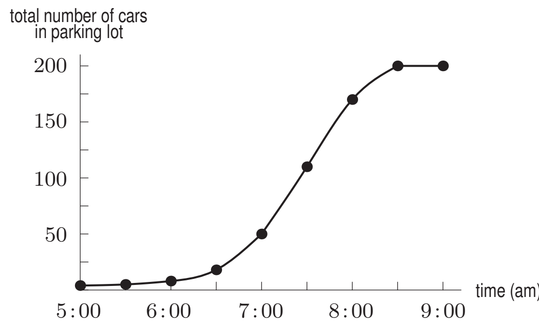


Figure 4.163

(b) The rate of arrival between 5:00 and 5:30 is $(5-4)/0.5 = 2$ cars/hours. We assign this the time 5:15 (rather than either 5:00 or 5:30). The other values in Table 4.7 are calculated in a similar way. The data is plotted in Figure 4.164.

Table 4.7

Time (am)	5:15	5:45	6:15	6:45	7:15	7:45	8:15	8:45
Rate of arrival (cars/hour)	2	6	20	64	120	120	60	0

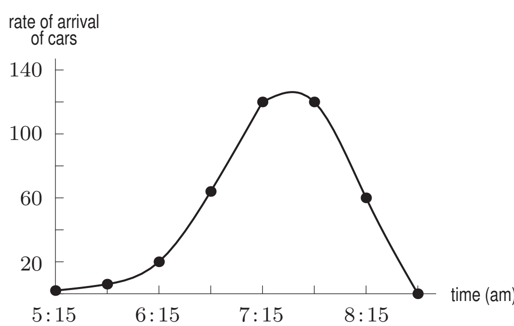


Figure 4.164

(c) Rush hour occurs around the time when the rate of arrival of cars is maximum, namely, about 7:30 am.

(d) The maximum of the rate of arrival occurs at the inflection point of the total number of cars in the lot.

57. For $x > 0$, the line in Figure 4.165 has

$$\text{Slope} = \frac{y}{x} = \frac{x^2e^{-3x}}{x} = xe^{-3x}.$$

If the slope has a maximum, it occurs where

$$\begin{aligned}\frac{d}{dx}\,(\text{Slope}) &= 1\cdot e^{-3x} - 3xe^{-3x} = 0\\ e^{-3x}\,(1-3x) &= 0\\ x &= \frac{1}{3}.\end{aligned}$$

For this x-value,

$$\text{Slope } = \frac{1}{3}e^{-3(1/3)} = \frac{1}{3}e^{-1} = \frac{1}{3e}.$$

Figure 4.165 shows that the slope tends toward 0 as $x \to \infty$; the formula for the slope shows that the slope tends toward 0 as $x \to 0$. Thus the only critical point, $x = 1/3$, must give a local and global maximum.

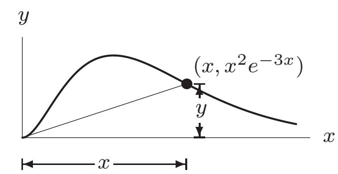


Figure 4.165

58. (a) The rectangle in Figure 4.166 has area, A, given by

$$A = xy = xe^{-2x}.$$

At a critical point, we have

$$\begin{aligned}\frac{dA}{dx} &= 1\cdot e^{-2x} - 2xe^{-2x} = 0\\ e^{-2x}\,(1-2x) &= 0\\ x &= \frac{1}{2}.\end{aligned}$$

Since $A = 0$ when $x = 0$ and $A \to 0$ as $x \to \infty$, the critical point $x = 1/2$ is a local and global maximum. Thus the maximum area is

$$A = \frac{1}{2}e^{-2(1/2)} = \frac{1}{2e}.$$

(b) The rectangle in Figure 4.166 has perimeter, P, given by

$$P = 2x + 2y = 2x + 2e^{-2x}.$$

At a critical point, we have

$$\begin{aligned}\frac{dP}{dx} &= 2 - 4e^{-2x} = 0\\ e^{-2x} &= \frac{1}{2}\\ -2x &= \ln\frac{1}{2}\\ x &= -\frac{1}{2}\ln\frac{1}{2} = \frac{1}{2}\ln 2.\end{aligned}$$

To see if this critical point gives a maximum or minimum, we find

$$\frac{d^2P}{dx^2} = 8e^{-2x}.$$

Since $d^2P/dx^2 > 0$ for all x, including $x = \frac{1}{2}\ln 2$, the critical point is a local and global minimum. Thus, the minimum perimeter is

$$P = 2\left(\frac{1}{2}\ln 2\right) + 2e^{-2\left(\frac{1}{2}\ln 2\right)} = \ln 2 + 2e^{-\ln 2} = \ln 2 + 2\cdot\frac{1}{2} = \ln 2 + 1.$$

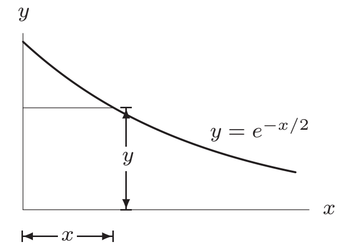


Figure 4.166

59. **(a)** To minimize A with h and s fixed we have to find $\dfrac{dA}{d\theta}$.

$$\begin{aligned}\frac{dA}{d\theta} &= \frac{3}{2}s^2\frac{d}{d\theta}\left(\frac{\sqrt{3}-\cos\theta}{\sin\theta}\right)\\ &= \frac{3}{2}s^2\left(\frac{\sin^2\theta-\cos\theta(\sqrt{3}-\cos\theta)}{\sin^2\theta}\right)\\ &= \frac{3}{2}s^2\left(\frac{\sin^2\theta-\sqrt{3}\cos\theta+\cos^2\theta}{\sin^2\theta}\right)\\ &= \frac{3}{2}s^2\left(\frac{1-\sqrt{3}\cos\theta}{\sin^2\theta}\right)\end{aligned}$$

Set $\dfrac{dA}{d\theta} = \dfrac{3}{2}s^2\left(\dfrac{1-\sqrt{3}\cos\theta}{\sin^2\theta}\right) = 0$. Then $1-\sqrt{3}\cos\theta = 0$ and $\cos\theta = \frac{1}{\sqrt{3}}$, so $\theta \approx 54.7^\circ$ is a critical point of A. Since

$$\begin{aligned}\left.\frac{d^2A}{d\theta^2}\right|_{\theta=54.7} &= \left.\frac{3}{2}s^2\left(\frac{\sqrt{3}\sin^3\theta-2\sin\theta\cos\theta(1-\sqrt{3}\cos\theta)}{\sin^4\theta}\right)\right|_{\theta=54.7}\\ &\approx \frac{3}{2}s^2(2.122) > 0,\end{aligned}$$

$\theta = 54.7$ is indeed a minimum.

(b) Since 55° is very close to 54.7°, we conclude that bees attempt to minimize the surface areas of their honey combs.

60. **(a)** The vertical intercept is $W = Ae^{-e^{b-c\cdot 0}} = Ae^{-e^b}$. There is no horizontal intercept since the exponential function is always positive. There is a horizontal asymptote. As $t \to \infty$, we see that $e^{b-ct} = e^b/e^{ct} \to 0$, since t is positive. Therefore $W \to Ae^0 = A$, so there is a horizontal asymptote at $W = A$.

(b) The derivative is

$$\frac{dW}{dt} = Ae^{-e^{b-ct}}(-e^{b-ct})(-c) = Ace^{-e^{b-ct}}e^{b-ct}.$$

Thus, dW/dt is always positive, so W is always increasing and has no critical points. The second derivative is

$$\begin{aligned}\frac{d^2W}{dt^2} &= \frac{d}{dt}(Ace^{-e^{b-ct}})e^{b-ct} + Ace^{-e^{b-ct}}\frac{d}{dt}(e^{b-ct})\\ &= Ac^2e^{-e^{b-ct}}e^{b-ct}e^{b-ct} + Ace^{-e^{b-ct}}(-c)e^{b-ct}\\ &= Ac^2e^{-e^{b-ct}}e^{b-ct}(e^{b-ct}-1).\end{aligned}$$

Now e^{b-ct} decreases from $e^b > 1$ when $t = 0$ toward 0 as $t \to \infty$. The second derivative changes sign from positive to negative when $e^{b-ct} = 1$, i.e., when $b - ct = 0$, or $t = b/c$. Thus the curve has an inflection point at $t = b/c$, where $W = Ae^{-e^{b-(b/c)c}} = Ae^{-1}$.

(c) See Figure 4.167.

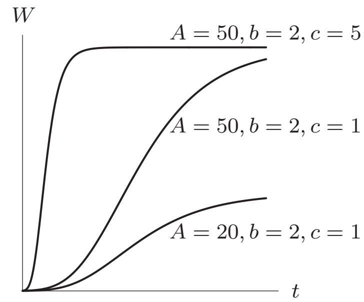


Figure 4.167

(d) The final size of the organism is given by the horizontal asymptote $W = A$. The curve is steepest at its inflection point, which occurs at $t = b/c$, $W = Ae^{-1}$. Since $e = 2.71828\ldots \approx 3$, the size the organism when it is growing fastest is about $A/3$, one third its final size. So yes, the Gompertz growth function is useful in modeling such growth.

CHECK YOUR UNDERSTANDING

1. False. The function f has a local *minimum* at p if $f(p) \leq f(x)$ for points x near p.
2. True—according to the definition, critical points of f occur where $f'(x) = 0$ or $f'(x)$ is undefined.
3. False. The function f can also have critical points where f' is undefined.
4. True, since f has lower values to the left of p and to the right of p than it does at p.
5. False. If $f'(p) = 0$ and $f''(p) > 0$ then f has a local minimum at p.
6. True. If $f'(p) = 0$ and $f''(p) > 0$ then by the second derivative test f has a local minimum at p.
7. False. If $f''(p) > 0$, the function is concave up, but p must be a critical point before we can conclude that f has a local minimum at p.
8. False. Consider $f(x) = x^3$. The function has a critical point at $x = 0$, but $x = 0$ is neither a local maximum nor a local minimum.
9. False. Consider $f(x) = e^x$. The function and its derivative are defined for all x, and $f'(x)$ is never equal to zero. Thus, f has no critical points.
10. False. Consider $f(x) = |x|$. The function has a local minimum at $x = 0$, but $f'(0)$ is undefined.
11. True, by the definition of inflection point.
12. False. Consider $f(x) = x^4$. The second derivative is zero at $x = 0$, but the function is concave up for all x.
13. True. Consider $f(x) = x^3$. The point $x = 0$ is both a critical point and an inflection point of f.
14. True. The graph of $f(x) = x^3$ is concave down to the left of $x = 0$ and concave up to the right of $x = 0$, so there is an inflection point at $x = 0$.
15. False. The function $H(x) = x^4$ is concave up for all x, so H has no inflection points.
16. True. For example, the function $f(x) = x^3 + x$ has no critical points and has one inflection point, at $x = 0$.
17. True. For example, you can sketch the graph of such a function with a local maximum at $(-1, 1)$, a local minimum at $(1, -1)$, and horizontal asymptote $y = 0$ as $x \to \pm\infty$.
18. True. The second derivative is equal to zero at $x = -1$ and is positive to the left of $x = -1$ and negative to the right of $x = -1$. Thus, f has an inflection point at $x = -1$.
19. False. The second derivative is equal to zero at $x = -1$ but the sign of f'' is negative both to the left and right of $x = -1$. Thus, there is no change of concavity at $x = -1$.
20. True. Note that e^x is positive for all x and that the factor $(x + 1)$ changes sign from negative to positive at $x = -1$. Thus the function changes concavity at $x = -1$ and $x = -1$ is an inflection point.
21. True. Consider $f(x) = -x^2$. The function has a local and global maximum at $x = 0$.
22. False. The global maximum may also occur at an endpoint $x = 1$ or $x = 2$.
23. False. Consider $f(x) = -x^2$ over the entire real line. There is no global minimum.
24. True. If f is increasing on the interval,then the largest value occurs at the right endpoint.

25. False. If $f(x) < 0$ then f is decreasing on the interval, and the largest value occurs at the *left* endpoint. The global maximum will be at $x = a$.

26. True. For example, let $S(x) = x$. On the interval $1 \leq x \leq 2$, the global maximum is 2 and occurs at the right-hand endpoint $x = 2$. On the interval $2 \leq x \leq 3$, the global maximum of 3 occurs at $x = 3$.

27. False. The function k has no critical points. As $x \to 0$ from the right, the function increases without bound. Thus, k does not have a global maximum when $x > 0$.

28. False. According to the definition, the function f has a global *minimum* at p on the interval given that $f(p) \leq f(x)$ for all $-5 \leq x \leq 5$.

29. True, as defined in the text.

30. False. By the second derivative test, we have that p is a local minimum. However, a global minimum could occur at another critical point or at an endpoint if the function is defined on a closed interval.

31. True. Cost is increasing at a slower rate than revenue–thus the cost of the next item will be less than the revenue for that item.

32. True, since for the last item made and sold, we have received less in revenue than it cost to make, so by not making and selling this item, profit would increase.

33. False, since a point where marginal revenue equals marginal cost may also indicate minimum profit.

34. True.

35. False. Maximum profit occurs at a critical point of the profit function—or at an endpoint of the profit function if there are constraints on the number of items that can be made or sold.

36. True. When marginal profit is zero we have a critical point. The point must be tested to see if it gives a maximum of the profit function.

37. False. The cost and revenue functions cross when a profit is turning into a loss, in which case profit is decreasing, or when a loss is turning into a profit, in which case profit is increasing. In neither case is profit a maximum.

38. True. Since Profit = Revenue - Cost, when the cost and revenue curves are equal (and therefore the graphs cross), the profit is zero.

39. True. Revenue is price times quantity. If the price is constant at price p, the graph of marginal revenue is the horizontal line $y = p$.

40. False. If prices are constant at price p, the graph of revenue is a line with slope p.

41. True, as specified in the text.

42. True. The units of marginal cost and average cost have the same units—for example, if the units of cost are dollars, the units of marginal cost and average cost are $/item.

43. False. Marginal cost gives the rate at which cost is changing—or, once q items have been produced, the approximate cost of the next item. Average cost gives the average cost of producing q items. In general, these are not the same.

44. False. Average cost of q items can be visualized as the slope of a line from the origin to the point $(q, C(q))$ on the graph of the cost function C.

45. False. If marginal cost is less than average cost, increasing production decreases average cost.

46. False. Marginal cost equals average cost at critical points of average cost.

47. True. If marginal cost is greater than average cost, increasing production increases average cost.

48. True, since the example in the section shows that the average cost function has critical points exactly where average cost (which is the slope of the line from the origin to the cost function) is equal to the marginal cost (which is the slope of the tangent to the cost function).

49. False. Not necessarily. Average cost may be increasing or decreasing as a functions of quantity, depending on the concavity of the cost function.

50. False. Average cost is minimized when equal to marginal cost, but this is not the same quantity at which marginal cost is minimized.

51. False. The elasticity of demand is given by $E = |p/q \cdot dq/dp|$.

52. False. If $E > 1$ then we say that demand is *elastic*.

53. False. If $0 \leq E < 1$ then we say that demand is *inelastic*.

54. True. This is what elasticity tells us.

55. False. If a product is considered a necessity, the demand is generally inelastic.

56. False. An increase in price may cause an increase or a decrease in revenue, depending on elasticity.

57. False. If elasticity $E > 1$, demand is elastic and revenue increases when price decreases.

58. True. At a critical point of the revenue function $E = 1$, and the demand is neither elastic nor inelastic.

59. True, since by increasing the price, demand drops, so costs are lower, but revenue is higher, so profit increases.

60. False. If $E > 1$ then increasing price causes a decrease in revenue, demand, and costs. We have seen examples where profits decrease because revenue decreases more than costs.

61. True. The function P fits the model $f(t) = L/(1 + Ce^{-kt})$ with $L = 1000, C = 2$, and $k = 3$.

62. False. Note that the end behavior of P does not approach the limiting value L. Instead, as $t \to \infty$, the function values approach zero. The fact that the exponent of e is not negative causes this behavior.

63. False. The function P has an inflection point when $P = 1000/2 = 500$. This occurs at approximately $t = 0.2310$.

64. True, since the logistic function $P = L/(1 + Ce^{-kt})$ has an inflection point at $L/2$.

65. True. Note that as t increases, the denominator $(1 + Ce^{-kt})$ approaches 1, so $P(t)$ approaches L.

66. True, since the slope of the dose/response curve is the rate of change of the response as a function of the dose.

67. True. The logistic model approaches the carrying capacity as time increases.

68. True. If k is large the function approaches L more rapidly than when k is small.

69. True. At the inflection point, $P = L/2$, the function changes from concave up to concave down.

70. True. We can show this analytically by noting that $P'(t) = LkCe^{-kt}(1 + Ce^{-kt})^{-2} > 0$ for all t.

71. False. The formula for a surge function is $y = ate^{-bt}$ while the formula for an exponential decay function is $y = ae^{-bt}$.

72. True.

73. False. The function is concave down and then concave up as it approaches the t-axis asymptotically.

74. True, the surge function will cross the line when it is increasing and again when it is decreasing.

75. False. The drug will never be effective since the concentration will always be below the minimum effective concentration.

76. True. There is exactly one inflection point, and it is always to the right of the one critical point.

77. True, as specified in the text.

78. True. Note that y can be written as $y = (1/12)te^{-3t}$. This is a surge function with $a = 1/12, b = 3$.

79. False. The function y can be rewritten as $y = (1/12)te^{3t}$. This function does not fit the model of a surge function because the exponent of e is positive. Note that $y \to \infty$ as t increases.

80. True, since the surge function $y = ate^{-bt}$ has maximum at $t = 1/b$.

PROJECTS FOR CHAPTER FOUR

1. Since $a(q) = C(q)/q$, we have $C(q) = a(q) \cdot q$. Thus $C'(q) = a'(q)q + a(q)$, and so

$$C'(q_0) = a'(q_0)q_0 + a(q_0).$$

Since t_1 is the line tangent to $a(q)$ at $q = q_0$, the slope of t_1 is $a'(q_0)$, and the equation of t_1 is

$$y = a(q_0) + a'(q_0) \cdot (q - q_0) = a'(q_0) \cdot q + (a(q_0) - a'(q_0) \cdot q_0)\,.$$

Thus the y-intercept of t_1 is given by $a(q_0) - a'(q_0)q_0$, and the equation of the line t_2 is

$$y = 2 \cdot a'(q_0) \cdot q + (a(q_0) - a'(q_0) \cdot q_0)$$

since t_2 has twice the slope of t_1. Let's compute the y-value on t_2 when $q = q_0$:

$$y = 2 \cdot a'(q_0) \cdot q_0 + (a(q_0) - a'(q_0) \cdot q_0) = a'(q_0)q_0 + a(q_0) = C'(q_0).$$

Hence $C'(q_0)$ is given by the point on t_2 where $q = q_0$.

2. (a) (i) We want to minimize A, the total area lost to the forest, which is made up of n firebreaks and 1 stand of trees lying between firebreaks. The area of each firebreak is $(50\text{ km})(0.01\text{ km}) = 0.5\text{ km}^2$, so the total area lost to the firebreaks is $0.5n\text{ km}^2$. There are n total stands of trees between firebreaks. The area of a single stand of trees can be found by subtracting the firebreak area from the forest and dividing by n, so

$$\text{Area of one stand of trees} = \frac{2500 - 0.5n}{n}.$$

Thus, the total area lost is

$$\begin{aligned} A &= \text{Area of one stand} + \text{Area lost to firebreaks} \\ &= \frac{2500 - 0.5n}{n} + 0.5n = \frac{2500}{n} - 0.5 + 0.5n. \end{aligned}$$

We assume that A is a differentiable function of a continuous variable, n. Differentiating this function yields

$$\frac{dA}{dn} = -\frac{2500}{n^2} + 0.5.$$

At critical points, $dA/dn = 0$, so $0.5 = 2500/n^2$ or $n = \sqrt{2500/0.5} \approx 70.7$. Since n must be an integer, we check that when $n = 71$, $A = 70.211$ and when $n = 70$, $A = 70.214$. Thus, $n = 71$ gives a smaller area lost.

We can check that this is a local minimum since the second derivative is positive everywhere

$$\frac{d^2A}{dn^2} = \frac{5000}{n^3} > 0.$$

Finally, we check the endpoints: $n = 1$ yields the entire forest lost after a fire, since there is only one stand of trees in this case and it all burns. The largest n is 5000, and in this case the firebreaks remove the entire forest. Both of these cases maximize the area of forest lost. Thus, $n = 71$ is a global minimum. So 71 firebreaks minimizes the area of forest lost.

(ii) Repeating the calculation using b for the width gives

$$A = \frac{2500}{n} - 50b + 50bn,$$

and

$$\frac{dA}{dn} = \frac{-2500}{n^2} + 50b,$$

with a critical point when $b = 50/n^2$ so $n = \sqrt{50/b}$. So, for example, if we make the width b four times as large we need half as many firebreaks.

(b) We want to minimize A, the total area lost to the forest, which is made up of n firebreaks in one direction, n firebreaks in the other, and one square of trees surrounded by firebreaks. The area of each firebreak is 0.5 km^2, and there are $2n$ of them, giving a total of $0.5 \cdot 2n$. But this is larger than the total area covered by the firebreaks, since it counts the small intersection squares, of size $(0.01)^2$, twice. Since there are n^2 intersections, we must subtract $(0.01)^2n^2$ from the total area of the $2n$ firebreaks. Thus,

$$\text{Area covered by the firebreaks} = 0.5 \cdot 2n - (0.01)^2n^2.$$

To this we must add the area of one square patch of trees lost in a fire. These are squares of side $(50 - 0.01n)/n = 50/n - 0.01$. Thus the total area lost is

$$A = n - 0.0001n^2 + (50/n - 0.01)^2$$

Treating n as a continuous variable and differentiating this function yields

$$\frac{dA}{dn} = 1 - 0.0002n + 2\left(\frac{50}{n} - 0.01\right)\left(\frac{-50}{n^2}\right).$$

Using a computer algebra system to find critical points we find that $dA/dn = 0$ when $n \approx 17$ and $n = 5000$. Thus $n = 17$ gives a minimum lost area, since the endpoints of $n = 1$ and $n = 5000$ both yield $A = 2500$ or the entire forest lost. So we use 17 firebreaks in each direction.

3. (a) Since the company can produce more goods if it has more raw materials to use, the function $f(x)$ is increasing. Thus, we expect the derivative $f'(x)$ to be positive.

(b) The cost to the company of acquiring x units of raw material is wx, and the revenue from the sale of $f(x)$ units of the product is $pf(x)$. The company's profit is $\pi(x) = \text{Revenue} - \text{Cost} = pf(x) - wx$.

(c) Since profit $\pi(x)$ is maximized at $x = x^*$, we have $\pi'(x^*) = 0$. From $\pi'(x) = pf'(x) - w$, we have $pf'(x^*) - w = 0$. Thus $f'(x^*) = w/p$.

(d) Computing the second derivative of $\pi(x)$ gives $\pi''(x) = pf''(x)$. Since $\pi(x)$ has a maximum at $x = x^*$, the second derivative $\pi''(x^*) = pf''(x^*)$ is negative. Thus $f''(x^*)$ is negative.

(e) Differentiate both sides of $pf'(x^*) - w = 0$ with respect to w. The chain rule gives

$$
\begin{aligned}
p\frac{d}{dw}f'(x^*) - 1 &= 0 \\
pf''(x^*)\frac{dx^*}{dw} - 1 &= 0 \\
\frac{dx^*}{dw} &= \frac{1}{pf''(x^*)}.
\end{aligned}
$$

Since $f''(x^*) < 0$, we see dx^*/dw is negative.

(f) Since $dx^*/dw < 0$, the quantity x^* is a decreasing function of w. If the price w of the raw material goes up, the company should buy less.