CHAPTER SIX

Solutions for Section 6.1

1. By counting grid squares, we find $\int_1^6 f(x)\,dx = 8.5$, so the average value of f is $\dfrac{8.5}{6-1} = \dfrac{8.5}{5} = 1.7$.

2. **(a)** Counting the squares yields an estimate of 25 squares, each with area $= 1$, so we conclude that

$$\int_0^5 f(x)\,dx \approx 25.$$

(b) The average height appears to be around 5.

(c) Using the formula, we get

$$\text{Average value} = \frac{\int_0^5 f(x)\,dx}{5-0} \approx \frac{25}{5} \approx 5,$$

which is consistent with (b).

3. Average value $= \dfrac{1}{2-0}\displaystyle\int_0^2 (1+t)\,dt = \dfrac{1}{2}(4) = 2.$

4. The average value equals

$$\frac{1}{3}\int_0^3 f(x)\,dx = \frac{24}{3} = 8.$$

5. By a visual estimate, the average value is ≈ -3.

6. By a visual estimate, the average value is ≈ 8.

7. It appears that the area under a line at about $y = 8.5$ is approximately the same as the area under $f(x)$ on the interval $x = a$ to $x = b$, so we estimate that the average value is about 8.5. See Figure 6.1.
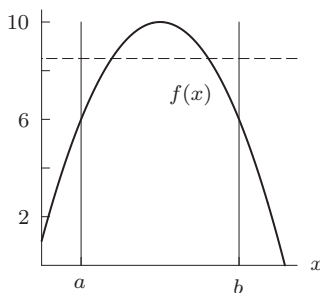

Figure 6.1

8. It appears that the area under a line at about $y = 45$ is approximately the same as the area under $f(x)$ on the interval $x = a$ to $x = b$, so we estimate that the average value is about 45. See Figure 6.2.
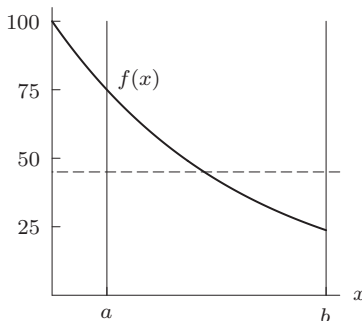

Figure 6.2

9. **(a)** Let $f(t)$ be the annual income at age t. Between ages 25 and 85,

$$\text{Average annual income} = \frac{1}{85 - 25} \int_{25}^{85} f(t)\, dt.$$

The integral equals the area of the region under the graph of $f(t)$. The region is a rectangle of height 40,000 dollars/year and width 40 years, so we have

$$\int_{25}^{85} f(t)\, dt = \text{Height} \times \text{Width} = 1{,}600{,}000 \text{ dollars.}$$

Therefore

$$\text{Average annual income} = \frac{1{,}600{,}000}{85 - 25} = \$26{,}667 \text{ per year.}$$

(b) This person spends less than their income for their entire working life, age 25 to 65. In retirement, ages 65 to 85, they have no income but can continue spending at the same rate because they saved in their working years.

10. **(a)** Let $f(t)$ be the annual income at age t. Between ages 25 and 85,

$$\text{Average annual income} = \frac{1}{85 - 25} \int_{25}^{85} f(t)\, dt.$$

The integral equals the area of the region under the graph of $f(t)$. The region consists of 10 full grid rectangles and 4 half grid rectangles, or 12 full rectangles in all. Each grid rectangle has an area of 10 years $\cdot$ 20,000 dollars/year $=$ 200,000 dollars. Thus, we have

$$\int_{25}^{85} f(t)\, dt = 12(200{,}000) = 2{,}400{,}000 \text{ dollars,}$$

so

$$\text{Average annual income} = \frac{2{,}400{,}000}{85 - 25} = \$40{,}000 \text{ per year.}$$

(b) This person's income increases steadily over forty years from \$20,000/year to \$100,000/year. Every working year the income increases by \$2,000/year. Their income reaches \$40,000, their spending rate, when they turn 35. They spend less than their income every year from age 35 to 65. They spend more than their income every year from age 25 to 35 and from age 65 to 85.

11. Since $t = 0$ in 1975 and $t = 35$ in 2010, we want:

$$\text{Average Value} = \frac{1}{35 - 0} \int_{0}^{35} 225(1.15)^t\, dt$$

$$= \frac{1}{35}(212{,}787) = \$6080.$$

12. **(a)** The average inventory is given by the formula

$$\frac{1}{90 - 0} \int_{0}^{90} 5000(0.9)^t\, dt.$$

Using a calculator yields

$$\int_{0}^{90} 5000(0.9)^t\, dt \approx 47452.5$$

so the average inventory is

$$\frac{47452.5}{90} \approx 527.25.$$

(b) The function is graphed in Figure 6.3. The area of a rectangle of height 527.25 is equal to the area under the curve.
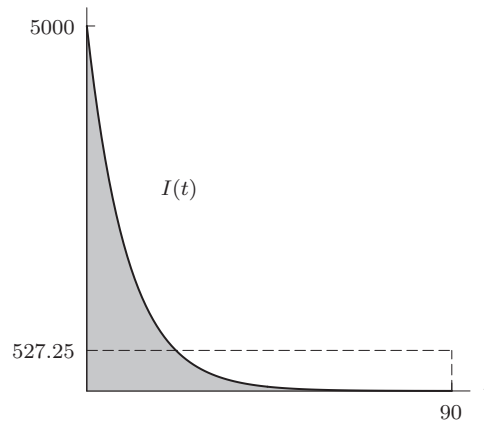
Figure 6.3

13. **(a)** Systolic, or maximum, blood pressure $= 120$ mm Hg.
    (b) Diastolic, or minimum, blood pressure $= 80$ mm Hg.
    (c) Average $= (120 + 80)/2 = 100$ mm Hg
    (d) The pressure is at the diastolic level for a longer time than it is at or near the systolic level. Thus the average arterial pressure over the cardiac cycle is closer to the diastolic pressure than to the systolic pressure. The average pressure over the cycle is less than the average that was computed in part (c).

14. The shaded region under the pressure curve in Figure 6.4 is well approximated by a rectangle, of width 1 and height 80, surmounted by a triangle of base 0.45 and height 40 (See Figure 6.5). Thus

$$\text{Average value} = \frac{1}{1 - 0} \int_0^1 f(x)\, dx = \text{ Area of the shaded region } \approx 1 \cdot 80 + \frac{1}{2}(0.45)(40) = 89 \text{ mm Hg}.$$

Thus, the average blood pressure is about 90 mm Hg. To check the answer graphically, draw the horizontal line at average pressure 89 mm Hg as in Figure 6.6. Observe that the area of the region above the average line but below the pressure curve appears equal to the area of the region below the average line but above the pressure curve.
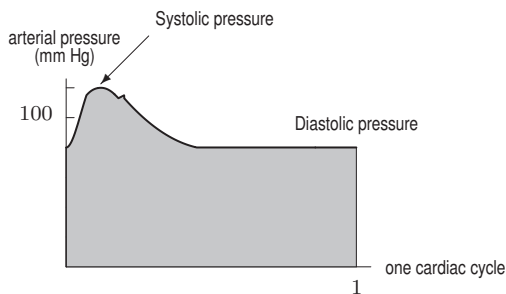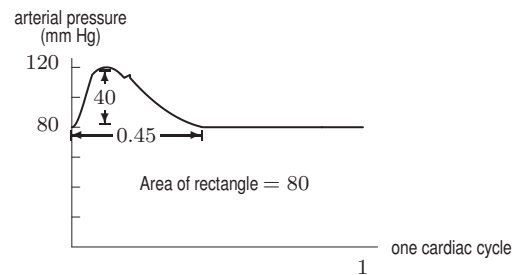

Figure 6.4: Total area under pressure curve

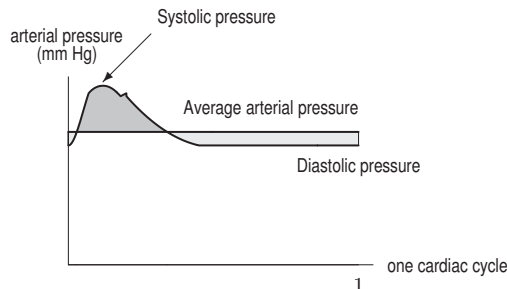**Figure 6.5**: Calculating approximate area under pressure curve

Figure 6.6: Average arterial pressure $= 89$ mm Hg

15. (a) Over the interval $[-1, 3]$, we estimate that the total change of the population is about 1.5, by counting boxes between the curve and the x-axis; we count about 1.5 boxes below the x-axis from $x = -1$ to $x = 1$ and about 3 above from $x = 1$ to $x = 3$. So the average rate of change is just the total change divided by the length of the interval, that is $1.5/4 = 0.375$ thousand/hour.

(b) We can estimate the total change of the algae population by counting boxes between the curve and the x-axis. Here, there is about 1 box above the x-axis from $x = -3$ to $x = -2$, about 0.75 of a box below the x-axis from $x = -2$ to $x = -1$, and a total change of about 1.5 boxes thereafter (as discussed in part (a)). So the total change is about $1 - 0.75 + 1.5 = 1.75$ thousands of algae.

16. (a) When $t = 10$ we have $P = 6.1e^{0.0125(10)} = 6.9$. The predicted population of the world in the year 2010 is 6.9 billion people.

(b) We find the average value of P on the interval $t = 0$ to $t = 10$:

$$\text{Average value} = \frac{1}{10 - 0} \int_0^{10} 6.1e^{0.0125t}\, dt = \frac{1}{10}(64.976) = 6.5.$$

The average population of the world over this decade is 6.5 billion people.

17. (a) Since $t = 0$ to $t = 31$ covers January:

$$\begin{array}{l}\text{Average number of} \\ \text{daylight hours in January}\end{array} = \frac{1}{31} \int_0^{31} [12 + 2.4\sin(0.0172(t - 80))]\ dt.$$

Using left and right sums with $n = 100$ gives

$$\text{Average} \approx \frac{306}{31} \approx 9.9 \text{ hours.}$$

(b) Assuming it is not a leap year, the last day of May is $t = 151 (= 31 + 28 + 31 + 30 + 31)$ and the last day of June is $t = 181 (= 151 + 30)$. Again finding the integral numerically:

$$\begin{array}{l}\text{Average number of} \\ \text{daylight hours in June}\end{array} = \frac{1}{30} \int_{151}^{181} [12 + 2.4\sin(0.0172(t - 80))]\ dt$$
$$\approx \frac{431}{30} \approx 14.4 \text{ hours.}$$

(c)

$$\text{Average for whole year} = \frac{1}{365} \int_0^{365} [12 + 2.4\sin(0.0172(t - 80))]\ dt$$
$$\approx \frac{4381}{365} \approx 12.0 \text{ hours.}$$

(d) The average over the whole year should be 12 hours, as computed in (c). Since Madrid is in the northern hemisphere, the average for a winter month, such as January, should be less than 12 hours (it is 9.9 hours) and the average for a summer month, such as June, should be more than 12 hours (it is 14.4 hours).

18. (a) At the end of one hour $t = 60$, and $H = 22°$C.

(b)

$$\text{Average temperature} = \frac{1}{60} \int_0^{60} (20 + 980e^{-0.1t})\,dt$$
$$= \frac{1}{60}(10976) = 183°\text{C.}$$

(c) Average temperature at beginning and end of hour $= (1000 + 22)/2 = 511°$C. The average found in part (b) is smaller than the average of these two temperatures because the bar cools quickly at first and so spends less time at high temperatures. Alternatively, the graph of H against t is concave up.

$H(°C)$

$1000°$

$511°$

$22°$

$H = 20 + 980e^{-0.1t}$

60

t (mins)
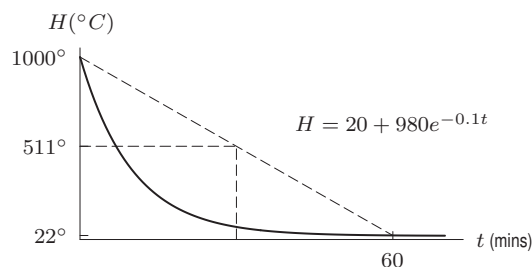
19. (a) See Figure 6.7. More sales were made in the second half of the year, because the area under the second half of the curve is greater.

sales/month

1200

800

400

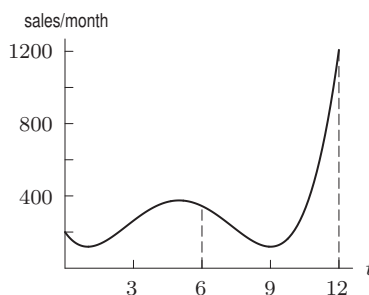3 6 9 12

t

Figure 6.7

(b) The total sales for the first six months amount to

$$\int_0^6 r(t)dt = \$1531.20$$

The total sales for the last 6 months of the year amount to

$$\int_6^{12} r(t)dt = \$1963.20$$

(c) Thus the total sales for the year amount to

$$\int_0^{12} r(t)dt = \int_0^6 r(t)dt + \int_6^{12} r(t)dt = \$1531.20 + \$1963.20 = \$3494.40$$

(d) The average sales per month is the quotient of the total sales with 12 months giving

$$\frac{\text{Total sales}}{12 \text{ months}} = \frac{3494.4}{12} = \$291.20/\text{month}.$$

20. (a) $E(t) = 1.4e^{0.07t}$

(b)

$$\text{Average Yearly Consumption} = \frac{\text{Total Consumption for the Century}}{100 \text{ years}}$$

$$= \frac{1}{100} \int_0^{100} 1.4e^{0.07t} \, dt$$

$$\approx 219 \text{ million megawatt-hours.}$$

(c) We are looking for t such that $E(t) \approx 219$:
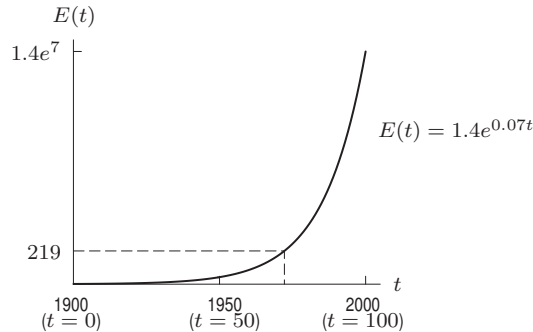
$$1.4e^{0.07t} \approx 219$$

$$e^{0.07t} = 156.4.$$

Taking natural logs,

$$0.07t = \ln 156.4$$
$$t \approx \frac{5.05}{0.07} \approx 72.18.$$

Thus, consumption was closest to the average during 1972.

(d) Between the years 1900 and 2000 the graph of $E(t)$ looks like

From the graph, we can see the t value such that $E(t) = 219$. It lies to the right of $t = 50$, and is thus in the second half of the century.

21. Looking at the graph of $f(x)$, we see that the curve has a horizontal tangent at $x = 1$. Thus $f'(1) = 0$.

Next we must determine if the average value of $f(x)$ is positive or negative on $0 \le x \le 4$. The average value of the function from $x = 0$ to $x = 4$ will have the same sign as the integral of the function over the same interval. Because the area between the curve and the x-axis that is above the x-axis is greater than the area that is below the x-axis, the integral is positive. Thus the average value is positive.

The area between the graph of $f(x)$ and the x-axis between $x = 0$ and $x = 1$ lies entirely below the x-axis. Hence $\int_0^1 f(x)\,dx$ is negative.

We can now arrange the quantities in order from least to greatest: $(c) < (a) < (b)$.

22. In (a), $f'(1)$ is the slope of a tangent line at $x = 1$, which is negative. As for (c), the rate of change in $f(x)$ is given by $f'(x)$, and the average value of this over $0 \le x \le a$ is

$$\frac{1}{a - 0} \int_0^a f'(x)\,dx = \frac{f(a) - f(0)}{a - 0}.$$

This is the slope of the line through the points $(0, 1)$ and $(a, 0)$, which is less negative that the tangent line at $x = 1$. Therefore, $(a) < (c) < 0$. The quantity (b) is $\left(\int_0^a f(x)\,dx\right)/a$ and (d) is $\int_0^a f(x)\,dx$, which is the net area under the graph of f (counting the area as negative for f below the x-axis). Since $a > 1$ and $\int_0^a f(x)\,dx > 0$, we have $0 < (b) < (d)$. Therefore

$$(a) < (c) < (b) < (d).$$

Solutions for Section 6.2

1. (a) The equilibrium price is \$30 per unit, and the equilibrium quantity is 6000.

(b) The region representing the consumer surplus is the shaded triangle in Figure 6.8 with area $\frac{1}{2} \cdot 6000 \cdot 70 = 210,000$. The consumer surplus is \$210,000.

The area representing the producer surplus, shaded in Figure 6.9, is about 7 grid squares, each of area 10,000. The producer surplus is about \$70,000.
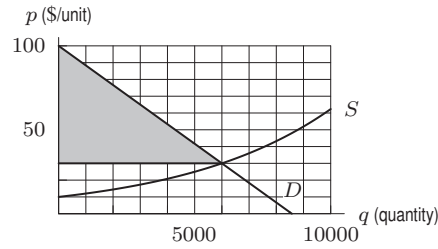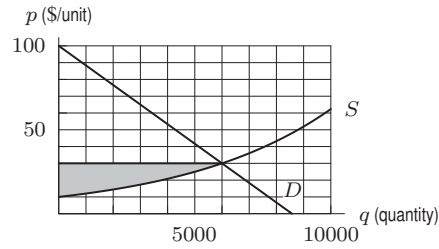
Figure 6.8

Figure 6.9

2. Looking at the graph we see that the supply and demand curves intersect at roughly the point $(345, 8)$. Thus the equilibrium price is $8 per unit and the equilibrium quantity is 345 units. Figures 6.10 and 6.11 show the shaded areas corresponding to the consumer surplus and the producer surplus. Counting grid squares we see that the consumer surplus is roughly $2000 while the producer surplus is roughly $1400.
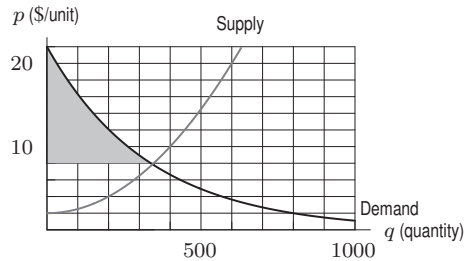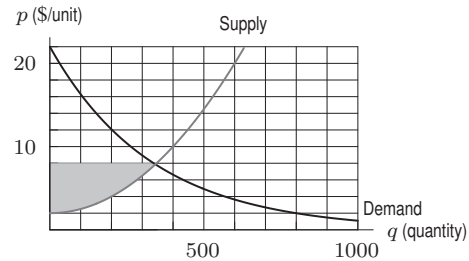

Figure 6.10: Consumer surplus

Figure 6.11: Producer surplus

3. When $q = 5$, the price is $p = 100 - 3 \cdot 5^2 = 25$. From Figure 6.18 on page 283 of the text, with $q^+ = 5$,

$$\text{Consumer Surplus} = \int_0^5 (100 - 3q^2)\, dq - 5 \cdot 25$$

Using a calculator or computer to evaluate the integral we get

$$\text{Consumer Surplus} = 250.$$

4. To calculate the producer surplus we need to find the market equilibrium price, p^*, and quantity q^*. In equilibrium, when supply equals demand, $35 - q^2 = 3 + q^2$ so $q^2 = 16$. We need only consider the positive solution $q^* = 4$. The corresponding equilibrium price is $p^* = 35 - 4^2 = 19$. From Figure 6.13 on page 282 of the text,

$$\text{Producer surplus} = q^* p^* - \int_0^{q^*} (3 + q^2)\, dq = 4 \cdot 19 - \int_0^4 (3 + q^2)\, dq$$

Using a calculator or computer to evaluate the integral we get

$$\text{Producer surplus} = 42.667.$$

5. When $q = 10$, the price is $p = 100 - 4 \cdot 10 = 60$. From Figure 6.18 on page 283 of the text, with $q^+ = 10$,

$$\text{Consumer surplus} = \int_0^{10} (100 - 4q)\, dq - 60 \cdot 10$$

Using a calculator or computer to evaluate the integral we get

$$\text{Consumer surplus} = 200.$$

6. (a) It appears that the equilibrium price is $p^* = 6$ dollars per unit and the equilibrium quantity is $q^* = 400$ units. See Figure 6.12.
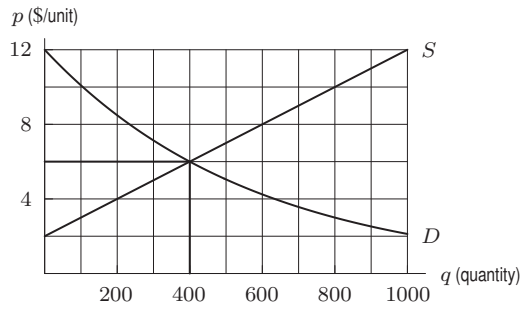

Figure 6.12

(b) To find the consumer surplus, we estimate the area shaded in Figure 6.13 by counting grid squares. There appear to be about 5.5 grid squares in this shaded area, and each grid square has area 200, so the total area is about 1100. The consumer surplus is about $1100.

Similarly, to find the producer surplus, we estimate the area shaded in Figure 6.14 to be about 4 grid squares, for a total area of 800. The producer surplus is about $800.
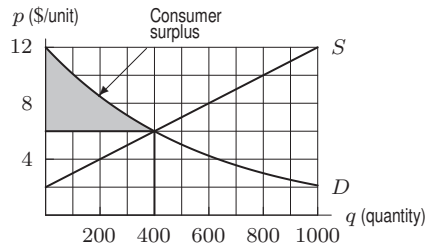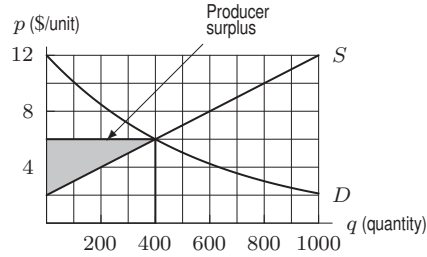

Figure 6.13

Figure 6.14

(c) The consumer surplus is given by the area shaded in Figure 6.15. This area is about 6 grid squares, for a total area of about 1200. The consumer surplus is about $1200, which is larger than the consumer surplus of $1100 at the equilibrium price.

The producer surplus is the area shaded in Figure 6.16. This area is about 200, so the producer surplus is about $200. This is less than the producer surplus of $800 at the equilibrium price. Notice that the sum of the consumer surplus and the producer surplus is $1900 at the equilibrium price and is $1400 at the artificial price. The total gains from trade are always lower at an artificial price.
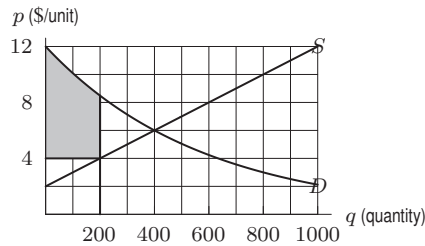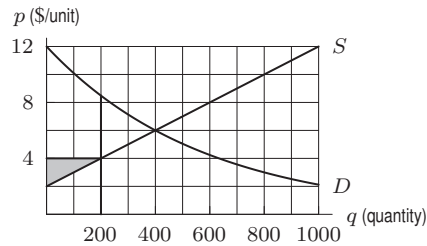

Figure 6.15

Figure 6.16

7. (a) Looking at the figure in the problem we see that the equilibrium price is roughly $30 giving an equilibrium quantity of 125 units.

(b) Consumer surplus is the area above p^* and below the demand curve. Graphically this is represented by the shaded area in Figure 6.17. From the graph we can estimate the shaded area to be roughly 14 squares where each square represents ($25/unit)·(10 units). Thus the consumer surplus is approximately
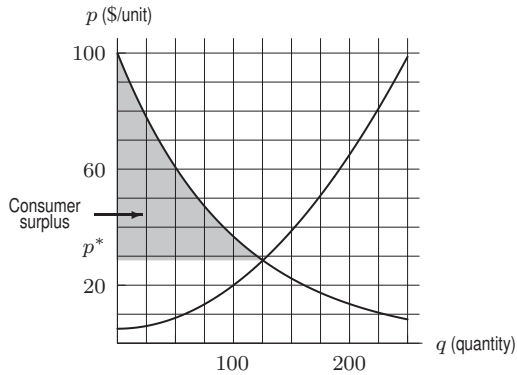
$$14 \cdot \$250 = \$3500.$$
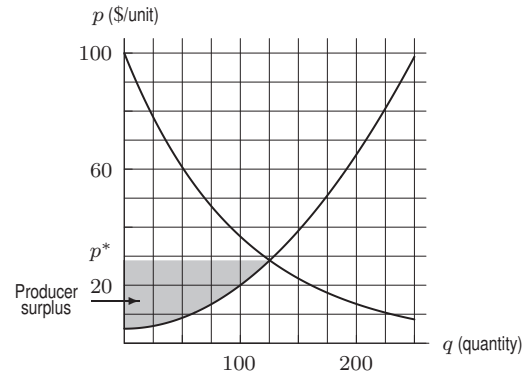

Figure 6.17

Figure 6.18

Producer surplus is the area under p^* and above the supply curve. Graphically this is represented by the shaded area in Figure 6.18. From the graph we can estimate the shaded area to be roughly 8 squares where each square represents ($25/unit)·(10 units). Thus the producer surplus is approximately

$$8 \cdot \$250 = \$2000$$

(c) We have

$$\text{Total gains from trade} = \text{Consumer surplus} + \text{producer surplus}$$
$$= \$3500 + \$2000$$
$$= \$5500.$$

8. (a) The consumer surplus is the area the between demand curve and the price $40—roughly 9 squares. See Figure 6.19. Since each square represents ($25/unit)·(10 units), the total area is

$$9 \cdot \$250 = \$2250.$$

At a price of $40, about 90 units are sold. The producer surplus is the area under $40, above the supply curve, and to the left of $q = 90$. See Figure 6.19. The area is 10.5 squares or

$$10.5 \cdot \$250 = \$2625.$$

The total gains from the trade is

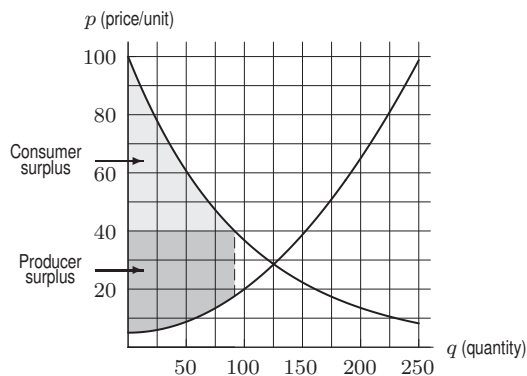$$\text{Total gain} = \text{Consumer surplus} + \text{Producer surplus} = \$4875.$$

Figure 6.19

(b) The consumer surplus is less with the price control.
The producer surplus is greater with the price control.
The total gains from trade are less with the price control.

9. (a) The price at which 300 units are demanded $= 20e^{-0.002 \cdot 300} \approx \$11$.
The price at which 300 units are supplied $= 0.02 \cdot 300 + 1 = \$7$. Therefore, the price at which 300 units are demanded is higher. See Figure 6.20.

(b) Looking at Figure 6.20 we see that

$$p^* \approx \$9 = \text{equilibrium price} \quad \text{and} \quad q^* \approx 400 = \text{equilibrium quantity.}$$
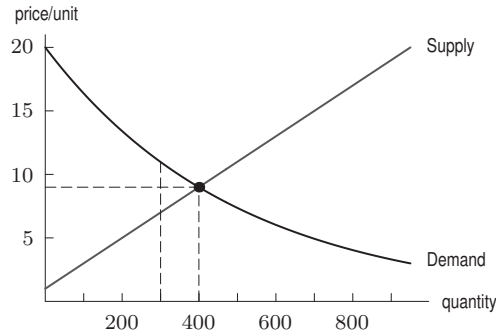

Figure 6.20

(c) Consumer surplus is the area under the demand curve and above $9. Thus

$$\text{Consumer surplus} = \int_0^{400} f(q)dq - 400 \cdot 9 \approx 1907 \text{ dollars.}$$

See Figure 6.21. Consumers gain $1907 in buying goods at the equilibrium price instead of at the price they would be willing to pay.
Producer surplus is the area under $9 and above the supply curve. Thus

$$\text{Producer surplus} = 400 \cdot 9 - \int_0^{400} g(q)dq = 1600 \text{ dollars.}$$

See Figure 6.22. Producers gained $1600 in supplying goods at the equilibrium price instead of the price at which they would have been willing to supply the goods.
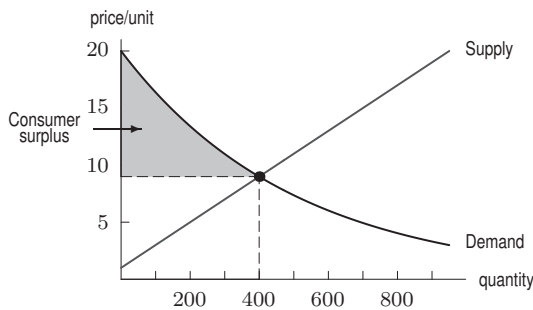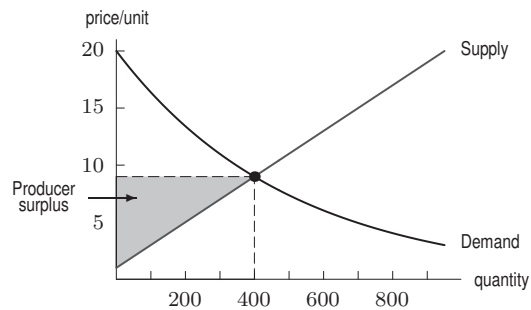

Figure 6.21

Figure 6.22

10. The total gains from trade at the equilibrium price is shaded in Figure 6.23. We see in Figures 6.24 and 6.25 that if the price is artificially high or low, the quantity sold is less than q^*. Thus, the total gains from trade are reduced.
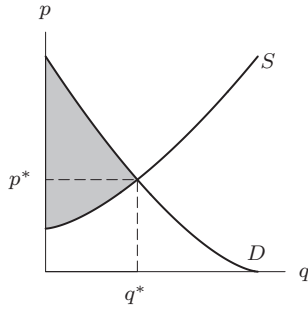
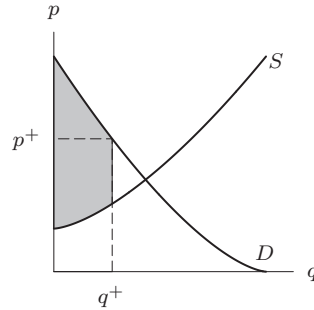**Figure 6.23**: Shade area: Total gains from trade at equilibrium price

Figure 6.24: Shade area: Gains when price is artificially high
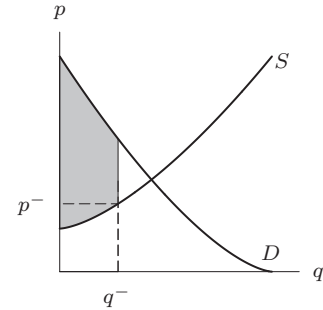
Figure 6.25: Shade area: Gains when price is artificially low

11. Figure 6.26 shows the consumer and producer surplus for the price, p^-. For comparison, Figure 6.27 shows the consumer and producer surplus at the equilibrium price.
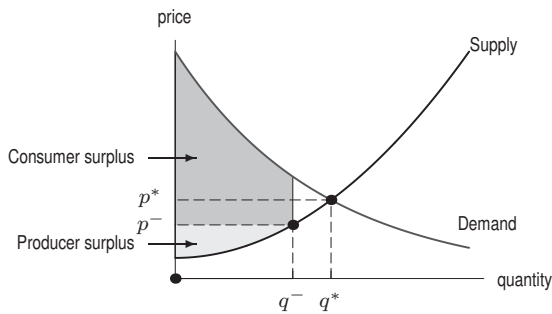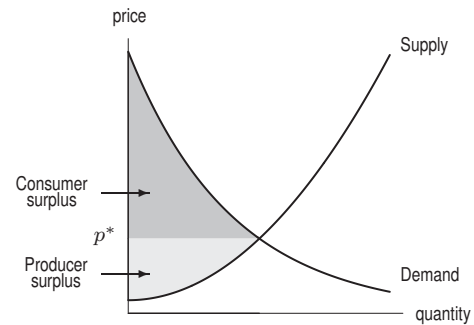

Figure 6.26

Figure 6.27

(a) The producer surplus is the area on the graph between p^- and the supply curve. Lowering the price also lowers the producer surplus.

(b) The consumer surplus — the area between the supply curve and the line p^- — may increase or decrease depends on the functions describing the supply and demand, and the lowered price. (For example, the consumer surplus seems to be increased in Figure 6.26 but if the price were brought down to $0 then the consumer surplus would be zero, and hence clearly less than the consumer surplus at equilibrium.)

(c) Figure 6.26 shows that the total gains from the trade are decreased.

12. (a) In Table 6.1, the quantity q increases as the price p decreases, while in Table 6.2, q increases as p increases. Therefore, the demand data is in Table 6.1 and the supply data is in Table 6.2.

(b) It appears that the equilibrium price is $p^* = 25$ dollars per unit and the equilibrium quantity is $q^* = 400$ units sold at this price.

(c) To estimate the consumer surplus, we use the demand data in Table 6.1. We use a Riemann sum using the price from the demand data minus the equilibrium price of 25.

$$\text{Left sum} = (60 - 25) \cdot 100 + (50 - 25) \cdot 100 + (41 - 25) \cdot 100 + (32 - 25) \cdot 100 = 8300.$$

$$\text{Right sum} = (50 - 25) \cdot 100 + (41 - 25) \cdot 100 + (32 - 25) \cdot 100 + (25 - 25) \cdot 100 = 4800.$$

We average the two to estimate that

$$\text{Consumer surplus} \approx \frac{8300 + 4800}{2} = 6550.$$

To estimate the producer surplus, we use the supply data in Table 6.2. We use a Riemann sum using the equilibrium price of 25 minus the price from the supply data.

$$\text{Left sum} = (25 - 10) \cdot 100 + (25 - 14) \cdot 100 + (25 - 18) \cdot 100 + (25 - 22) \cdot 100 = 3600.$$

$$\text{Right sum } = (25 - 14) \cdot 100 + (25 - 18) \cdot 100 + (25 - 22) \cdot 100 + (25 - 25) \cdot 100 = 2100.$$

We average the two to estimate that

$$\text{Producer surplus } \approx \frac{3600 + 2100}{2} = 2850.$$

13. (a) If the price is artificially high, the consumer surplus at the artificial price is always less than the consumer surplus at the equilibrium price, but the producer surplus may be larger or smaller. See Figure 6.28.
    (b) If the price is artificially low, the producer surplus at the artificial price is always less than the producer surplus at the equilibrium price, but the consumer surplus may be larger or smaller. See Figure 6.29.
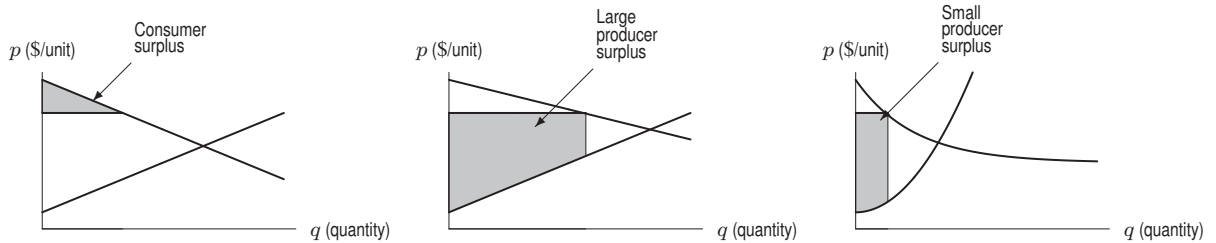
Figure 6.28
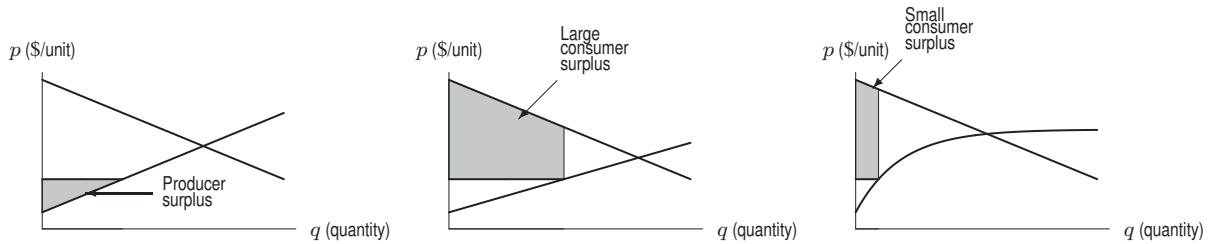
Figure 6.29

14. The supply curve, $S(q)$, represents the minimum price p per unit that the suppliers will be willing to supply some quantity q of the good for. See Figure 6.30. If the suppliers have q^* of the good and q^* is divided into subintervals of size Δq, then if the consumers could offer the suppliers for each Δq a price increase just sufficient to induce the suppliers to sell an additional Δq of the good, the consumers' total expenditure on q^* goods would be

$$p_1 \Delta q + p_2 \Delta q + \cdots = \sum p_i \Delta q.$$

As $\Delta q \to 0$ the Riemann sum becomes the integral $\displaystyle\int_0^{q^*} S(q)\, dq$. Thus $\displaystyle\int_0^{q^*} S(q)\, dq$ is the amount the consumers would pay if suppliers could be forced to sell at the lowest price they would be willing to accept.

Figure 6.30

15.
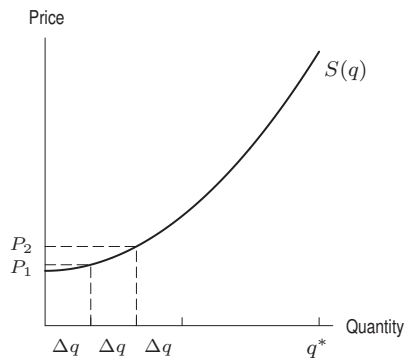
$$\int_0^{q^*} (p^* - S(q)) \, dq = \int_0^{q^*} p^* \, dq - \int_0^{q^*} S(q) \, dq$$

$$= p^* q^* - \int_0^{q^*} S(q) \, dq.$$

Using Problem 14, this integral is the extra amount consumers pay (i.e., suppliers earn over and above the minimum they would be willing to accept for supplying the good). It results from charging the equilibrium price.

16. (a) $p^* q^*$ = the total amount paid for q^* of the good at equilibrium. See Figure 6.31.

(b) $\int_0^{q^*} D(q) \, dq$ = the maximum consumers would be willing to pay if they had to pay the highest price acceptable to them for each additional unit of the good. See Figure 6.32.
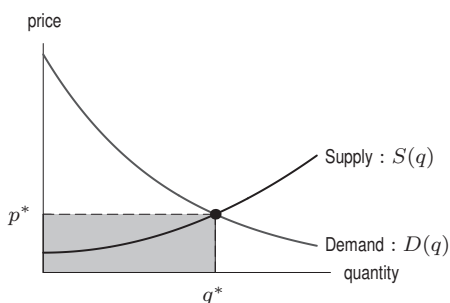
Figure 6.31
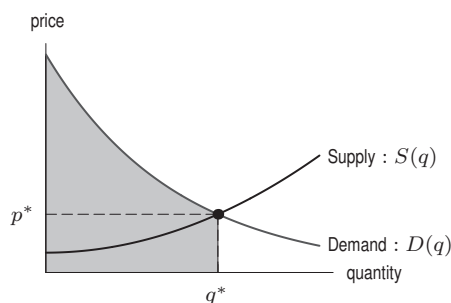
Figure 6.32

(c) $\int_0^{q^*} S(q) \, dq$ = the minimum suppliers would be willing to accept if they were paid the minimum price acceptable to them for each additional unit of the good. See Figure 6.33.

(d) $\int_0^{q^*} D(q) \, dq - p^* q^*$ = consumer surplus. See Figure 6.34.
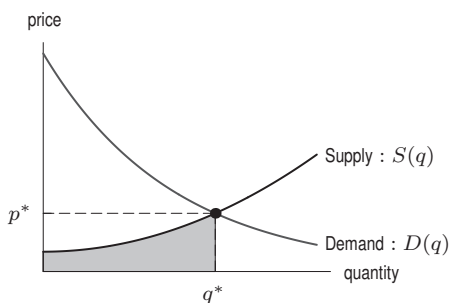
Figure 6.33
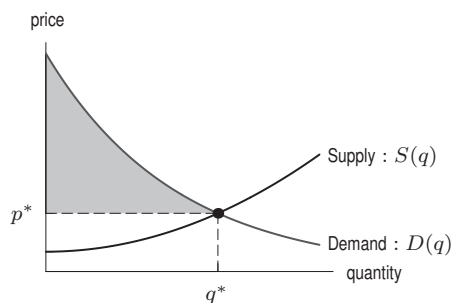
Figure 6.34

(e) $p^* q^* - \int_0^{q^*} S(q) \, dq$ = producer surplus. See Figure 6.35.

(f) $\int_0^{q^*} (D(q) - S(q)) \, dq$ = producer surplus and consumer surplus. See Figure 6.36.
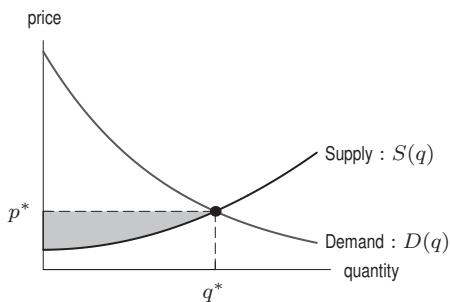
Figure 6.35
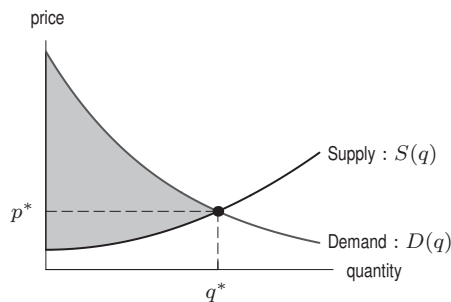
Figure 6.36

Solutions for Section 6.3

1.

$$\text{Future Value} = \int_0^{15} 3000e^{0.06(15-t)}\, dt$$
$$\approx \$72{,}980.16$$

$$\text{Present Value} = \int_0^{15} 3000e^{-0.06t}\, dt$$
$$\approx \$29{,}671.52.$$

There's a quicker way to calculate the present value of the income stream, since the future value of the income stream is (as we've shown) $72,980.16, the present value of the income stream must be the present value of $72,980.16. Thus,

$$\text{Present Value} = \$72{,}980.16(e^{-.06\cdot 15})$$
$$\approx \$29{,}671.52,$$

which is what we got before.

2.
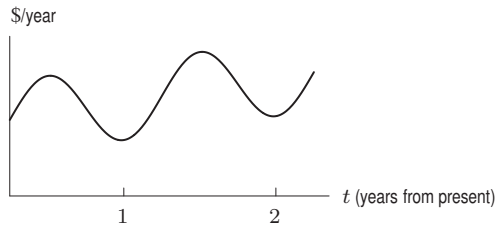
The graph reaches a peak each summer, and a trough each winter. The graph shows sunscreen sales increasing from cycle to cycle. This gradual increase may be due in part to inflation and to population growth.

3. (a) We first find the present value, P, of the income stream:

$$P = \int_0^{10} 6000e^{-0.05t}\, dt = \$47{,}216.32.$$

We use the present value to find the future value, F:

$$F = Pe^{rt} = 47126.32e^{0.05(10)} = \$77{,}846.55.$$

(b) The income stream contributed $6000 per year for 10 years, or $60,000. The interest earned was $77{,}846.55 - 60{,}000 = \$17{,}846.55$.

4. We first find the present value:

$$\text{Present value} = \int_0^{20} 12000e^{-0.06t}\, dt = \$139{,}761.16.$$

We use the present value to find the future value:

$$\text{Future value} = Pe^{rt} = 139761.16e^{0.06(20)} = \$464{,}023.39.$$

5. (a) (i) If the interest rate is 3%, we have

$$\text{Present value} = \int_0^4 5000e^{-0.03t}\, dt = \$18{,}846.59.$$

(ii) If the interest rate is 10%, we have

$$\text{Present value} = \int_0^4 5000e^{-0.10t}\, dt = \$16,484.00.$$

(b) At the end of the four-year period, if the interest rate is 3%,

$$\text{Value} = 18,846.59e^{0.03(4)} = \$21,249.47.$$

At 10%,

$$\text{Value} = 16,484.00e^{0.10(4)} = \$24,591.24.$$

6.

$$\text{Present Value} = \int_0^{10} (100 + 10t)e^{-.05t}\, dt$$
$$= \$1,147.75.$$

7. (a) The present value of the revenue during the first year is the sum of the present value during the first six months (from $t = 0$ to $t = 1/2$) and the present value during the second six months (from $t = 1/2$ to $t = 1$).

$$\begin{array}{l}\text{Present value of revenue} \\ \text{during the first year}\end{array} = \int_0^{1/2} (45,000 + 60,000t)e^{-0.07t}\, dt + \int_{1/2}^1 75,000e^{-0.07t}\, dt$$
$$= 29,438.08 + 35,583.85 = 65,021.93.$$

The present value of the revenue earned by the machine during the first year of operation is about 65,022 dollars.

(b) The present value over a time interval $[0, T]$ with $T > 1/2$ is

$$\begin{array}{l}\text{Present value during} \\ \text{first } T \text{ years}\end{array} = \int_0^{1/2} (45,000 + 60,000t)e^{-0.07t}\, dt + \int_{1/2}^T 75,000e^{-0.07t}\, dt$$
$$= 29,438.08 + \int_{1/2}^T 75,000e^{-0.07t}\, dt$$

If the present value of revenue equals the cost, then

$$150,000 = 29,438.08 + \int_{1/2}^T 75,000e^{-0.07t}\, dt.$$

Solving for the integral, we get

$$120,561.92 = \int_{1/2}^T 75,000e^{-0.07t}\, dt$$

Trying a few values for T gives

$$T \approx 2.27.$$

It will take approximately 2.27 years for the present value of the revenue to equal the cost of the machine.

8. The future value is $20,000 in 5 years. We first find the present value. Since

$$20,000 = Pe^{0.06(5)} = Pe^{0.3}.$$

Solving for P gives

$$P = \frac{20,000}{e^{0.3}} = \$14,816.36.$$

Now we solve for the income stream S which gives a present value of $14,816.36:

$$14,816.36 = \int_0^5 Se^{-0.06t}\, dt.$$

Since S is a constant, we bring it outside the integral sign:

$$14,816.36 = S\int_0^5 e^{-0.06t}\, dt = S(4.320)$$

Solving for S gives

$$S = \frac{14816.36}{4.320} \approx 3430.$$

Money must be deposited at a rate of about $3430 per year, or about $66 per week.

9. (a) If P is the present value, then the value in two years at 9% interest is $Pe^{0.09(2)}$:

$$500{,}000 = Pe^{0.09(2)}$$
$$P = \frac{500{,}000}{e^{0.09(2)}} = 417{,}635.11$$

The present value of the renovations is $417,635.11.

(b) If money is invested at a constant rate of $\$S$ per year, then

$$\text{Present value of deposits} = \int_0^2 Se^{-0.09t}\, dt$$

Since S is constant, we can take it out in front of the integral sign:

$$\text{Present value of deposits} = S\int_0^2 e^{-0.09t}\, dt$$

We want the rate S so that the present value is 417,635.11:

$$417{,}635.11 = S\int_0^2 e^{-0.09t}\, dt$$
$$417{,}635.11 = S(1.830330984)$$
$$S = \frac{417{,}635.11}{1.830330984} = 228{,}174.64.$$

Money deposited at a continuous rate of 228,174.64 dollars per year and earning 9% interest per year has a value of $500,000 after two years.

10. We compute the present value of the company's earnings over the next 8 years:

$$\text{Present value of earnings} = \int_0^8 50{,}000e^{-0.07t}\, dt = \$306{,}279.24.$$

If you buy the rights to the earnings of the company for $350,000, you expect that the earnings will be worth more than $350,000. Since the present value of the earnings is less than this amount, you should not buy.

11. At any time t, the company receives income of $s(t) = 50e^{-t}$ thousands of dollars per year. Thus the present value is

$$\text{Present value} = \int_0^2 s(t)e^{-0.06t}\, dt$$
$$= \int_0^2 (50e^{-t})e^{-0.06t}\, dt$$
$$= \$41{,}508.$$

12. (a) Since the income stream is $7.5 billion per year and the interest rate is 8.5%,

$$\text{Present value} = \int_0^1 7.5e^{-0.085t}\, dt$$
$$= 7.19 \text{ billion dollars.}$$

The present value of Intel's profits over the one-year time period is about 7.19 billion dollars.

(b) The value at the end of the year is $7.19e^{0.085(1)} = 7.83$, or about 7.83 billion dollars.

13. (a) Net revenue in 2003, when $t = 0$, is 4.6 billion dollars. Net revenue in 2013, when $t = 10$, is projected to be $4.6 + 0.4(10) = 8.6$ billion dollars.

(b) January 1, 2003 through January 1, 2013 is a ten-year time period, and $t = 0$ corresponds to January 1, 2003, so the value on January 1, 2003 of the revenue over this ten-year time period is

$$\text{Value on Jan. 1, 2003} = \int_0^{10} (4.6 + 0.4t)e^{-0.035t}\, dt = 54.7 \text{ billion dollars.}$$

The value, on January 1, 2003, of Harley-Davidson revenue over the ten-year time period is about 54.7 billion dollars.

(c) We have

$$\text{Value on Jan. 1, 2013} = 54.7e^{0.035(10)} = 77.6 \text{ billion dollars.}$$

14. (a) Since the rate at which revenue is generated is at least 17.9 billion dollars per year, the present value of the revenue over a five-year time period is at least

$$\int_0^5 17.9e^{-0.045t} \, dt = 80.1 \text{ billion dollars.}$$

Since the rate at which revenue is generated is at most 22.8 billion dollars per year, the present value of the revenue over a five-year time period is at most

$$\int_0^5 22.8e^{-0.045t} \, dt = 102.1 \text{ billion dollars.}$$

The present value of McDonald's revenue over a five year time period is between 80.1 and 102.1 billion dollars.

(b) The present value of the revenue over a twenty-five year time period is at least

$$\int_0^{25} 17.9e^{-0.045t} \, dt = 268.6 \text{ billion dollars.}$$

The present value of the revenue over a twenty-five year time period is at most

$$\int_0^{25} 22.8e^{-0.045t} \, dt = 342.2 \text{ billion dollars.}$$

The present value of McDonald's revenue over a twenty-five year time period is between 268.6 and 342.2 billion dollars.

15. We want to find the value of T making the present value of income stream ($80,000/year) equal to $130,000. Thus we want to find the time T at which

$$130,000 = \int_0^T 80,000e^{-0.085t} \, dt.$$

Trying a few values of T, we get $T \approx 1.75$. It takes approximately one year and nine months for the present value of the profit generated by the new machinery to equal the cost of the machinery.

16. (a) Suppose the oil extracted over the time period $[0, M]$ is S. (See Figure 6.37.) Since $q(t)$ is the rate of oil extraction, we have:

$$S = \int_0^M q(t)dt = \int_0^M (a - bt)dt = \int_0^M (10 - 0.1t) \, dt.$$

To calculate the time at which the oil is exhausted, set $S = 100$ and try different values of M. We find $M = 10.6$ gives

$$\int_0^{10.6} (10 - 0.1t) \, dt = 100,$$

so the oil is exhausted in 10.6 years.

Extraction Curve

$q(t)$

Area below
the extraction curve
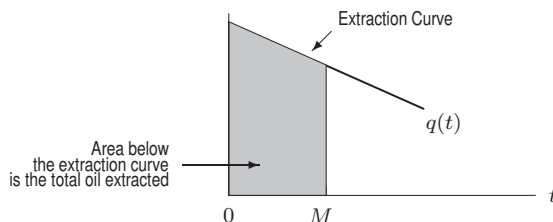is the total oil extracted

$0 \qquad M$

t

Figure 6.37

(b) Suppose p is the oil price, C is the extraction cost per barrel, and r is the interest rate. We have the present value of the profit as

$$\text{Present value of profit} = \int_0^M (p - C)q(t)e^{-rt}dt$$

$$= \int_0^{10.6} (20 - 10)(10 - 0.1t)e^{-0.1t} \, dt$$

$$= 624.9 \text{ million dollars.}$$

17. Price in future $= P(1 + 20\sqrt{t})$.

The present value V of price satisfies $V = P(1 + 20\sqrt{t})e^{-0.05t}$.

We want to maximize V. To do so, we find the critical points of $V(t)$ for $t \geq 0$. (Recall that $\sqrt{t}$ is nondifferentiable at $t = 0$.)

$$\frac{dV}{dt} = P\left[\frac{20}{2\sqrt{t}}e^{-0.05t} + (1 + 20\sqrt{t})(-0.05e^{-0.05t})\right]$$

$$= Pe^{-0.05t}\left[\frac{10}{\sqrt{t}} - 0.05 - \sqrt{t}\right].$$

Setting $\dfrac{dV}{dt} = 0$ gives $\dfrac{10}{\sqrt{t}} - 0.05 - \sqrt{t} = 0$. Using a calculator, we find $t \approx 10$ years. Since $V'(t) > 0$ for $0 < t < 10$ and $V'(t) < 0$ for $t > 10$, we confirm that this is a maximum. Thus, the best time to sell the wine is in 10 years.

Solutions for Section 6.4

1. (a) The town is growing by 50 people per year. The population in year 1 is thus $1000 + 50 = 1050$. Continuing to add 50 people per year to the population of the town produces the data in Table 6.1.

Table 6.1

Year	0	1	2	3	4	...	10
Population	1000	1050	1100	1150	1200	...	1500

(b) The town is growing at 5% per year. The population in year 1 is thus $1000 + 1000(0.05) = 1000(1.05) = 1050$. Continuing to multiply the town's population by 1.05 each year yields the data in Table 6.2.

Table 6.2

Year	0	1	2	3	4	...	10
Population	1000	1050	1103	1158	1216	...	1629

2. The relative birth rate is $\frac{30}{1000} = 0.03$ and the relative death rate is $\frac{20}{1000} = 0.02$. The relative rate of growth is $0.03 - 0.02 = 0.01$

3. The change in $\ln P(t)$ is the area under the curve, which is $\frac{1}{2} \cdot 10 \cdot (0.02) = 0.1$. So

$$\ln P(10) - \ln P(0) = \int_0^{10} \frac{P'(t)}{P(t)}\,dt = 0.1$$

$$\ln\left(\frac{P(10)}{P(0)}\right) = 0.1$$

$$\frac{P(10)}{P(0)} = e^{0.1} \approx 1.11.$$

The population has increased by about 11% over the 10-year period.

4. (a) The 14 is the relative change, as it is the absolute change divided by the initial figure.

The absolute change is $144.7 - 127 = 17.7$ million.

$$\text{The absolute rate of change} = \frac{144.7 - 127}{2005 - 1995}$$

$$= \frac{17.7}{10}$$

$$= 1.77 \text{ million per year.}$$

$$\text{The relative rate of change} = \frac{\text{absolute rate of change}}{\text{initial figure}} = \frac{1.77}{127}$$

$$= 1.4\% \text{ per year.}$$

(b) The 24 is a relative change.
   The 16.2 is an absolute change.
   The 9.1 is an absolute change.
   The 1.3 is an absolute change.
   The 28 is a relative change.

5. Between 2003 and 2004, absolute increase $= 33.3 - 30.2 = 3.1$ million. Between 2006 and 2007, absolute increase $= 39.6 - 37.6 = 2.0$ million.

   Between 2003 and 2004, relative increase $= 3.1/30.2 = 10.3\%$. Between 2006 and 2007, relative increase $= 2.0/37.6 = 5.3\%$.

6. (a) Since the absolute growth rate for one year periods is constant, the function is linear, and the formula is

$$P = 100 + 10t.$$

(b) Since the relative growth rate for one year periods is constant, the function is exponential, and the formula is

$$P = 100(1.10)^t.$$

(c) See Figure 6.38.
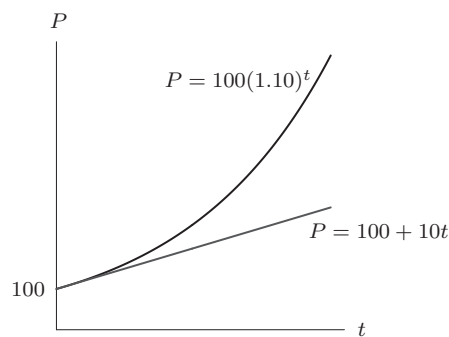
Figure 6.38: Two kinds of population growth

7. (a) If population is decreasing at 100 per hour,

$$\text{Population} = 4000 - 100(\text{Number of hours}).$$

So

$$P = 4000 - 100t.$$

(b) If population is decreasing at 5% per hour, then it is multiplied by $(1 - 0.05) = 0.95$ for each additional hour. This means

$$P = 4000(1 - 0.05)^t = 4000(0.95)^t.$$

The linear function $(P = 4000 - 100t)$ reaches zero first.

8. The area under the curve is about 4.5 grid squares, and each grid square has area $(0.05)(2) = 0.1$, so the area under the curve is about 0.45. The change in $\ln P$ is the area under the curve, so

$$\ln P(8) - \ln P(0) = \int_0^8 \frac{P'(t)}{P(t)} \, dt = 0.45$$

$$\ln\left(\frac{P(8)}{P(0)}\right) = 0.45$$

$$\frac{P(8)}{P(0)} = e^{0.45} = 1.57$$

The population grows by 57% over the 8-year period. Since

$$\frac{P(8)}{P(0)} = 1.57,$$

solving for $P(8)$ and substituting $P(0) = 10{,}000$ gives

$$P(8) = 1.57 \cdot P(0)$$
$$= 1.57(10{,}000)$$
$$= 15{,}700.$$

The population is about 15,700 after 8 years.

9. The relative growth rate is a constant $0.02 = 2\%$. The change in $\ln P(t)$ is the area under the curve which is $10(0.02) = 0.2$. So

$$\ln P(10) - \ln P(0) = \int_0^{10} \frac{P'(t)}{P(t)}\, dt = 0.2$$

$$\ln\left(\frac{P(10)}{P(0)}\right) = 0.2$$

$$\frac{P(10)}{P(0)} = e^{0.2} \approx 1.22.$$

The population has increased by about 22% over the 10-year period.

Another way of looking at this problem is to say that since $P(t)$ is growing at a constant 2% rate, it is growing exponentially, so

$$P(t) = P(0)e^{0.02t}.$$

Substituting $t = 10$ gives the same result as before:

$$\frac{P(10)}{P(0)} = e^{0.02(10)} = e^{0.2} \approx 1.22.$$

10. The change in $\ln P$ is the area under the curve, which is $\frac{1}{2} \cdot 10 \cdot (0.08) = 0.4$. So

$$\ln P(10) - \ln P(0) = \int_0^{10} \frac{P'(t)}{P(t)}\, dt = 0.4$$

$$\ln\left(\frac{P(10)}{P(0)}\right) = 0.4$$

$$\frac{P(10)}{P(0)} = e^{0.4} \approx 1.49.$$

The population has increased by about 49% over the 10-year period.

11. The area between the graph of the relative rate of change and the t-axis is 0 (because the areas above and below are equal). Since the change in $\ln P(t)$ is the area under the curve

$$\ln P(10) - \ln P(10) = \int_0^{10} \frac{P'(t)}{P(t)}\, dt = 0$$

$$\ln\left(\frac{P(10)}{P(0)}\right) = 0$$

$$\frac{P(10)}{P(0)} = e^0 = 1$$

$$P(10) = P(0).$$

The population at $t = 0$ and at $t = 10$ are equal.

12. The area of the triangle is $\frac{1}{2} \cdot 10 \cdot (0.04) = 0.2$. The change in $\ln P(t)$ is $\int_0^{10} \frac{P'(t)}{P(t)}\, dt$, which is the negative of the area, or -0.2. So

$$\ln P(10) - \ln P(0) = \int_0^{10} \frac{P'(t)}{P(t)}\, dt = -0.2$$

$$\ln\left(\frac{P(10)}{P(0)}\right) = -0.2$$

$$\frac{P(10)}{P(0)} = e^{-0.2} \approx 0.82.$$

The population has decreased by about 18% over the 10-year period.

13. Although the relative rate of change is decreasing, it is everywhere positive, so f is an increasing function for $0 \le t \le 10$.

14. Since the relative rate of change is positive for $0 \le t < 5$ and negative for $5 < t \le 10$, the function f is increasing for $0 \le t < 5$ and decreasing for $5 < t \le 10$.

15. Since the relative rate of change is negative for $0 < t < 5$ and positive for $5 < t < 10$, the function f is decreasing for $0 < t < 5$ and increasing for $5 < t < 10$.

16. Although the relative rate of change is increasing, it is negative everywhere, so f is a decreasing function for $0 \le t \le 10$.

17. (a) The population of Nicaragua is growing exponentially and we use $P = P_0 a^t$. In 2008 (at $t = 0$), we know $P = 5.78$, so $P_0 = 5.78$. Nicaragua is growing at a rate of 1.8% per year, so $a = 1.018$. We have

$$P = 5.78(1.018)^t.$$

(b) We first find the average rate of change between $t = 0$ (2008) and $t = 1$ (2009). At $t = 0$, we know $P = 5.78$ million. At $t = 1$, we find that $P = 5.78(1.018)^1 = 5.88404$ million. We have

$$\text{Average rate of change} = \frac{P(1) - P(0)}{1 - 0} = \frac{5.88404 - 5.78}{1} = 0.10404 \text{ million people/yr.}$$

Between 2004 and 2005, the population of Nicaragua grew by 0.10404 million people per year, or 104,040 people per year.

When $t = 2$ (2010), we have $P = 5.78(1.018)^2 = 5.98995$. Between 2009 and 2010, we have

$$\text{Average rate of change} = \frac{P(2) - P(1)}{2 - 1} = \frac{5.98995 - 5.88404}{1} = 0.10591 \text{ million people/yr.}$$

Between 2009 and 2010, the population of Nicaragua grew by 0.10591 million people per year, or 105,910 people per year. Notice that the population increase was greater between 2009 and 2010.

(c) The relative rate of change between 2008 and 2009 is the absolute rate of change divided by the population in 2008, which is $0.10404/5.78 = 0.018$. The relative rate of change is 1.8% per year, as we expected. Between 2009 and 2010, the relative rate of change is $0.10591/5.88404 = 0.018$. Over small intervals, the relative rate of change will always be approximately 1.8% per year.

18. The population increases during this 10-year period since the birth rate is greater than the death rate. The area between the two curves is a triangle with area $\frac{1}{2} \cdot 10 \cdot 0.01 = 0.05$, so we have

$$\ln\left(\frac{P(10)}{P(0)}\right) = \int_0^{10} \frac{P'(t)}{P(t)} dt = 0.05.$$

Therefore,

$$\frac{P(10)}{P(0)} = e^{0.05} = 1.0513, \quad \text{so} \quad P(10) = 1.0513 \cdot P(0).$$

The population grew by about 5.13% during this time.

19. The population decreases during this 10-year period since the birth rate is less than the death rate. The area between the two curves is a rectangle with area $10 \cdot 0.01 = 0.1$, so we have

$$\ln\left(\frac{P(10)}{P(0)}\right) = \int_0^{10} \frac{P'(t)}{P(t)} dt = -0.1.$$

Therefore,

$$\frac{P(10)}{P(0)} = e^{-0.1} = 0.9048, \quad \text{so} \quad P(10) = 0.9048 \cdot P(0).$$

The population decreased by about 9.52% during this time.

Solutions for Chapter 6 Review

1. (a) Since $f(x)$ is positive on the interval from 0 to 6, the integral is equal to the area under the curve. By examining the graph, we can measure and see that the area under the curve is 20 square units, so

$$\int_0^6 f(x)dx = 20.$$

(b) The average value of $f(x)$ on the interval from 0 to 6 equals the definite integral we calculated in part (a) divided by the size of the interval. Thus

$$\text{Average Value} = \frac{1}{6}\int_0^6 f(x)dx = 3\frac{1}{3}.$$

2. Average value $= \dfrac{1}{10 - 0} \displaystyle\int_0^{10} e^t dt \approx 2202.55$

3. (a) Average value $= \displaystyle\int_0^1 \sqrt{1 - x^2}\, dx = 0.79.$

(b) The area between the graph of $y = 1 - x$ and the x-axis is 0.5. Because the graph of $y = \sqrt{1 - x^2}$ is concave down, it lies above the line $y = 1 - x$, so its average value is above 0.5. See Figure 6.39.
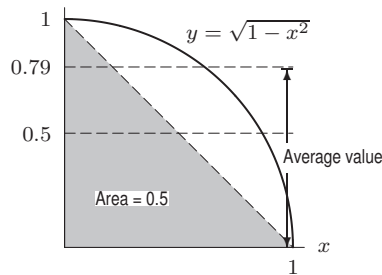

Figure 6.39

4. (a) When $t = 0$, $f(0) = 100e^0 = 100$. So there are 100 cases at the start of the six months. When $t = 6$, $f(6) = 100e^{-3} = 4.98$, so there are almost 5 cases left at the end of the half year.

(b) Average number of cases in the inventory can be given by

$$\left(\frac{1}{6 - 0}\right)\int_0^6 f(t)\, dt \approx \frac{190}{6} \approx 32 \text{ cases.}$$

5. (a) $Q(10) = 4(0.96)^{10} \approx 2.7.$ $Q(20) = 4(0.96)^{20} \approx 1.8$

(b)

$$\frac{Q(10) + Q(20)}{2} \approx 2.21$$

(c) The average value of Q over the interval is about 2.18.

(d) Because the graph of Q is concave up between $t = 10$ and $t = 20$, the area under the curve is less than what is obtained by connecting the endpoints with a straight line.

6. It appears that the area under a line at about $y = 2.5$ is approximately the same as the area under $f(x)$ on the interval $x = a$ to $x = b$, so we estimate that the average value is about 2.5. See Figure 6.40.
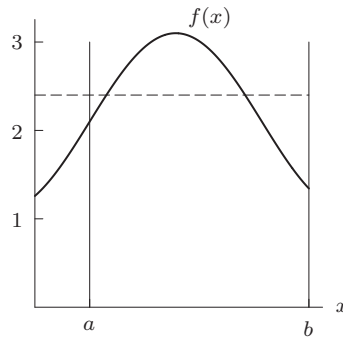
Figure 6.40

7. It appears that the area under a line at about $y = 17$ is approximately the same as the area under $f(x)$ on the interval $x = a$ to $x = b$, so we estimate that the average value is about 17. See Figure 6.41.
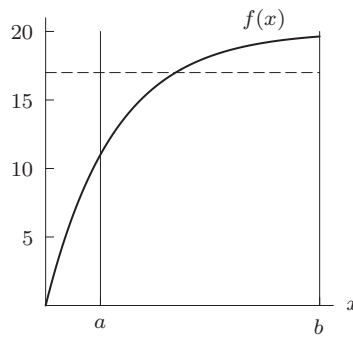

Figure 6.41

8. We'll show that in terms of the average value of f,

$$I > II = IV > III$$

Using the definition of average value and the fact that f is even, we have

$$\begin{aligned}\text{Average value} \\ \text{of } f \text{ on II}\end{aligned} = \frac{\int_0^2 f(x)dx}{2} = \frac{\frac{1}{2}\int_{-2}^2 f(x)dx}{2}$$

$$= \frac{\int_{-2}^2 f(x)dx}{4}$$

$$= \text{Average value of } f \text{ on IV}.$$

Since f is decreasing on [0,5], the average value of f on the interval $[0, c]$, where $0 \le c \le 5$, is decreasing as a function of c. The larger the interval the more low values of f are included. Hence

$$\begin{aligned}\text{Average value of } f \\ \text{on } [0, 1]\end{aligned} > \begin{aligned}\text{Average value of } f \\ \text{on } [0, 2]\end{aligned} > \begin{aligned}\text{Average value of } f \\ \text{on } [0, 5]\end{aligned}$$

9. **(a)** Average value of $f = \frac{1}{5}\int_0^5 f(x)\,dx$.
   (b) Average value of $|f| = \frac{1}{5}\int_0^5 |f(x)|\,dx = \frac{1}{5}(\int_0^2 f(x)\,dx - \int_2^5 f(x)\,dx)$.

10. **(a)** The supply and demand curves are shown in Figure 6.42. Tracing along the graphs, we find that they intersect approximately at the point $(322, 11.43)$. Thus, the equilibrium price is about \$11.43, and the equilibrium quantity is about 322 units.
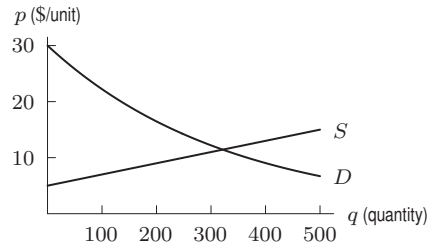
p ($/unit)

Figure 6.42

(b) The consumer surplus is shown in Figure 6.43. This is an area between two curves and we have

$$\text{Consumer surplus} = \int_0^{322} (30e^{-0.003q} - 11.43)dq = \$2513.52.$$

The producer surplus is shown in Figure 6.44. This is an area between two curves and we have

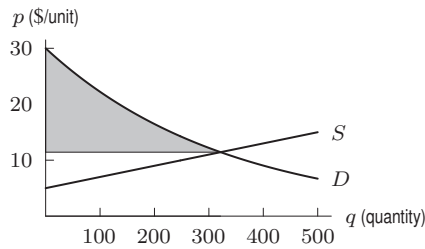$$\text{Producer surplus} = \int_0^{322} (11.43 - (5 + 0.02q))dq = \$1033.62.$$
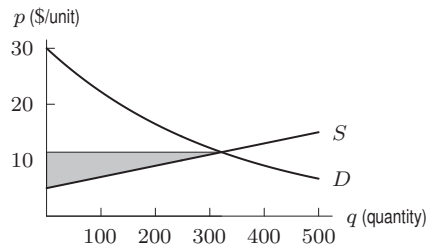

Figure 6.43

Figure 6.44

11. **(a)** Consumer surplus is greater than producer surplus in Figure 6.45.
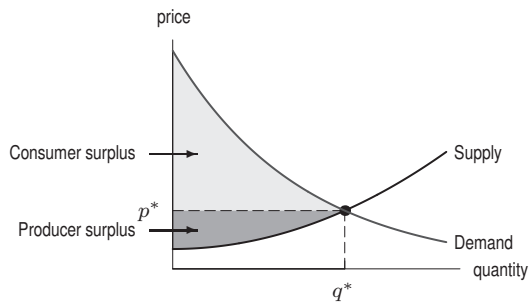
Figure 6.45
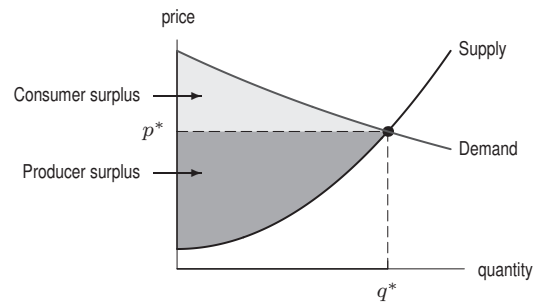
Figure 6.46

(b) Producer surplus is greater than consumer surplus in Figure 6.46.

12. Measuring money in thousands of dollars, the equation of the line representing the demand curve passes through (50, 980) and (350, 560). Its slope is $(560 - 980)/(350 - 50) = -420/300$. See Figure 6.47. So the equation is $y - 560 = -\frac{420}{300}(x - 350)$, i.e. $y - 560 = -\frac{7}{5}x + 490$. Thus

$$\text{Consumer surplus} = \int_0^{350} \left(-\frac{7}{5}x + 1050\right) dx - 350 \cdot 560 = 85{,}750.$$

(Note that $85{,}750 = \frac{1}{2} \cdot 490 \cdot 350$, the area of the triangle in Figure 6.47. We could have used this instead of the integral to find the consumer surplus.)

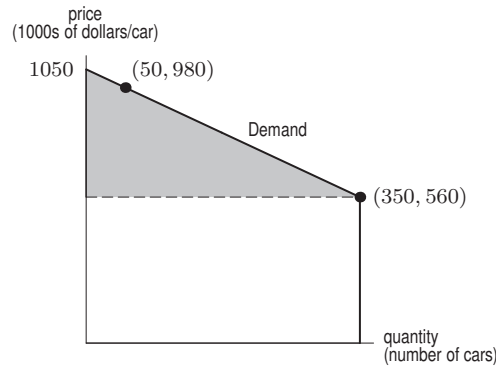Recalling that our unit measure for the price axis is $1000/car, the consumer surplus is $85,750,000.

Figure 6.47

13. **(a)** The quantity demanded at a price of $50 is calculated by substituting $p = 50$ into the demand equation $p = 100e^{-0.008q}$. Solving $50 = 100e^{-0.008q}$ for q gives $q \approx 86.6$. In other words, at a price of $50, consumer demand is about 87 units. The quantity supplied at a price of $50 is calculated by substituting by $p = 50$ into the supply equation $p = 4\sqrt{q} + 10$. Solving $50 = 4\sqrt{q} + 10$ for q gives $q = 100$. So at a price of $50, producers supply about 100 units. At a price of $50, the supply is larger than the demand, so some goods remain unsold. We can expect prices to be pushed down.

(b) The supply and demand curves are shown in Figure 6.48. The equilibrium price is about $p^* = \$48$ and the equilibrium quantity is about $q^* = 91$ units. The market will push prices downward from $50 toward the equilibrium price of $48. This agrees with the conclusion to part (a) that prices will drop.
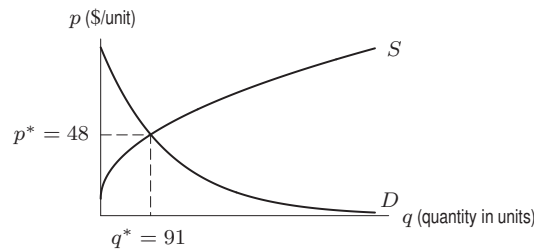

Figure 6.48: Demand and supply curves for a product

(c) See Figure 6.49. We have

$$\text{Consumer surplus} = \text{Area between demand curve and horizontal line } p = p^*.$$

The demand curve has equation $p = 100e^{0.008q}$, so

$$\text{Consumer surplus} = \int_0^{91} 100e^{-0.008q}\, dq - p^* q^* = 6464 - 48 \cdot 91 = 2096.$$

Consumers gain $2096 by buying goods at the equilibrium price instead of the price they would have been willing to pay.

For producer surplus, see Figure 6.50. We have

$$\text{Producer surplus} = \text{Area between supply curve and horizontal line } p = p^*$$

The supply curve has equation $p = 4\sqrt{q} + 10$, so

$$\text{Producer surplus} = p^* q^* - \int_0^{91} (4\sqrt{q} + 10)\, dq = 48 \cdot 91 - 3225 = 1143.$$

Producers gain $1143 by supplying goods at the equilibrium price instead of the price at which they would have been willing to provide the goods.
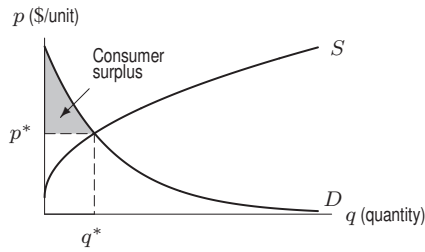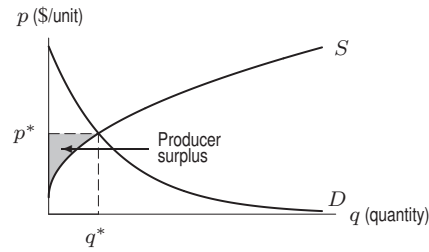

Figure 6.49: Consumer surplus

Figure 6.50: Producer surplus

14. The present value is given by

$$\text{Present value} = \int_0^M Se^{-rt}\,dt,$$

with $S = 1000$, $r = 0.09$ and $M = 5$. Hence,

$$\text{Present value} = \int_0^5 1000e^{-0.09t}\,dt = \$4026.35.$$

15. (a) The future value in 10 years is $100,000. We first find the present value, P:

$$100000 = Pe^{0.10(10)}$$
$$P = \$36{,}787.94$$

We solve for the income stream S:

$$36{,}787.94 = \int_0^{10} Se^{-0.10t}\,dt$$
$$36{,}787.94 = S\int_0^{10} e^{-0.10t}\,dt$$
$$36{,}787.94 = S(6.321)$$
$$S = \$5820.00 \text{ per year.}$$

The income stream required is about $5820 per year (or about $112 per week).

(b) The present value is $36,787.94. This is the amount that would have to be deposited now.

16. (a) The present value of the net sales over this five-year time period is given by

$$\text{Present value} = \int_0^5 (4.8 + 0.1t)e^{-0.02t}\,dt = 24.0 \text{ billion dollars.}$$

(b) The future value, on January 1, 2010 is given by

$$\text{Value 5 years later} = 24.0e^{0.02(5)} = 26.5 \text{ billion dollars.}$$

The value, on January 1, 2010, of Hershey's net sales during this time period is 26.5 billion dollars.

17. The present value, P, is given by

$$P = \int_0^M Se^{-rt}\,dt,$$

where S is the income stream, $r = 0.05$ and $M = 10$. We have

$$P = \int_0^{10} Se^{-0.05t}\,dt = S\int_0^{10} e^{-0.05t}\,dt = 7.8694S.$$

Solving $5000 = 7.8694S$ for S gives an income stream of $S = \$635.37$ per year.

18. One good way to approach the problem is in terms of present values. In 1980, the present value of Germany's loan was 20 billion DM. Now let's figure out the rate that the Soviet Union would have to give money to Germany to pay off 10% interest on the loan by using the formula for the present value of a continuous stream. Since the Soviet Union sends gas at a constant rate, the rate of deposit, $P(t)$, is a constant c. Since they don't start sending the gas until after 5 years have passed, the present value of the loan is given by:

$$\text{Present Value} = \int_5^\infty P(t)e^{-rt}\, dt.$$

We want to find c so that

$$20{,}000{,}000{,}000 = \int_5^\infty ce^{-rt}\, dt = c\int_5^\infty e^{-rt}\, dt$$

$$= c\int_5^\infty e^{-0.10t}\, dt$$

$$\approx 6.065c.$$

Dividing, we see that c should be about 3.3 billion DM per year. At 0.10 DM per m^3 of natural gas, the Soviet Union must deliver gas at the constant, continuous rate of about 33 billion m^3 per year.

19. (a) Let $P(t)$ be the population at time t in years. The population is increasing on the interval $0 \le t \le 10$. We have

$$\ln\left(\frac{P(10)}{P(0)}\right) = \int_0^{10} \frac{P'(t)}{P(t)}\, dt = \frac{1}{2}\cdot 10\cdot 0.02 = 0.1.$$

Therefore,

$$\frac{P(10)}{P(0)} = e^{0.1} = 1.105, \quad \text{so} \quad P(10) = 1.105\cdot P(0).$$

The population grew by about 10.5% during this time.

(b) The population is decreasing on the interval $10 \le t \le 15$. We have

$$\ln\left(\frac{P(15)}{P(10)}\right) = \int_{10}^{15} \frac{P'(t)}{P(t)}\, dt = -\frac{1}{2}\cdot 5\cdot 0.01 = -0.025.$$

Therefore,

$$\frac{P(15)}{P(10)} = e^{-0.025} = 0.975, \quad \text{so} \quad P(15) = 0.975\cdot P(10).$$

The population decreased by about 2.5% during this time.

(c) On the entire interval, we have

$$\ln\left(\frac{P(15)}{P(0)}\right) = \int_0^{15} \frac{P'(t)}{P(t)}\, dt = 0.1 - 0.025 = 0.075.$$

Therefore,

$$\frac{P(15)}{P(0)} = e^{0.075} = 1.078, \quad \text{so} \quad P(15) = 1.078\cdot P(0).$$

The population grew about 7.8% during this 15-year period.

20. (a) We compute the left- and right-hand sums:

$$\text{Left sum} = -0.03 + 0.03 - 0.06\ldots + 0.03 = -0.38$$

and

$$\text{Right sum} = 0.03 - 0.06\ldots + 0.03 + 0.02 = -0.33,$$

and average these to get our best estimate

$$\text{Average} = \frac{-0.38 + (-0.33)}{2} = -0.355.$$

We estimate that

$$\int_{1990}^{2002} \frac{P'(t)}{P(t)}\, dt \approx -0.36.$$

(b) The integral found in part (a) is the change in $\ln P(t)$, so we have

$$\ln P(2002) - \ln P(1990) = \int_{1990}^{2002} \frac{P'(t)}{P(t)} \, dt = -0.36$$

$$\ln \left(\frac{P(2002)}{P(1990)} \right) = -0.36$$

$$\frac{P(2002)}{P(1990)} = e^{-0.36} = 0.70.$$

Burglaries went down by a factor of 0.70 during this time, which means that the number of burglaries in 2002 was 0.70 times the number of burglaries in 1990. This is a decrease of about 30% during this period.

21. The rate at which petroleum is being used t years after 1990 is given by

$$r(t) = 1.4 \cdot 10^{20} (1.02)^t \text{ joules/year.}$$

Between 1990 and M years later

$$\text{Total quantity of petroleum used} = \int_0^M 1.4 \cdot 10^{20} (1.02)^t \, dt = 1.4 \cdot 10^{20} \left. \frac{(1.02)^t}{\ln(1.02)} \right|_0^M$$

$$= \frac{1.4 \cdot 10^{20}}{\ln(1.02)} ((1.02)^M - 1) \text{ joules.}$$

Setting the total quantity used equal to 10^{22} gives

$$\frac{1.4 \cdot 10^{20}}{\ln(1.02)} \left((1.02)^M - 1 \right) = 10^{22}$$

$$(1.02)^M = \frac{100 \ln(1.02)}{1.4} + 1 = 2.41$$

$$M = \frac{\ln(2.41)}{\ln(1.02)} \approx 45 \text{ years.}$$

So we will run out of petroleum in 2035.

22. (a) Since $f(x) = x^n$, we have $f'(x) = nx^{n-1}$, so

$$\text{Relative growth rate} = \frac{f'(x)}{f(x)} = \frac{nx^{n-1}}{x^n} = \frac{n}{x}.$$

For $x > 0$, the relative growth rate is decreasing to zero.

(b) The relative growth rate of $g(x) = e^{kx}$ is the constant k because

$$\text{Relative growth rate} = \frac{g'(x)}{g(x)} = \frac{ke^{kx}}{e^{kx}} = k.$$

Since k is a positive constant, the relative growth rate of $f(x)$ will, for large x, be smaller than k. So, for large x, the exponential function is growing faster than any power function.

CHECK YOUR UNDERSTANDING

1. False. The average value of $f(x)$ on the interval $0 \le x \le 10$ is $\frac{1}{10} \int_0^{10} f(x) \, dx$.

2. False, since f could have values much larger than 50 or much less than 0 between 0 and 50.

3. True, since $0 < \int_0^{10} f(x) \, dx \le \int_0^{10} 50 \, dx = 500$, so the average value of f is between 0 and 50.

4. True, since the units of $\int_a^b f(x) \, dx$ are the units of $f(x)$ multiplied by the units of x, and the average value is $1/(b - a) \int_a^b f(x) \, dx$ so we divide the integral units by the units of x, giving the same units as f.

5. False. A counterexample is $f(x) = x^2$, since $(1/10) \int_0^{10} x^2 \, dx = 100/3$, while $(1/20) \int_0^{20} x^2 \, dx = 400/3$.

6. True, since all of the function values are doubled for $2f(x)$. Also, we see that $(1/10) \int_0^{10} 2f(x) \, dx = 2 \cdot (1/10) \int_0^{10} f(x) \, dx$.

7. False, since, for example, $f(x)$ could be $+1$ over half of the interval $0 \leq x \leq 10$ and -1 over the other half.

8. True. If $f(x) > g(x) \geq 0$, then $\int_0^{10} f(x)\,dx > \int_0^{10} g(x) \geq 0$, so $(1/10) \int_0^{10} f(x)\,dx > (1/10) \int_0^{10} g(x)$.

9. True. This follows from the definition.

10. False. For example, consider the cost function $C(q) = 5q$. The average cost of $q = 10$ items is $\frac{C(q)}{q} = \frac{50}{10} = 5$ dollars per item. The average value of the cost function from $q = 0$ to $q = 10$ is $\frac{1}{10} \int_0^{10} C(q)\,dq = \frac{1}{10} \frac{500}{2} = 25$ dollars.

11. True, as specified in the text.

12. True, as specified in the text.

13. False. The equilibrium price is the price where the supply curve crosses the demand curve.

14. False. The consumer surplus is the total amount gained by consumers by purchasing items at prices *lower* than they are willing to pay.

15. True, they both have units of dollars.

16. False. The units of producer surplus are dollars/quantity $\cdot$ quantity $=$ dollars.

17. False. Total gains from trade is the *sum* of consumer and producer surplus.

18. False. The integral is the amount the consumers who bought would have spent if they had paid as much as they were willing to pay. The consumers actually spend $p^* q^*$.

19. True, as specified in the text.

20. False. Producer surplus at price p^* is the area between the *supply* curve and the horizontal line at p^*.

21. True. In M years, future value $=$ present value $\cdot e^{rM}$, and since the interest rate r is positive, the future value is greater.

22. True. The amount deposited is $5000 and the future value also includes the accumulated interest.

23. False, since if there were no interest, a deposit of $5000 today would pay for an income stream of $1000 per year for 5 years. With an interest rate of 2% less than $5000 would have to be deposited today to generate the same income stream.

24. False, since the single deposit grows to $1000e^{0.03} = 1030.45$ dollars in one year, while the income stream has a future value in one year of $e^{0.03} \int_0^1 1000e^{-0.03t}\,dt = 1015.15$.

25. False. An income stream has units of dollars/year.

26. False. The future value is $3000e^{5r}$.

27. True, since present value $=$ future value$/e^{5r}$ $=$ future value $\cdot e^{-5r}$.

28. True. Over the five year period, the amount deposited is $10,000 so the rest is the amount of interest earned.

29. False. The present value of an income stream of 2000 dollars per year that starts now and pays out over 6 years with a continuous interest rate of 2% is $\int_0^6 2000e^{-.02t}\,dt$.

30. True, since the amount deposited is $18,000 and the future value also includes the compounded interest.

31. False. As we saw in Chapter 2, the relative rate of change of population P with respect to time t is $\frac{1}{P}dP/dt$.

32. False. For example, if f gives population as a function of the number of years, the rate of change has units of people per year. The relative rate of change has units of percent per year.

33. False, since this is a relative growth rate. The correct statement is $(1/P)dP/dt = 0.02$.

34. True. Since the relative growth rate is $\frac{1}{P} \cdot \frac{dP}{dt} = \frac{P'(t)}{P(t)}$, the relative growth rate of P is the derivative of $\ln P(t)$.

35. True, since people are being added to the population faster than they're being removed.

36. False. If percent birth rate is above the percent death rate, the population is increasing.

37. False. It is the integral of the growth rate, not the relative growth rate, of a population P that gives the total change in P.

38. False. If the integral of the relative growth rate of a population P over 30 years is 0.5, the original population has been multiplied by $e^{0.5} = 1.65$, so the population has increased by approximately 65%.

39. True. If the integral of the relative growth rate is A, then the population at the end is e^A times the population at the beginning. When $A < 0$, we have $0 < e^A < 1$, so the population decreased.

40. True, since the natural logarithm of the ratio of the two populations is the value of the integral, A.

PROJECTS FOR CHAPTER SIX

1. (a) If the poorest $p\%$ of the population has exactly $p\%$ of the goods, then $F(x) = x$.

(b) Any such F is increasing. For example, the poorest 50% of the population includes the poorest 40%, and so the poorest 50% must own more than the poorest 40%. Thus $F(0.4) \leq F(0.5)$, and so, in general, F is increasing. In addition, it is clear that $F(0) = 0$ and $F(1) = 1$.

The graph of F is concave up by the following argument. Consider $F(0.05) - F(0.04)$. This is the fraction of resources the fifth poorest percent of the population has. Similarly, $F(0.20) - F(0.19)$ is the fraction of resources that the twentieth poorest percent of the population has. Since the twentieth poorest percent owns more than the fifth poorest percent, we have

$$F(0.05) - F(0.04) \leq F(0.20) - F(0.19).$$
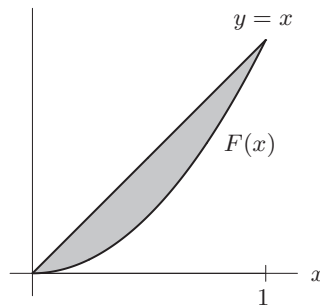
More generally, we can see that

$$F(x_1 + \Delta x) - F(x_1) \leq F(x_2 + \Delta x) - F(x_2)$$

for any x_1 smaller than x_2 and for any increment Δx. Dividing this inequality by Δx and taking the limit as $\Delta x \to 0$, we get

$$F'(x_1) \leq F'(x_2).$$

So, the derivative of F is an increasing function, i.e. F is concave up.

(c) G is twice the shaded area below in the following figure.

(d) The most equitable distribution is the one modeled by the equation $F(x) = x$, as this tells us that the wealth is evenly distributed. The less equitable the distribution the further the function is from resembling $F(x) = x$. Thus in our case, Country A has more equitable distribution.

(e) The function F goes through the points $(0,0)$ and $(1,1)$ and is increasing and concave up. Graphical representations of the two extremes for G are shown in Figures 6.51 and 6.52. Since Gini's index is twice the area shown, the maximum possible value is 1 and the minimum possible value is 0. If Gini's index is 1, the distribution of resources is as inequitable as it can get, with one person holding all resources and everyone else having none. (See Figure 6.51.) If Gini's index is 0, the distribution of resources is totally equitable, with $F(x) = x$ as in part (a). (See Figure 6.52.) The closer Gini's index is to 0, the more equitable the distribution.
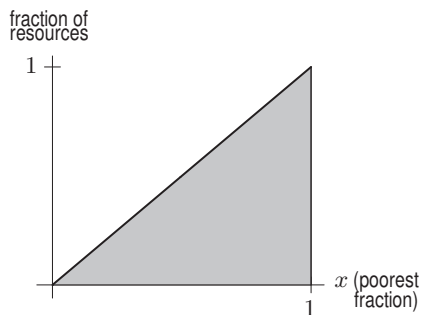

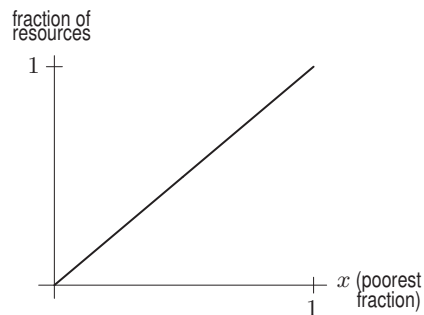**Figure 6.51**: Maximum value of Gini's index: Function F with $G = 1$

Figure 6.52: Minimum value of Gini's index: Function F with $G = 0$

2. (a) In Figure 6.53, the area of the shaded region is $F(M)$. Thus, $F(M) = \int_0^M y(t)\,dt$ and, by the Fundamental Theorem, $F'(M) = y(M)$.
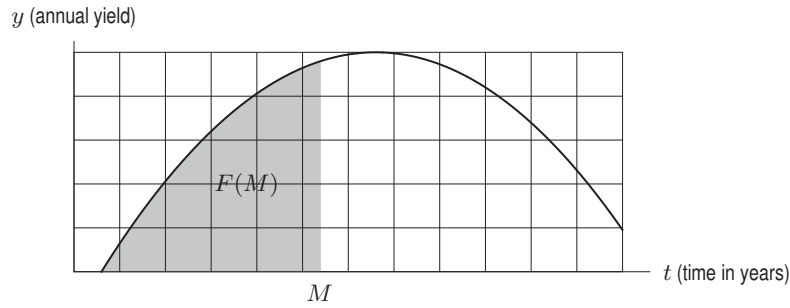

Figure 6.53

(b) Figure 6.54 is a graph of $F(M)$. Note that the graph of y looks like the graph of a quadratic function. Thus, the graph of F looks like a cubic.
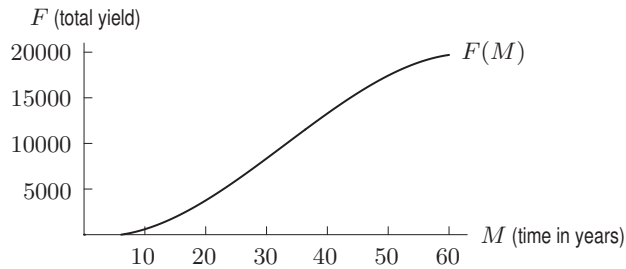

Figure 6.54

(c) We have

$$a(M) = \frac{1}{M}F(M) = \frac{1}{M}\int_0^M y(t)\,dt.$$

(d) If the function $a(M)$ takes on its maximum at some point M, then $a'(M) = 0$. Since

$$a(M) = \frac{1}{M}F(M),$$

differentiating using the quotient rule gives

$$a'(M) = \frac{MF'(M) - F(M)}{M^2} = 0,$$

so $MF'(M) = F(M)$. Since $F'(M) = y(M)$, the condition for a maximum may be written as

$$My(M) = F(M)$$

or as

$$y(M) = a(M).$$

To estimate the value of M which satisfies $My(M) = F(M)$, use the graph of $y(t)$. Notice that $F(M)$ is the area under the curve from 0 to M, and that $My(M)$ is the area of a rectangle of base M and height $y(M)$. Thus, we want the area under the curve to be equal to the area of the rectangle, or $A = B$ in Figure 6.55. This happens when $M \approx 50$ years. In other words, the orchard should be cut down after about 50 years.
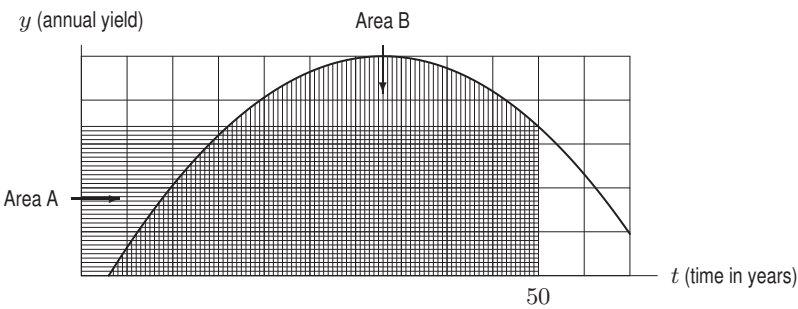
Figure 6.55