CHAPTER SEVEN

Solutions for Section 7.1

1. $5x$

2. $\frac{5}{2}t^2$

3. $\frac{1}{3}t^3 + \frac{1}{2}t^2$

4. $\frac{1}{3}x^3$

5. $\dfrac{x^5}{5}$.

6. $\dfrac{t^8}{8} + \dfrac{t^4}{4}$.

7. $6(\dfrac{x^4}{4}) + 4x = \dfrac{3x^4}{2} + 4x$.

8. $\dfrac{5q^3}{3}$.

9. We break the antiderivative into two terms. Since y^3 is an antiderivative of $3y^2$ and $-y^4/4$ is an antiderivative of $-y^3$, an antiderivative of $3y^2 - y^3$ is

$$y^3 - \frac{y^4}{4}.$$

10. $10x + 8(\dfrac{x^4}{4}) = 10x + 2x^4$.

11. $x^3 + 5x$.

12. Antiderivative $F(x) = \dfrac{x^2}{2} + \dfrac{x^6}{6} - \dfrac{x^{-4}}{4} + C$

13. $\dfrac{x^3}{3} - 6(\dfrac{x^2}{2}) + 17x = \dfrac{x^3}{3} - 3x^2 + 17x$.

14. $\frac{2}{3}z^{\frac{3}{2}}$

15. $\dfrac{5}{2}x^2 - \dfrac{2}{3}x^{\frac{3}{2}}$

16. Since $(\sqrt{z})^3 = z^{3/2}$, an antiderivative of $(\sqrt{z})^3$ is

$$\frac{z^{(3/2)+1}}{(3/2)+1} = \frac{2}{5}z^{5/2}.$$

17. $\ln|z|$

18. $\dfrac{t^4}{4} - \dfrac{t^3}{6} - \dfrac{t^2}{2}$

19. $-\dfrac{1}{2z^2}$

20. $\dfrac{y^5}{5} + \ln|y|$

21. $F(x) = \dfrac{x^7}{7} - \dfrac{1}{7}(\dfrac{x^{-5}}{-5}) + C = \dfrac{x^7}{7} + \dfrac{1}{35}x^{-5} + C$

22. $\ln|x| - \dfrac{1}{x} - \dfrac{1}{2x^2} + C$

23. $\dfrac{e^{-3t}}{-3} = \dfrac{-e^{-3t}}{3}$.

24. $\sin t$

25. $G(t) = 5t + \sin t + C$

26. $G(\theta) = -\cos\theta - 2\sin\theta + C$

27. $f(x) = 3$, so $F(x) = 3x + C$. $F(0) = 0$ implies that $3 \cdot 0 + C = 0$, so $C = 0$. Thus $F(x) = 3x$ is the only possibility.

28. $f(x) = -7x$, so $F(x) = \frac{-7x^2}{2} + C$. $F(0) = 0$ implies that $-\frac{7}{2} \cdot 0^2 + C = 0$, so $C = 0$. Thus $F(x) = -7x^2/2$ is the only possibility.

29. $f(x) = 2 + 4x + 5x^2$, so $F(x) = 2x + 2x^2 + \frac{5}{3}x^3 + C$. $F(0) = 0$ implies that $C = 0$. Thus $F(x) = 2x + 2x^2 + \frac{5}{3}x^3$ is the only possibility.

30. $f(x) = x^2$, so $F(x) = \dfrac{x^3}{3} + C$. $F(0) = 0$ implies that $\dfrac{0^3}{3} + C = 0$, so $C = 0$. Thus $F(x) = \dfrac{x^3}{3}$ is the only possibility.

31. $f(x) = x^{1/2}$, so $F(x) = \frac{2}{3}x^{3/2} + C$. $F(0) = 0$ implies that $\frac{2}{3} \cdot 0^{3/2} + C = 0$, so $C = 0$. Thus $F(x) = \frac{2}{3}x^{3/2}$ is the only possibility.

32. Since $\dfrac{d}{dx}(e^x) = e^x$, we take $F(x) = e^x + C$. Now

$$F(0) = e^0 + C = 1 + C = 0,$$

so

$$C = -1$$

and

$$F(x) = e^x - 1.$$

33. $\frac{5}{2}x^2 + 7x + C$

34. $3x^3 + C.$

35. $2x^3 + C.$

36. $\dfrac{t^{13}}{13} + C.$

37. $\displaystyle\int (x+1)^2\,dx = \frac{(x+1)^3}{3} + C.$

Another way to work the problem is to expand $(x+1)^2$ to $x^2 + 2x + 1$ as follows:

$$\int (x+1)^2\,dx = \int (x^2 + 2x + 1)\,dx = \frac{x^3}{3} + x^2 + x + C.$$

These two answers are the same, since $\dfrac{(x+1)^3}{3} = \dfrac{x^3 + 3x^2 + 3x + 1}{3} = \dfrac{x^3}{3} + x^2 + x + \dfrac{1}{3}$, which is $\dfrac{x^3}{3} + x^2 + x$, plus a constant.

38. $\displaystyle\int (x^2 + x^{-2})\,dx = \frac{x^3}{3} + \frac{x^{-1}}{-1} + C = \frac{x^3}{3} - \frac{1}{x} + C$

39. $\displaystyle\int (t^2 + 5t + 1)\,dt = \frac{t^3}{3} + 5 \cdot \frac{t^2}{2} + t + C$

40. $5e^z + C$

41. $\displaystyle\int (t^3 + 6t^2)\,dt = \frac{t^4}{4} + 6 \cdot \frac{t^3}{3} + C = \frac{t^4}{4} + 2t^3 + C$

42. $\dfrac{x^6}{6} - 3x^4 + C$

43. $\displaystyle\int 3w^{1/2}\,dw = 3 \cdot \frac{w^{3/2}}{3/2} + C = 2w^{3/2} + C$

44. $\dfrac{x^3}{3} + 2x^2 - 5x + C$

45. $3\ln|t| + \dfrac{2}{t} + C$

46. $\dfrac{e^{2t}}{2} + C.$

47. $\dfrac{x^2}{2} + 2x^{1/2} + C$

48. $\displaystyle\int (x^3 + 5x^2 + 6)dx = \frac{x^4}{4} + \frac{5x^3}{3} + 6x + C$

49. $e^x + 5x + C$

50. $\dfrac{x^3}{3} + \ln|x| + C.$

51. $\displaystyle\int e^{3r}\,dr = \frac{1}{3}e^{3r} + C$

52. $\sin\theta + C$

53. $-\cos t + C$

54. $\displaystyle\int 25e^{-0.04q}\,dq = 25\left(\frac{1}{-0.04}e^{-0.04q}\right) + C = -625e^{-0.04q} + C$

55. $25e^{4x} + C$

56. $2e^x - 8\sin x + C$

57. $3\sin x + 7\cos x + C$

58. $-\dfrac{1}{3}\cos(3x) + C$

59. We use the substitution $w = x^2 + 4$, $dw = 2xdx$:

$$\int x\cos(x^2+4)dx = \frac{1}{2}\int \cos w\,dw = \frac{1}{2}\sin w + C = \frac{1}{2}\sin(x^2+4) + C.$$

60. $2\sin(3x) + C$

61. $10x - 4\cos(2x) + C$

62. $-6\cos(2x) + 3\sin(5x) + C$

63. An antiderivative is $F(x) = 3x^2 - 5x + C$. Since $F(0) = 5$, we have $5 = 0 + C$, so $C = 5$. The answer is $F(x) = 3x^2 - 5x + 5$.

64. An antiderivative is $F(x) = \dfrac{x^3}{3} + x + C$. Since $F(0) = 5$, we have $5 = 0 + C$, so $C = 5$. The answer is $F(x) = x^3/3 + x + 5$.

65. An antiderivative is $F(x) = -4\cos(2x) + C$. Since $F(0) = 5$, we have $5 = -4\cos 0 + C = -4 + C$, so $C = 9$. The answer is $F(x) = -4\cos(2x) + 9$.

66. An antiderivative is $F(x) = 2e^{3x} + C$. Since $F(0) = 5$, we have $5 = 2e^0 + C = 2 + C$, so $C = 3$. The answer is $F(x) = 2e^{3x} + 3$.

67. The marginal cost, MC, is given by differentiating the total cost function, C, with respect to q so

$$\frac{dC}{dq} = MC.$$

Therefore,

$$\begin{aligned} C &= \int MC\,dq \\ &= \int \left(3q^2 + 4q + 6\right)\,dq \\ &= q^3 + 2q^2 + 6q + D, \end{aligned}$$

where D is a constant. We can check this by noting

$$\frac{dC}{dq} = \frac{d}{dq}\left(q^3 + 2q^2 + 6q + D\right) = 3q^2 + 4q + 6 = MC.$$

The fixed costs are given to be 200 so $C = 200$ when $q = 0$, thus $D = 200$. The total cost function is

$$C = q^3 + 2q^2 + 6q + 200.$$

68. (a) The marginal revenue, MR, is given by differentiating the total revenue function, R, with respect to q so

$$\frac{dR}{dq} = MR.$$

Therefore,

$$\begin{aligned} R &= \int MR\,dq \\ &= \int (20 - 4q)\;dq \\ &= 20q - 2q^2 + C. \end{aligned}$$

We can check this by noting

$$\frac{dR}{dq} = \frac{d}{dq}\left(20q - 2q^2 + C\right) = 20 - 4q = MR.$$

When no goods are produced the total revenue is zero so $C = 0$ and the total revenue is $R = 20q - 2q^2$.

(b) The total revenue, R, is given by pq where p is the price, so the demand curve is

$$p = \frac{R}{q} = 20 - 2q.$$

Solutions for Section 7.2

1. We use the substitution $w = x^3 + 1$, $dw = 3x^2dx$.

$$\int 3x^2(x^3+1)^4dx = \int w^4dw = \frac{w^5}{5} + C = \frac{1}{5}(x^3+1)^5 + C.$$

2. We use the substitution $w = x^2 + 1$, $dw = 2xdx$.

$$\int \frac{2x}{x^2+1}dx = \int \frac{1}{w}dw = \ln|w| + C = \ln(x^2+1) + C.$$

3. We use the substitution $w = x + 10$, $dw = dx$.

$$\int (x+10)^3dx = \int w^3dw = \frac{w^4}{4} + C = \frac{1}{4}(x+10)^4 + C.$$

Check: $\frac{d}{dx}(\frac{1}{4}(x+10)^4 + C) = (x+10)^3$.

4. We use the substitution $w = x^2 + 9$, $dw = 2xdx$:

$$\int x(x^2+9)^6dx = \frac{1}{2}\int w^6dw = \frac{1}{2}\frac{w^7}{7} + C = \frac{1}{14}(x^2+9)^7 + C.$$

5. We use the substitution $w = q^2 + 1$, $dw = 2qdq$.

$$\int 2qe^{q^2+1}dq = \int e^w dw = e^w + C = e^{q^2+1} + C.$$

6. We use the substitution $w = 5t + 2$, $dw = 5dt$.

$$\int 5e^{5t+2}dt = \int e^w dw = e^w + C = e^{5t+2} + C.$$

Check: $\frac{d}{dt}(e^{5t+2} + C) = 5e^{5t+2}$.

7. Make the substitution $w = t^2$, $dw = 2t\,dt$. The general antiderivative is $\int te^{t^2}\,dt = (1/2)e^{t^2} + C$.

8. We use the substitution $w = -x$, $dw = -\,dx$.

$$\int e^{-x}dx = -\int e^w dw = -e^w + C = -e^{-x} + C.$$

Check: $\frac{d}{dx}(-e^{-x} + C) = -(-e^{-x}) = e^{-x}$.

9. We use the substitution $w = t^3 - 3$, $dw = 3t^2\,dt$.

$$\int t^2(t^3-3)^{10}\,dt = \frac{1}{3}\int (t^3-3)^{10}(3t^2dt) = \int w^{10}\left(\frac{1}{3}\,dw\right)$$
$$= \frac{1}{3}\frac{w^{11}}{11} + C = \frac{1}{33}(t^3-3)^{11} + C.$$

Check: $\frac{d}{dt}[\frac{1}{33}(t^3-3)^{11} + C] = \frac{1}{3}(t^3-3)^{10}(3t^2) = t^2(t^3-3)^{10}$.

10. We use the substitution $w = 1 + 2x^3$, $dw = 6x^2\,dx$.

$$\int x^2(1+2x^3)^2\,dx = \int w^2(\frac{1}{6}\,dw) = \frac{1}{6}(\frac{w^3}{3}) + C = \frac{1}{18}(1+2x^3)^3 + C.$$

Check: $\frac{d}{dx}\left[\frac{1}{18}(1+2x^2)^3 + C\right] = \frac{1}{18}[3(1+2x^3)^2(6x^2)] = x^2(1+2x^3)^2$.

11. We use the substitution $w = x^2 - 4$, $dw = 2x\,dx$.

$$\int x(x^2-4)^{7/2}\,dx = \frac{1}{2}\int (x^2-4)^{7/2}(2xdx) = \frac{1}{2}\int w^{7/2}\,dw$$
$$= \frac{1}{2}\left(\frac{2}{9}w^{9/2}\right) + C = \frac{1}{9}(x^2-4)^{9/2} + C.$$

Check: $\frac{d}{dx}\left(\frac{1}{9}(x^2-4)^{9/2} + C\right) = \frac{1}{9}\left(\frac{9}{2}(x^2-4)^{7/2}\right)2x = x(x^2-4)^{7/2}$.

12. We use the substitution $w = x^2 + 3$, $dw = 2x\,dx$.

$$\int x(x^2+3)^2\,dx = \int w^2(\frac{1}{2}\,dw) = \frac{1}{2}\frac{w^3}{3} + C = \frac{1}{6}(x^2+3)^3 + C.$$

Check: $\frac{d}{dx}\left[\frac{1}{6}(x^2+3)^3 + C\right] = \frac{1}{6}\left[3(x^2+3)^2(2x)\right] = x(x^2+3)^2$.

13. We use the substitution $w = 4 - x$, $dw = -dx$.

$$\int \frac{1}{\sqrt{4-x}}\,dx = -\int \frac{1}{\sqrt{w}}\,dw = -2\sqrt{w} + C = -2\sqrt{4-x} + C.$$

Check: $\frac{d}{dx}(-2\sqrt{4-x} + C) = -2\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{4-x}}\cdot -1 = \frac{1}{\sqrt{4-x}}$.

14. We use the substitution $w = y + 5$, $dw = dy$, to get

$$\int \frac{dy}{y+5} = \int \frac{dw}{w} = \ln|w| + C = \ln|y+5| + C.$$

Check: $\dfrac{d}{dy}(\ln|y+5| + C) = \dfrac{1}{y+5}$.

15. We use the substitution $w = x^3$, $dw = 3x^2 dx$.

$$12 \int x^2 \cos(x^3) dx = \frac{12}{3} \int \cos(w) dw = 4\sin(w) + C = 4\sin(x^3) + C.$$

16. We use the substitution $w = 2t - 7,\ dw = 2\,dt$.

$$\int (2t-7)^{73}\,dt = \frac{1}{2}\int w^{73}\,dw = \frac{1}{(2)(74)} w^{74} + C = \frac{1}{148}(2t-7)^{74} + C.$$

Check: $\dfrac{d}{dt}\left[\dfrac{1}{148}(2t-7)^{74} + C\right] = \dfrac{74}{148}(2t-7)^{73}(2) = (2t-7)^{73}$.

17. In this case, it seems easier not to substitute.

$$\int (x^2+3)^2\,dx = \int (x^4 + 6x^2 + 9)\,dx = \frac{x^5}{5} + 2x^3 + 9x + C.$$

Check: $\dfrac{d}{dx}\left[\dfrac{x^5}{5} + 2x^3 + 9x + C\right] = x^4 + 6x^2 + 9 = (x^2+3)^2$.

18. In this case, it seems easier not to substitute.

$$\int y^2(1+y)^2\,dy = \int y^2(y^2 + 2y + 1)\,dy = \int (y^4 + 2y^3 + y^2)\,dy$$
$$= \frac{y^5}{5} + \frac{y^4}{2} + \frac{y^3}{3} + C.$$

Check: $\dfrac{d}{dy}\left(\dfrac{y^5}{5} + \dfrac{y^4}{2} + \dfrac{y^3}{3} + C\right) = y^4 + 2y^3 + y^2 = y^2(y+1)^2$.

19. We use the substitution $w = 3 - t$, $dw = -\,dt$.

$$\int \sin(3-t)dt = -\int \sin(w)dw = -(-\cos(w)) + C = \cos(3-t) + C.$$

Check: $\frac{d}{dt}(\cos(3-t) + C) = -\sin(3-t)(-1) = \sin(3-t)$.

20. We use the substitution $w = \cos\theta + 5,\ dw = -\sin\theta\,d\theta$.

$$\int \sin\theta(\cos\theta + 5)^7\,d\theta = -\int w^7\,dw = -\frac{1}{8}w^8 + C$$
$$= -\frac{1}{8}(\cos\theta + 5)^8 + C.$$

Check:

$$\frac{d}{d\theta}\left[-\frac{1}{8}(\cos\theta + 5)^8 + C\right] = -\frac{1}{8}\cdot 8(\cos\theta + 5)^7 \cdot (-\sin\theta)$$
$$= \sin\theta(\cos\theta + 5)^7$$

21. We use the substitution $w = \cos 3t,\ dw = -3\sin 3t\,dt$.

$$\int \sqrt{\cos 3t}\sin 3t\,dt = -\frac{1}{3}\int \sqrt{w}\,dw$$
$$= -\frac{1}{3}\cdot\frac{2}{3}w^{\frac{3}{2}} + C = -\frac{2}{9}(\cos 3t)^{\frac{3}{2}} + C.$$

Check:

$$\frac{d}{dt}\left[-\frac{2}{9}(\cos 3t)^{\frac{3}{2}} + C\right] = -\frac{2}{9}\cdot\frac{3}{2}(\cos 3t)^{\frac{1}{2}}\cdot(-\sin 3t)\cdot 3$$
$$= \sqrt{\cos 3t}\sin 3t.$$

22. We use the substitution $w = 1 + 3t^2,\ dw = 6t\,dt$.

$$\int \frac{t}{1+3t^2}\,dt = \int \frac{1}{w}\left(\frac{1}{6}\,dw\right) = \frac{1}{6}\ln|w| + C = \frac{1}{6}\ln(1+3t^2) + C.$$

(We can drop the absolute value signs since $1 + 3t^2 > 0$ for all t).

Check: $\frac{d}{dt}\left[\frac{1}{6}\ln(1+3t^2) + C\right] = \frac{1}{6}\frac{1}{1+3t^2}(6t) = \frac{t}{1+3t^2}$.

23. We use the substitution $w = \sin\theta,\ dw = \cos\theta\,d\theta$.

$$\int \sin^6\theta\cos\theta\,d\theta = \int w^6\,dw = \frac{w^7}{7} + C = \frac{\sin^7\theta}{7} + C.$$

Check: $\frac{d}{d\theta}\left[\frac{\sin^7\theta}{7} + C\right] = \sin^6\theta\cos\theta$.

24. We use the substitution $w = x^3 + 1$, $dw = 3x^2\,dx$, to get

$$\int x^2 e^{x^3+1}\,dx = \frac{1}{3}\int e^w\,dw = \frac{1}{3}e^w + C = \frac{1}{3}e^{x^3+1} + C.$$

Check: $\frac{d}{dx}\left(\frac{1}{3}e^{x^3+1} + C\right) = \frac{1}{3}e^{x^3+1}\cdot 3x^2 = x^2 e^{x^3+1}$.

25. We use the substitution $w = \sin x$, $dw = \cos x dx$:

$$\int \sin^2 x\cos x dx = \int w^2 dw = \frac{w^3}{3} + C = \frac{(\sin x)^3}{3} + C.$$

26. We use the substitution $w = \sin\alpha,\ dw = \cos\alpha\,d\alpha$.

$$\int \sin^3\alpha\cos\alpha\,d\alpha = \int w^3\,dw = \frac{w^4}{4} + C = \frac{\sin^4\alpha}{4} + C.$$

Check: $\frac{d}{d\alpha}\left(\frac{\sin^4\alpha}{4} + C\right) = \frac{1}{4}\cdot 4\sin^3\alpha\cdot\cos\alpha = \sin^3\alpha\cos\alpha$.

27. We use the substitution $w = 4x^2$, $dw = 8xdx$.

$$\int x\sin(4x^2)dx = \frac{1}{8}\int \sin(w)dw = -\frac{1}{8}\cos(w) + C = -\frac{1}{8}\cos(4x^2) + C.$$

28. We use the substitution $w = 3x - 4$, $dw = 3dx$.

$$\int e^{3x-4}dx = \frac{1}{3}\int e^w dw = \frac{1}{3}e^w + C = \frac{1}{3}e^{3x-4} + C.$$

Check: $\frac{d}{dx}\left(\frac{1}{3}e^{3x-4} + C\right) = \frac{1}{3}e^{3x-4}\cdot 3 = e^{3x-4}$.

29. We use the substitution $w = 3x^2$, $dw = 6x dx$.

$$\int xe^{3x^2} dx = \frac{1}{6}\int e^w dw = \frac{1}{6}e^w + C = \frac{1}{6}e^{3x^2} + C.$$

Check: $\frac{d}{dx}\left(\frac{1}{6}e^{3x^2} + C\right) = \frac{1}{6}e^{3x^2} \cdot 6x = xe^{3x^2}$.

30. We use the substitution $w = 3x^2 + 4$, $dw = 6x dx$.

$$\int x\sqrt{3x^2 + 4}dx = \frac{1}{6}\int w^{1/2} dw = \frac{1}{6}\frac{w^{3/2}}{3/2} + C = \frac{1}{9}(3x^2 + 4)^{3/2} + C.$$

31. We use the substitution $w = 5q^2 + 8$, $dw = 10q dq$.

$$\int \frac{q}{5q^2 + 8} dq = \frac{1}{10}\int \frac{1}{w} dw = \frac{1}{10}\ln|w| + C = \frac{1}{10}\ln(5q^2 + 8) + C.$$

Check: $\frac{d}{dq}\left(\frac{1}{10}\ln(5q^2 + 8) + C\right) = \frac{1}{10} \cdot \frac{1}{5q^2 + 8} \cdot 10q = \frac{q}{5q^2 + 8}$.

32. We use the substitution $w = \ln z$, $dw = \frac{1}{z}\,dz$.

$$\int \frac{(\ln z)^2}{z}\,dz = \int w^2\,dw = \frac{w^3}{3} + C = \frac{(\ln z)^3}{3} + C.$$

Check: $\frac{d}{dz}\left[\frac{(\ln z)^3}{3} + C\right] = 3 \cdot \frac{1}{3}(\ln z)^2 \cdot \frac{1}{z} = \frac{(\ln z)^2}{z}$.

33. We use the substitution $w = y^2 + 4$, $dw = 2y\,dy$.

$$\int \frac{y}{y^2 + 4}\,dy = \frac{1}{2}\int \frac{dw}{w} = \frac{1}{2}\ln|w| + C = \frac{1}{2}\ln(y^2 + 4) + C.$$

(We can drop the absolute value signs since $y^2 + 4 \geq 0$ for all y.)

Check: $\frac{d}{dy}\left[\frac{1}{2}\ln(y^2 + 4) + C\right] = \frac{1}{2} \cdot \frac{1}{y^2 + 4} \cdot 2y = \frac{y}{y^2 + 4}$.

34. We use the substitution $w = e^t + t$, $dw = (e^t + 1)\,dt$.

$$\int \frac{e^t + 1}{e^t + t}\,dt = \int \frac{1}{w}\,dw = \ln|w| + C = \ln|e^t + t| + C.$$

Check: $\frac{d}{dt}(\ln|e^t + t| + C) = \frac{e^t + 1}{e^t + t}$.

35. We use the substitution $w = \sqrt{y}$, $dw = \frac{1}{2\sqrt{y}}\,dy$.

$$\int \frac{e^{\sqrt{y}}}{\sqrt{y}}\,dy = 2\int e^w\,dw = 2e^w + C = 2e^{\sqrt{y}} + C.$$

Check: $\frac{d}{dy}(2e^{\sqrt{y}} + C) = 2e^{\sqrt{y}} \cdot \frac{1}{2\sqrt{y}} = \frac{e^{\sqrt{y}}}{\sqrt{y}}$.

36. We use the substitution $w = \sqrt{x}$, $dw = \frac{1}{2\sqrt{x}}\,dx$.

$$\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx = \int \cos w(2\,dw) = 2\sin w + C = 2\sin\sqrt{x} + C.$$

Check: $\frac{d}{dx}(2\sin\sqrt{x} + C) = 2\cos\sqrt{x}\left(\frac{1}{2\sqrt{x}}\right) = \frac{\cos\sqrt{x}}{\sqrt{x}}$.

37. We use the substitution $w = x + e^x$, $dw = (1 + e^x)\,dx$.

$$\int \frac{1 + e^x}{\sqrt{x + e^x}}\,dx = \int \frac{dw}{\sqrt{w}} = 2\sqrt{w} + C = 2\sqrt{x + e^x} + C.$$

Check: $\frac{d}{dx}(2\sqrt{x + e^x} + C) = 2 \cdot \frac{1}{2}(x + e^x)^{-\frac{1}{2}} \cdot (1 + e^x) = \frac{1 + e^x}{\sqrt{x + e^x}}$.

38. We use the substitution $w = e^t + 1$, $dw = e^t dt$:

$$\int \frac{e^t}{e^t+1}dt = \int \frac{1}{w}dw = \ln|w| + C = \ln(e^t+1) + C.$$

39. We use the substitution $w = x^2 + 2x + 19,\ dw = 2(x+1)dx$.

$$\int \frac{(x+1)dx}{x^2+2x+19} = \frac{1}{2}\int \frac{dw}{w} = \frac{1}{2}\ln|w| + C = \frac{1}{2}\ln(x^2+2x+19) + C.$$

(We can drop the absolute value signs, since $x^2 + 2x + 19 = (x+1)^2 + 18 > 0$ for all x.)
Check: $\frac{1}{dx}[\frac{1}{2}\ln(x^2+2x+19)] = \frac{1}{2}\frac{1}{x^2+2x+19}(2x+2) = \frac{x+1}{x^2+2x+19}$.

40. We use the substitution $w = e^x + e^{-x}$, $dw = (e^x - e^{-x})\,dx$.

$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx = \int \frac{dw}{w} = \ln|w| + C = \ln(e^x + e^{-x}) + C.$$

(We can drop the absolute value signs since $e^x + e^{-x} > 0$ for all x).
Check: $\frac{d}{dx}[\ln(e^x+e^{-x}) + C] = \frac{1}{e^x+e^{-x}}(e^x - e^{-x})$.

41. **(a)** This integral can be evaluated using integration by substitution. We use $w = x^2$, $dw = 2xdx$.

$$\int x\sin x^2 dx = \frac{1}{2}\int \sin(w)dw = -\frac{1}{2}\cos(w) + C = -\frac{1}{2}\cos(x^2) + C.$$

(b) This integral cannot be evaluated using a simple integration by substitution.
(c) This integral cannot be evaluated using a simple integration by substitution.
(d) This integral can be evaluated using integration by substitution. We use $w = 1 + x^2$, $dw = 2xdx$.

$$\int \frac{x}{(1+x^2)^2}dx = \frac{1}{2}\int \frac{1}{w^2}dw = \frac{1}{2}(\frac{-1}{w}) + C = \frac{-1}{2(1+x^2)} + C.$$

(e) This integral cannot be evaluated using a simple integration by substitution.
(f) This integral can be evaluated using integration by substitution. We use $w = 2 + \cos x$, $dw = -\sin x dx$.

$$\int \frac{\sin x}{2+\cos x}dx = -\int \frac{1}{w}dw = -\ln|w| + C = -\ln|2+\cos x| + C.$$

42. **(a)** (i) Multiplying out gives

$$\int (x^2 + 10x + 25)\,dx = \frac{x^3}{3} + 5x^2 + 25x + C.$$

(ii) Substituting $w = x + 5$, so $dw = dx$, gives

$$\int (x+5)^2\,dx = \int w^2\,dw = \frac{w^3}{3} + C = \frac{(x+5)^3}{3} + C.$$

(b) The results of the two calculations are not the same since

$$\frac{(x+5)^3}{3} + C = \frac{x^3}{3} + \frac{15x^2}{3} + \frac{75x}{3} + \frac{125}{3} + C.$$

However they differ only by a constant, $125/3$, as guaranteed by the Fundamental Theorem of Calculus.

43. **(a)** $\int 4x(x^2+1)\,dx = \int (4x^3 + 4x)\,dx = x^4 + 2x^2 + C$.
(b) If $w = x^2 + 1$, then $dw = 2x\,dx$.

$$\int 4x(x^2+1)\,dx = \int 2w\,dw = w^2 + C = (x^2+1)^2 + C.$$

(c) The expressions from parts (a) and (b) look different, but they are both correct. Note that $(x^2+1)^2 + C = x^4 + 2x^2 + 1 + C$. In other words, the expressions from parts (a) and (b) differ only by a constant, so they are both correct antiderivatives.

Solutions for Section 7.3

1. Since $F'(x) = 5$, we use $F(x) = 5x$. By the Fundamental Theorem, we have

$$\int_1^3 5dx = 5x\Big|_1^3 = 5(3) - 5(1) = 15 - 5 = 10.$$

2. Since $F'(x) = 6x$, we use $F(x) = 3x^2$. By the Fundamental Theorem, we have

$$\int_0^4 6xdx = 3x^2\Big|_0^4 = 3 \cdot 4^2 - 3 \cdot 0^2 = 48 - 0 = 48.$$

3. Since $F'(x) = 2x + 3$, we use $F(x) = x^2 + 3x$. By the Fundamental Theorem, we have

$$\int_1^2 (2x+3)dx = (x^2 + 3x)\Big|_1^2 = (2^2 + 3 \cdot 2) - (1^2 + 3 \cdot 1) = 10 - 4 = 6.$$

4. If $f(t) = 3t^2 + 4t + 3$, then $F(t) = t^3 + 2t^2 + 3t$. By the Fundamental Theorem, we have

$$\int_0^2 (3t^2 + 4t + 3)\, dt = (t^3 + 2t^2 + 3t)\Big|_0^2 = 2^3 + 2(2^2) + 3(2) - 0 = 22.$$

5. Since $F'(t) = 1/t^2 = t^{-2}$, we take $F(t) = \dfrac{t^{-1}}{-1} = -1/t$. Then

$$\begin{aligned}\int_1^2 \frac{1}{t^2}\, dt &= F(2) - F(1)\\ &= -\frac{1}{2} - \left(-\frac{1}{1}\right)\\ &= \frac{1}{2}.\end{aligned}$$

6. Since $F'(x) = \dfrac{1}{\sqrt{x}} = x^{-1/2}$, we use $F(x) = 2x^{1/2} = 2\sqrt{x}$. By the Fundamental Theorem, we have

$$\int_1^4 \frac{1}{\sqrt{x}}dx = 2\sqrt{x}\Big|_1^4 = 2\sqrt{4} - 2\sqrt{1} = 4 - 2 = 2.$$

7. Since $F'(x) = 3x^2$, we take $F(x) = x^3$. Then

$$\begin{aligned}\int_0^5 3x^2\, dx &= F(5) - F(0)\\ &= 5^3 - 0^3\\ &= 125.\end{aligned}$$

8. If $F'(t) = t^3$, then $F(t) = \dfrac{t^4}{4}$. By the Fundamental Theorem, we have

$$\int_0^3 t^3\, dt = F(3) - F(0) = \frac{t^4}{4}\Big|_0^3 = \frac{3^4}{4} - \frac{0}{4} = \frac{81}{4}.$$

9. If $F'(x) = 6x^2$, then $F(x) = 2x^3$. By the Fundamental Theorem, we have

$$\int_1^3 6x^2\, dx = 2x^3\Big|_1^3 = 2(27) - 2(1) = 54 - 2 = 52.$$

10. Since $F'(t) = 5t^3$, we take $F(t) = \frac{5}{4}t^4$. Then

$$\begin{aligned}\int_1^2 5t^3\,dt &= F(2) - F(1)\\ &= \frac{5}{4}(2^4) - \frac{5}{4}(1^4)\\ &= \frac{5}{4}\cdot 16 - \frac{5}{4}\\ &= \frac{75}{4}\end{aligned}$$

11. Since $F'(x) = \sqrt{x}$, we take $F(x) = \dfrac{x^{3/2}}{3/2} = \dfrac{2}{3}x^{3/2}$. Then

$$\begin{aligned}\int_4^9 \sqrt{x}\,dx &= F(9) - F(4)\\ &= \frac{2}{3}\cdot 9^{3/2} - \frac{2}{3}\cdot 4^{3/2}\\ &= \frac{2}{3}\cdot 27 - \frac{2}{3}\cdot 8\\ &= \frac{38}{3}.\end{aligned}$$

12. Since $F'(y) = y^2 + y^4$, we take $F(y) = \dfrac{y^3}{3} + \dfrac{y^5}{5}$. Then

$$\begin{aligned}\int_0^1 (y^2 + y^4)\,dy &= F(3) - F(0)\\ &= \left(\frac{1^3}{3} + \frac{1^5}{5}\right) - \left(\frac{0^3}{3} + \frac{0^5}{5}\right)\\ &= \frac{1}{3} + \frac{1}{5} = \frac{8}{15}.\end{aligned}$$

13. Since $F'(t) = 1/(2t)$, we take $F(t) = \frac{1}{2}\ln|t|$. Then

$$\begin{aligned}\int_1^2 \frac{1}{2t}\,dt &= F(2) - F(1)\\ &= \frac{1}{2}\ln|2| - \frac{1}{2}\ln|1|\\ &= \frac{1}{2}\ln 2.\end{aligned}$$

14. $\displaystyle\int_2^5 (x^3 - \pi x^2)\,dx = \left(\frac{x^4}{4} - \frac{\pi x^3}{3}\right)\Bigg|_2^5 = \frac{609}{4} - 39\pi \approx 29.728.$

15. If $f(t) = e^{-0.2t}$, then $F(t) = -5e^{-0.2t}$. (This can be verified by observing that $\dfrac{d}{dt}(-5e^{-0.2t}) = e^{-0.2t}$.) By the Fundamental Theorem, we have

$$\int_0^1 e^{-0.2t}\,dt = (-5e^{-0.2t})\Big|_0^1 = -5(e^{-0.2}) - (-5)(1) = 5 - 5e^{-0.2} \approx 0.906.$$

16. $\displaystyle\int_0^1 2e^x\,dx = 2e^x\Big|_0^1 = 2e - 2 \approx 3.437.$

17. If $F'(t) = \cos t$, we can take $F(t) = \sin t$, so

$$\int_{-1}^{1} \cos t\, dt = \sin t \Big|_{-1}^{1} = \sin 1 - \sin(-1).$$

Since $\sin(-1) = -\sin 1$, we can simplify the answer and write

$$\int_{-1}^{1} \cos t\, dt = 2 \sin 1$$

18. $\displaystyle\int_0^{\pi/4} (\sin t + \cos t)\, dt = (-\cos t + \sin t)\Big|_0^{\pi/4} = \left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (-1 + 0) = 1.$

19. If $f(t) = e^{0.05t}$, then $F(t) = 20e^{0.05t}$ (you can check this by observing that $\frac{d}{dt}(20e^{0.05t}) = e^{0.05t}$). By the Fundamental Theorem, we have

$$\int e^{0.05t}\, dt = 20e^{0.05t}\Big|_0^3 = 20e^{0.15} - 20e^0 = 20(e^{0.15} - 1).$$

20. If $f(q) = 6q^2 + 4$, then $F(q) = 2q^3 + 4q$. By the Fundamental Theorem, we have

$$\int_0^1 (6q^2 + 4)\, dq = (2q^3 + 4q)\Big|_0^1 = 2(1) + 4(1) - (0 + 0) = 6.$$

21. **(a)** We substitute $w = 1 + x^2$, $dw = 2x\, dx$.

$$\int_{x=0}^{x=1} \frac{x}{1+x^2}\, dx = \frac{1}{2}\int_{w=1}^{w=2} \frac{1}{w}\, dw = \frac{1}{2}\ln|w|\Big|_1^2 = \frac{1}{2}\ln 2.$$

(b) We substitute $w = \cos x$, $dw = -\sin x\, dx$.

$$\int_{x=0}^{x=\frac{\pi}{4}} \frac{\sin x}{\cos x}\, dx = -\int_{w=1}^{w=\sqrt{2}/2} \frac{1}{w}\, dw$$

$$= -\ln|w|\Big|_1^{\sqrt{2}/2} = -\ln\frac{\sqrt{2}}{2} = \frac{1}{2}\ln 2.$$

22. We substitute $w = x^2 + 1$, so $dw = 2xdx$.

$$\int x(x^2+1)^2 dx = \frac{1}{2}\int w^2 dw = \frac{1}{2}\frac{w^3}{3} + C = \frac{1}{6}(x^2+1)^3 + C.$$

Using the Fundamental Theorem, we have

$$\int_0^2 x(x^2+1)^2 dx = \frac{1}{6}(x^2+1)^3\Big|_0^2 = \frac{1}{6}\cdot 5^3 - \frac{1}{6}\cdot 1^3 = \frac{125}{6} - \frac{1}{6} = \frac{124}{6} = \frac{62}{3}.$$

23. Let $w = x^2 + 1$, then $dw = 2xdx$. When $x = 0$, $w = 1$ and when $x = 3$, $w = 10$. Thus we have

$$\int_0^3 \frac{2x}{x^2+1}\, dx = \int_1^{10} \frac{dw}{w} = \ln|w|\Big|_1^{10} = \ln 10 - \ln 1 = \ln 10.$$

24. Let $w=-t^2$, then $dw=-2tdt$ so $tdt=-\frac{1}{2}dw$. When $t=0, w=0$ and when $t=1, w=-1$. Thus we have

$$\int_0^1 2te^{-t^2}\,dt=\int_0^{-1} 2e^w\left(-\frac{1}{2}\,dw\right)=-\int_0^{-1} e^w\,dw$$
$$=-e^w\Big|_0^{-1}=-e^{-1}-(-e^0)=1-e^{-1}.$$

25. We substitute $w=t+1$, so $dw=dt$.

$$\int\frac{1}{\sqrt{t+1}}dt=\int\frac{1}{\sqrt{w}}dw=\int w^{-1/2}dw=2w^{1/2}+C=2\sqrt{t+1}+C.$$

Using the Fundamental Theorem, we have

$$\int_0^3\frac{1}{\sqrt{t+1}}dt=2\sqrt{t+1}\Big|_0^3=2\sqrt{4}-2\sqrt{1}=4-2=2.$$

26. We have

$$\text{Area}=\int_1^4 x^2\,dx=\frac{x^3}{3}\Big|_1^4=\frac{4^3}{3}-\frac{1^3}{3}=\frac{64-1}{3}=21.$$

27. The integral which represents the area under this curve is

$$\text{Area}=\int_0^2(6x^2+1)\,dx.$$

Since $\dfrac{d}{dx}(2x^3+x)=6x^2+1$, we can evaluate the definite integral:

$$\int_0^2(6x^2+1)\,dx=(2x^3+x)\Big|_0^2=2(2^3)+2-(2(0)+0)=16+2=18.$$

28. We have

$$\text{Average value}=\frac{1}{10-0}\int_0^{10}(x^2+1)dx=\frac{1}{10}\left(\frac{x^3}{3}+x\right)\Big|_0^{10}=\frac{1}{10}\left(\frac{10^3}{3}+10-0\right)=\frac{103}{3}.$$

We see in Figure 7.1 that the average value of $103/3\approx 34.33$ for $f(x)$ looks right.

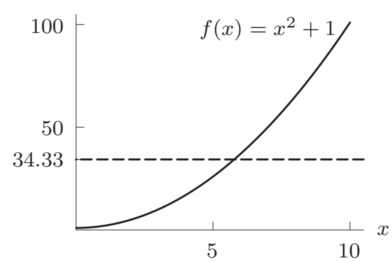

Figure 7.1

29. One antiderivative of $f(x)=e^{0.5x}$ is $F(x)=2e^{0.5x}$. Thus, the definite integral of $f(x)$ on the interval $0\le x\le 3$ is

$$\int_0^3 e^{0.5x}\,dx=F(3)-F(0)=2e^{0.5x}\Big|_0^3.$$

The average value of a function on a given interval is the definite integral over that interval divided by the length of the interval:

$$\text{Average value} = \left(\frac{1}{3-0}\right) \cdot \left(\int_0^3 e^{0.5x}\,dx\right) = \frac{1}{3}\left(2e^{0.5x}\Big|_0^3\right) = \frac{1}{3}(2e^{1.5} - 2e^0) \approx 2.32.$$

From the graph of $y = e^{0.5x}$ in Figure 7.2 we see that an average value of 2.32 on the interval $0 \le x \le 3$ does make sense.

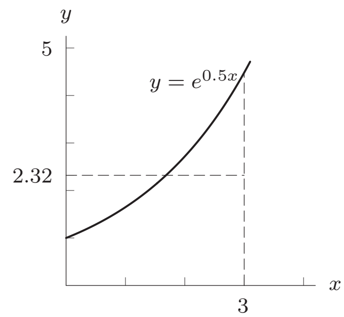

Figure 7.2

30. Since $y = x^3 - x = x(x-1)(x+1)$, the graph crosses the axis at the three points shown in Figure 7.3. The two regions have the same area (by symmetry). Since the graph is below the axis for $0 < x < 1$, we have

$$\begin{aligned} \text{Area } &= 2\left(-\int_0^1 \left(x^3 - x\right)\,dx\right) \\ &= -2\left[\frac{x^4}{4} - \frac{x^2}{2}\right]_0^1 = -2\left(\frac{1}{4} - \frac{1}{2}\right) = \frac{1}{2}. \end{aligned}$$

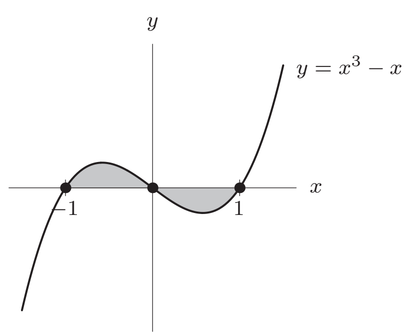

Figure 7.3

31. We have

$$\text{Area } = \int_1^b 4x\,dx = 2x^2\Big|_1^b = 2b^2 - 2.$$

We find the value of b making the area equal to 240:

$$\begin{aligned} 240 &= 2b^2 - 2 \\ 242 &= 2b^2 \\ 121 &= b^2 \\ b &= 11. \end{aligned}$$

32. The area under $f(x) = 8x$ between $x = 1$ and $x = b$ is given by $\int_1^b (8x)dx$. Using the Fundamental Theorem to evaluate the integral:

$$\text{Area } = 4x^2\Big|_1^b = 4b^2 - 4.$$

Since the area is 192, we have

$$\begin{aligned} 4b^2 - 4 &= 192 \\ 4b^2 &= 196 \\ b^2 &= 49 \\ b &= \pm 7. \end{aligned}$$

Since b is larger than 1, we have $b = 7$.

33. We have

$$\text{Area } = \int_0^b x^2 dx = \frac{x^3}{3}\Big|_0^b = \frac{b^3}{3}.$$

We find the value of b making the area equal to 100:

$$\begin{aligned} 100 &= \frac{b^3}{3} \\ 300 &= b^3 \\ b &= (300)^{1/3} = 6.694. \end{aligned}$$

34. **(a)** At time $t = 0$, the rate of oil leakage $= r(0) = 50$ thousand liters/minute.
At $t = 60$, rate $= r(60) = 15.06$ thousand liters/minute.
(b) To find the amount of oil leaked during the first hour, we integrate the rate from $t = 0$ to $t = 60$:

$$\begin{aligned} \text{Oil leaked} &= \int_0^{60} 50e^{-0.02t}\, dt = \left(-\frac{50}{0.02}e^{-0.02t}\right)\Big|_0^{60} \\ &= -2500e^{-1.2} + 2500e^0 = 1747 \text{ thousand liters.} \end{aligned}$$

35. **(a)** In 2010, we have $P = 6.1e^{0.012\cdot 10} = 6.9$ billion people.
In 2020, we have $P = 6.1e^{0.012\cdot 20} = 7.8$ billion people.
(b) We have

$$\begin{aligned} \text{Average population} &= \frac{1}{10-0}\int_0^{10} 6.1e^{0.012t}\, dt = \frac{1}{10}\cdot\frac{6.1}{0.012}e^{0.012t}\Big|_0^{10} \\ &= \frac{1}{10}\left(\frac{6.1}{0.012}(e^{0.12} - e^0)\right) = 6.5. \end{aligned}$$

The average population of the world between 2000 and 2010 is predicted to be 6.5 billion people.

36. **(a)** The graph of $y = e^{-x^2}$ is in Figure 7.4. The integral $\int_{-\infty}^{\infty} e^{-x^2}\, dx$ represents the entire area under the curve, which is shaded.

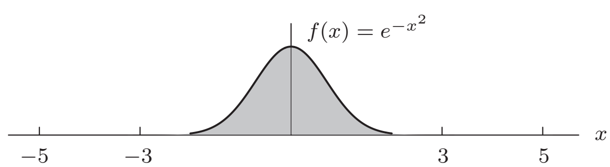

Figure 7.4

(b) Using a calculator or computer, we see that

$$\int_{-1}^{1} e^{-x^2}\, dx = 1.494, \quad \int_{-2}^{2} e^{-x^2}\, dx = 1.764, \quad \int_{-3}^{3} e^{-x^2}\, dx = 1.772, \quad \int_{-5}^{5} e^{-x^2}\, dx = 1.772$$

(c) From part (b), we see that as we extend the limits of integration, the area appears to get closer and closer to about 1.772. We estimate that

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = 1.772$$

.

37. Figure 7.5 shows the graphs of $y = 1/x^2$ and $y = 1/x^3$. We see that $\int_1^\infty \frac{1}{x^2}\,dx$ is larger, since the area under $1/x^2$ is larger than the area under $1/x^3$.

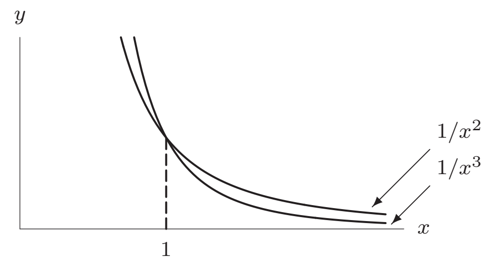

Figure 7.5

38. **(a)** An antiderivative of $F'(x) = \dfrac{1}{x^2}$ is $F(x) = -\dfrac{1}{x}$ $\left(\text{since}\dfrac{d}{dx}\left(\dfrac{-1}{x}\right) = \dfrac{1}{x^2}\right)$. So by the Fundamental Theorem we have:

$$\int_1^b \frac{1}{x^2}\,dx = -\frac{1}{x}\bigg|_1^b = -\frac{1}{b} + 1.$$

.

(b) Taking a limit, we have

$$\lim_{b\to\infty}\left(-\frac{1}{b} + 1\right) = 0 + 1 = 1.$$

Since the limit is 1, we know that

$$\lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,dx = 1.$$

So the improper integral converges to 1:

$$\int_1^\infty \frac{1}{x^2}\,dx = 1.$$

39. **(a)** Using a calculator or computer, we get

$$\int_0^3 e^{-2t}\,dt = 0.4988 \qquad \int_0^5 e^{-2t}\,dt = 0.49998$$
$$\int_1^7 e^{-2t}\,dt = 0.4999996 \qquad \int_0^{10} e^{-2t}\,dt = 0.499999999.$$

The values of these integrals are getting closer to 0.5. A reasonable guess is that the improper integral converges to 0.5.

(b) Since $-\frac{1}{2}e^{-2t}$ is an antiderivative of e^{-2t}, we have

$$\int_0^b e^{-2t}\,dt = -\frac{1}{2}e^{-2t}\bigg|_0^b = -\frac{1}{2}e^{-2b} - \left(-\frac{1}{2}e^0\right) = -\frac{1}{2}e^{-2b} + \frac{1}{2}.$$

(c) Since $e^{-2b} = 1/e^{2b}$, we have

$$e^{2b} \to \infty \qquad \text{as} \qquad b \to \infty, \qquad \text{so} \qquad e^{-2b} = \frac{1}{e^{2b}} \to 0.$$

Therefore,

$$\lim_{b\to\infty}\int_0^b e^{-2t}\,dt = \lim_{b\to\infty}\left(-\frac{1}{2}e^{-2b} + \frac{1}{2}\right) = 0 + \frac{1}{2} = \frac{1}{2}.$$

So the improper integral converges to $1/2 = 0.5$:

$$\int_0^\infty e^{-2t}\,dt = \frac{1}{2}.$$

40. **(a)** Evaluating the integrals with a calculator gives

$$\int_0^{10} xe^{-x/10}\,dx = 26.42$$
$$\int_0^{50} xe^{-x/10}\,dx = 95.96$$
$$\int_0^{100} xe^{-x/10}\,dx = 99.95$$
$$\int_0^{200} xe^{-x/10}\,dx = 100.00$$

(b) The results of part (a) suggest that

$$\int_0^{\infty} xe^{-x/10}\,dx \approx 100$$

41. **(a)** The total number of people that get sick is the integral of the rate. The epidemic starts at $t = 0$. Since the rate is positive for all t, we use ∞ for the upper limit of integration.

$$\text{Total number getting sick} = \int_0^{\infty} \left(1000te^{-0.5t}\right)\,dt$$

(b) The graph of $r = 1000te^{-0.5t}$ is shown in Figure 7.6. The shaded area represents the total number of people who get sick.

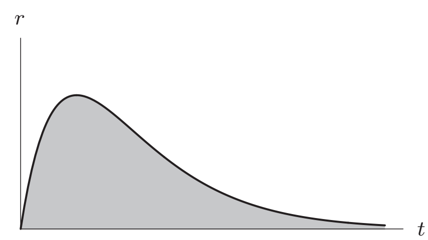

Figure 7.6

42. Since $y = x^3(1-x)$ is positive for $0 \le x \le 1$ and $y = 0$, when $x = 0, 1$, the area is given by

$$\text{Area } = \int_0^1 x^3(1-x)\,dx = \int_0^1 (x^3 - x^4)\,dx = \left.\frac{x^4}{4} - \frac{x^5}{5}\right|_0^1 = \frac{1}{20}.$$

43. Since $y = 0$ only when $x = 0$ and $x = 1$, the area lies between these limits and is given by

$$\begin{aligned}\text{Area } &= \int_0^1 x^2(1-x)^2 dx = \int_0^1 x^2(1-2x+x^2)\,dx = \int_0^1 (x^2 - 2x^3 + x^4)\,dx \\ &= \left.\frac{x^3}{3} - \frac{2}{4}x^4 + \frac{x^5}{5}\right|_0^1 = \frac{1}{30}.\end{aligned}$$

44. **(a)** Since $v(t) = 60/50^t$ is never 0, the car never stops.
(b) For time $t \ge 0$,

$$\text{Distance traveled} = \int_0^{\infty} \frac{60}{50^t}\,dt.$$

(c) Evaluating $\int_0^b \frac{60}{50^t}\,dt$ for $b = 1, 5, 10$ gives

$$\int_0^1 \frac{60}{50^t}\,dt = 15.0306 \qquad \int_0^5 \frac{60}{50^t}\,dt = 15.3373 \qquad \int_0^{10} \frac{60}{50^t}\,dt = 15.3373,$$

so the integral appears to converge to 15.3373; so we estimate the distance traveled to be 15.34 miles.

45. **(a)** No, it is not reached since

$$\text{Total number of rabbits} = \int_1^\infty \frac{1}{t^2}\,dt = 1.$$

Thus, the total number of rabbits is 1000.

(b) Yes, since $\displaystyle\int_1^\infty t\,dt$ does not converge to a finite value, which means that infinitely many rabbits could be produced, and therefore 1 million is certainly reached.

(c) Yes, since $\displaystyle\int_1^\infty \frac{1}{\sqrt{t}}\,dt$ does not converge to a finite value.

46. **(a)** In the first case, we are given that $R_0 = 1000$ widgets/year. So we have $R = 1000e^{0.15t}$. To determine the total number sold, we need to integrate this rate over the time period from 0 to 10. Therefore

$$\text{Total number of widgets sold } = \int_0^{10} 1000e^{0.15t}\,dt = 23{,}211 \text{ widgets}.$$

In the second case,

$$\text{Total number of widgets sold } = \int_0^{10} 150{,}000{,}000e^{0.15t}\,dt = 3.5 \text{ billionwidgets}.$$

(b) We want to determine T such that

$$\int_0^T 1000e^{0.15t}\,dt = \frac{23{,}211}{2}.$$

Trying a few values of T, we get

$$T \approx 6.7 \text{ years}.$$

Similarly, in the second case, we want T so that

$$\int_0^T 150{,}000{,}000e^{0.15t}\,dt = \frac{3,500,000,000}{2}$$

we get

$$T \approx 6.7 \text{ years}.$$

So the half way mark is reached at the same time regardless of the initial rate.

(c) Since half the widgets are sold in the last $3\frac{1}{2}$ years of the decade, if each widget is expected to last at least 3.5 years, their claim could easily be true.

Solutions for Section 7.4

1. Let $u = t$ and $v' = e^{5t}$, so $u' = 1$ and $v = \frac{1}{5}e^{5t}$.
Then $\int te^{5t}\,dt = \frac{1}{5}te^{5t} - \int \frac{1}{5}e^{5t}\,dt = \frac{1}{5}te^{5t} - \frac{1}{25}e^{5t} + C$.

2. Let $u = p$ and $v' = e^{(-0.1)p}$, $u' = 1$. Thus, $v = \int e^{(-0.1)p}\,dp = -10e^{(-0.1)p}$. With this choice of u and v, integration by parts gives:

$$\begin{aligned}\int pe^{(-0.1)p}\,dp &= p(-10e^{(-0.1)p}) - \int (-10e^{(-0.1)p})\,dp \\ &= -10pe^{(-0.1)p} + 10\int e^{(-0.1)p}\,dp \\ &= -10pe^{(-0.1)p} - 100e^{(-0.1)p} + C.\end{aligned}$$

3. Let $u = z + 1$, $v' = e^{2z}$. Thus, $v = \frac{1}{2}e^{2z}$ and $u' = 1$. Integrating by parts, we get:

$$\begin{aligned}\int (z+1)e^{2z}\,dz &= (z+1)\cdot\frac{1}{2}e^{2z} - \int \frac{1}{2}e^{2z}\,dz \\ &= \frac{1}{2}(z+1)e^{2z} - \frac{1}{4}e^{2z} + C \\ &= \frac{1}{4}(2z+1)e^{2z} + C.\end{aligned}$$

4. Let $u = \ln y$, $v' = y$. Then, $v = \frac{1}{2}y^2$ and $u' = \dfrac{1}{y}$. Integrating by parts, we get:

$$\begin{aligned}\int y\ln y\,dy &= \frac{1}{2}y^2\ln y - \int \frac{1}{2}y^2\cdot\frac{1}{y}\,dy \\ &= \frac{1}{2}y^2\ln y - \frac{1}{2}\int y\,dy \\ &= \frac{1}{2}y^2\ln y - \frac{1}{4}y^2 + C.\end{aligned}$$

5. Let $u = \ln x$ and $v' = x^3$, so $u' = \frac{1}{x}$ and $v = \frac{x^4}{4}$. Then

$$\int x^3\ln x\,dx = \frac{x^4}{4}\ln x - \int \frac{x^3}{4}\,dx = \frac{x^4}{4}\ln x - \frac{x^4}{16} + C.$$

6. Let $u = \ln 5q$, $v' = q^5$. Then $v = \frac{1}{6}q^6$ and $u' = \dfrac{1}{q}$. Integrating by parts, we get:

$$\begin{aligned}\int q^5\ln 5q\,dq &= \frac{1}{6}q^6\ln 5q - \int (5\cdot\frac{1}{5q})\cdot\frac{1}{6}q^6\,dq \\ &= \frac{1}{6}q^6\ln 5q - \frac{1}{36}q^6 + C.\end{aligned}$$

7. Let $u = y$ and $v' = (y+3)^{1/2}$, so $u' = 1$ and $v = \frac{2}{3}(y+3)^{3/2}$:

$$\int y\sqrt{y+3}\,dy = \frac{2}{3}y(y+3)^{3/2} - \int \frac{2}{3}(y+3)^{3/2}\,dy = \frac{2}{3}y(y+3)^{3/2} - \frac{4}{15}(y+3)^{5/2} + C.$$

8. Let $u = t + 2$ and $v' = \sqrt{2+3t}$, so $u' = 1$ and $v = \frac{2}{9}(2+3t)^{3/2}$. Then

$$\begin{aligned}\int (t+2)\sqrt{2+3t}\,dt &= \frac{2}{9}(t+2)(2+3t)^{3/2} - \frac{2}{9}\int (2+3t)^{3/2}\,dt \\ &= \frac{2}{9}(t+2)(2+3t)^{3/2} - \frac{4}{135}(2+3t)^{5/2} + C.\end{aligned}$$

9. Let $u = z$, $v' = e^{-z}$. Thus $v = -e^{-z}$ and $u' = 1$. Integration by parts gives:

$$\begin{aligned}\int ze^{-z}\,dz &= -ze^{-z} - \int (-e^{-z})\,dz \\ &= -ze^{-z} - e^{-z} + C \\ &= -(z+1)e^{-z} + C.\end{aligned}$$

10. Let $u = \ln x$, $v' = x^{-2}$. Then $v = -x^{-1}$ and $u' = x^{-1}$. Integrating by parts, we get:

$$\begin{aligned}\int x^{-2}\ln x\,dx &= -x^{-1}\ln x - \int(-x^{-1})\cdot x^{-1}\,dx\\ &= -x^{-1}\ln x - x^{-1} + C.\end{aligned}$$

11. Let $u = y$ and $v' = \frac{1}{\sqrt{5-y}}$, so $u' = 1$ and $v = -2(5-y)^{1/2}$.

$$\int \frac{y}{\sqrt{5-y}}\,dy = -2y(5-y)^{1/2} + 2\int(5-y)^{1/2}\,dy = -2y(5-y)^{1/2} - \frac{4}{3}(5-y)^{3/2} + C.$$

12. $\displaystyle\int \frac{t+7}{\sqrt{5-t}}\,dt = \int \frac{t}{\sqrt{5-t}}\,dt + 7\int(5-t)^{-1/2}\,dt.$

To calculate the first integral, we use integration by parts. Let $u = t$ and $v' = \frac{1}{\sqrt{5-t}}$, so $u' = 1$ and $v = -2(5-t)^{1/2}$. Then

$$\int \frac{t}{\sqrt{5-t}}\,dt = -2t(5-t)^{1/2} + 2\int(5-t)^{1/2}\,dt = -2t(5-t)^{1/2} - \frac{4}{3}(5-t)^{3/2} + C.$$

We can calculate the second integral directly: $7\displaystyle\int(5-t)^{-1/2} = -14(5-t)^{1/2} + C_1$. Thus

$$\int \frac{t+7}{\sqrt{5-t}}\,dt = -2t(5-t)^{1/2} - \frac{4}{3}(5-t)^{3/2} - 14(5-t)^{1/2} + C_2.$$

13. Let $u = t$, $v' = \sin t$. Thus, $v = -\cos t$ and $u' = 1$. With this choice of u and v, integration by parts gives:

$$\begin{aligned}\int t\sin t\,dt &= -t\cos t - \int(-\cos t)\,dt\\ &= -t\cos t + \sin t + C.\end{aligned}$$

14. $\displaystyle\int_3^5 x\cos x\,dx = (\cos x + x\sin x)\Big|_3^5 = \cos 5 + 5\sin 5 - \cos 3 - 3\sin 3 \approx -3.944.$

15. Let $u = t^2$ and $v' = e^{5t}$, so $u' = 2t$ and $v = \frac{1}{5}e^{5t}$.
Then $\int t^2e^{5t}\,dt = \frac{1}{5}t^2e^{5t} - \frac{2}{5}\int te^{5t}\,dt$.
Using Problem 1, we have $\int t^2e^{5t}\,dt = \frac{1}{5}t^2e^{5t} - \frac{2}{5}(\frac{1}{5}te^{5t} - \frac{1}{25}e^{5t}) + C$
$= \frac{1}{5}t^2e^{5t} - \frac{2}{25}te^{5t} + \frac{2}{125}e^{5t} + C.$

16. Let $u = (\ln t)^2$ and $v' = 1$, so $u' = \dfrac{2\ln t}{t}$ and $v = t$. Then

$$\int(\ln t)^2\,dt = t(\ln t)^2 - 2\int\ln t\,dt = t(\ln t)^2 - 2t\ln t + 2t + C.$$

(We use the fact that $\displaystyle\int \ln x\,dx = x\ln x - x + C$, a result which can be derived using integration by parts.)

17. $\displaystyle\int_1^5 \ln t\,dt = (t\ln t - t)\Big|_1^5 = 5\ln 5 - 4 \approx 4.047$

18. We use integration by parts. Let $u = z$ and $v' = e^{-z}$, so $u' = 1$ and $v = -e^{-z}$. Then

$$\begin{aligned}\int_0^{10} ze^{-z}\,dz &= -ze^{-z}\Big|_0^{10} + \int_0^{10} e^{-z}\,dz\\ &= -10e^{-10} + (-e^{-z})\Big|_0^{10}\\ &= -11e^{-10} + 1\\ &\approx 0.9995.\end{aligned}$$

19. $\displaystyle\int_1^3 t\ln t\,dt = \left(\frac{1}{2}t^2\ln t - \frac{1}{4}t^2\right)\Bigg|_1^3 = \frac{9}{2}\ln 3 - 2 \approx 2.944.$

20. $\displaystyle\int_0^5 \ln(1+t)\,dt = ((1+t)\ln(1+t) - (1+t))\Bigg|_0^5 = 6\ln 6 - 5 \approx 5.751.$

21. **(a)** We evaluate this integral using the substitution $w = 1 + x^3$.
(b) We evaluate this integral using the substitution $w = x^2$.
(c) We evaluate this integral using the substitution $w = x^3 + 1$.
(d) We evaluate this integral using the substitution $w = 3x + 1$.
(e) This integral can be evaluated using integration by parts with $u = \ln x$, $v' = x^2$.
(f) This integral can be evaluated using integration by parts with $u = \ln x$, $v' = 1$.

22. A calculator gives $\int_1^2 \ln x\,dx = 0.386$. An antiderivative of $\ln x$ is $x\ln x - 1$, so the Fundamental Theorem of Calculus gives

$$\int_1^2 \ln x\,dx = (x\ln x - x)\Big|_1^2 = 2\ln 2 - 1.$$

Since $2\ln 2 - 1 = 0.386$, the value from the Fundamental Theorem agrees with the numerical answer.

23. Using integration by parts with $u' = e^{-t}$, $v = t$, so $u = -e^{-t}$ and $v' = 1$, we have

$$\begin{aligned}\text{Area } &= \int_0^2 te^{-t}\,dt = -te^{-t}\Big|_0^2 - \int_0^2 -1\cdot e^{-t}\,dt \\ &= (-te^{-t} - e^{-t})\Big|_0^2 = -2e^{-2} - e^{-2} + 1 = 1 - 3e^{-2}.\end{aligned}$$

24. Since $\ln(x^2) = 2\ln x$ and $\int \ln x\,dx = x\ln x - x + C$, we have

$$\begin{aligned}\text{Area } &= \int_1^2 (\ln(x^2) - \ln x)\,dx = \int_1^2 (2\ln x - \ln x)\,dx \\ &= \int_1^2 \ln x\,dx = (x\ln x - x)\Big|_1^2 = 2\ln 2 - 2 - (1\ln 1 - 1) = 2\ln 2 - 1.\end{aligned}$$

25. Since the graph of $f(t) = \ln(t^2 - 1)$ is above the graph of $g(t) = \ln(t-1)$ for $t > 1$, we have

$$\text{Area } = \int_2^3 (\ln(t^2 - 1) - \ln(t-1))\,dt = \int_2^3 \ln\left(\frac{t^2-1}{t-1}\right)\,dt = \int_2^3 \ln(t+1)\,dt.$$

We can cancel the factor of $(t-1)$ in the last step above because the integral is over $2 \le t \le 3$, where $(t-1)$ is not zero.

We use $\int \ln x\,dx = x\ln x - 1$ with the substitution $x = t + 1$. The limits $t = 2$, $t = 3$ become $x = 3, x = 4$, respectively. Thus

$$\begin{aligned}\text{Area } &= \int_2^3 \ln(t+1)\,dt = \int_3^4 \ln x\,dx = (x\ln x - x)\Big|_3^4 \\ &= 4\ln 4 - 4 - (3\ln 3 - 3) = 4\ln 4 - 3\ln 3 - 1.\end{aligned}$$

26. Since $f'(x) = 2x$, integration by parts tells us that

$$\begin{aligned}\int_0^{10} f(x)g'(x)\,dx &= f(x)g(x)\Big|_0^{10} - \int_0^{10} f'(x)g(x)\,dx \\ &= f(10)g(10) - f(0)g(0) - 2\int_0^{10} xg(x)\,dx.\end{aligned}$$

We can use left and right Riemann Sums with $\Delta x = 2$ to approximate $\int_0^{10} xg(x)\,dx$:

$$\begin{aligned}\text{Left sum} &\approx 0\cdot g(0)\Delta x + 2\cdot g(2)\Delta x + 4\cdot g(4)\Delta x + 6\cdot g(6)\Delta x + 8\cdot g(8)\Delta x\\ &= (0(2.3)+2(3.1)+4(4.1)+6(5.5)+8(5.9))\,2 = 205.6.\\ \text{Right sum} &\approx 2\cdot g(2)\Delta x + 4\cdot g(4)\Delta x + 6\cdot g(6)\Delta x + 8\cdot g(8)\Delta x + 10\cdot g(10)\Delta x\\ &= (2(3.1)+4(4.1)+6(5.5)+8(5.9)+10(6.1))\,2 = 327.6.\end{aligned}$$

A good estimate for the integral is the average of the left and right sums, so

$$\int_0^{10} xg(x)\,dx \approx \frac{205.6+327.6}{2} = 266.6.$$

Substituting values for f and g, we have

$$\begin{aligned}\int_0^{10} f(x)g'(x)\,dx &= f(10)g(10) - f(0)g(0) - 2\int_0^{10} xg(x)\,dx\\ &\approx 10^2(6.1) - 0^2(2.3) - 2(266.6) = 76.8 \approx 77.\end{aligned}$$

27. We have

$$\text{Bioavailability} = \int_0^3 15te^{-0.2t}dt.$$

We first use integration by parts to evaluate the indefinite integral of this function. Let $u = 15t$ and $v' = e^{-0.2t}dt$, so $u' = 15dt$ and $v = -5e^{-0.2t}$. Then,

$$\begin{aligned}\int 15te^{-0.2t}dt &= (15t)(-5e^{-0.2t}) - \int(-5e^{-0.2t})(15dt)\\ &= -75te^{-0.2t} + 75\int e^{-0.2t}dt = -75te^{-0.2} - 375e^{-0.2t} + C.\end{aligned}$$

Thus,

$$\int_0^3 15te^{-0.2t}dt = (-75te^{-0.2t} - 375e^{-0.2t})\Big|_0^3 = -329.29 + 375 = 45.71.$$

The bioavailability of the drug over this time interval is 45.71 (ng/ml)-hours.

28. **(a)** We know that $\dfrac{dE}{dt} = r$, so the total energy E used in the first T hours is given by $E = \displaystyle\int_0^T te^{-at}\,dt$. We use integration by parts. Let $u = t$, $v' = e^{-at}$. Then $u' = 1$, $v = -\frac{1}{a}e^{-at}$.

$$\begin{aligned}E &= \int_0^T te^{-at}\,dt\\ &= -\frac{t}{a}e^{-at}\Big|_0^T - \int_0^T\left(-\frac{1}{a}e^{-at}\right)dt\\ &= -\frac{1}{a}Te^{-aT} + \frac{1}{a}\int_0^T e^{-at}\,dt\\ &= -\frac{1}{a}Te^{-aT} + \frac{1}{a^2}(1-e^{-aT}).\end{aligned}$$

(b)

$$\lim_{T\to\infty} E = -\frac{1}{a}\lim_{T\to\infty}\left(\frac{T}{e^{aT}}\right) + \frac{1}{a^2}\left(1 - \lim_{T\to\infty}\frac{1}{e^{aT}}\right).$$

Since $a > 0$, the second limit on the right hand side in the above expression is 0. In the first limit, although both the numerator and the denominator go to infinity, the denominator e^{aT} goes to infinity more quickly than T does. So in the end the denominator e^{aT} is much greater than the numerator T. Hence $\displaystyle\lim_{T\to\infty}\frac{T}{e^{aT}} = 0$. (You can check this by graphing $y = \dfrac{T}{e^{aT}}$ on a calculator or computer for some values of a.) Thus $\displaystyle\lim_{T\to\infty} E = \frac{1}{a^2}$.

29. We integrate by parts. Since we know what the answer is supposed to be, it's easier to choose u and v'. Let $u = x^n$ and $v' = e^x$, so $u' = nx^{n-1}$ and $v = e^x$. Then

$$\int x^n e^x \, dx = x^n e^x - n \int x^{n-1} e^x \, dx.$$

Solutions for Section 7.5

1. By the Fundamental Theorem,

$$\begin{aligned}
F(1) &= F(0) + \int_0^1 F'(t)\, dt \\
&= 5 - 1.5 = 3.5 \\
F(2) &= F(1) + \int_1^2 F'(t)\, dt \\
&= 3.5 - 1.5 = 2 \\
F(3) &= F(2) + \int_2^3 F'(t)\, dt \\
&= 2 - 0.5 = 1.5 \\
F(4) &= F(3) + \int_3^4 F'(t)\, dt \\
&= 1.5 + 0.5 = 2 \\
F(5) &= F(4) + \int_4^5 F'(t)\, dt \\
&= 2 + 0.5 = 2.5
\end{aligned}$$

Thus, our table is as follows:

Table 7.1

t	0	1	2	3	4	5
$F(t)$	5	3.5	2	1.5	2	2.5

2. First, we observe that
g is increasing when g' is positive, which is when $0 < x < 4$.
g is decreasing when g' is negative, which is when $4 < x < 6$.
Therefore, $x = 4$ is a local maximum. Table 7.2 shows the area between the curve and the x-axis for the intervals 0–1, 1–2, etc. It also shows the corresponding change in the value of g. These changes are used to compute the values of g using the Fundamental Theorem of Calculus:

$$g(1) - g(0) = \int_0^1 g'(x)\, dx = \frac{1}{2}.$$

Since $g(0) = 0$,

$$g(1) = \frac{1}{2}.$$

Similarly,

$$\begin{aligned}
g(2) - g(1) &= \int_1^2 g'(x)\, dx = 1 \\
g(2) &= g(1) + 1 = \frac{3}{2}.
\end{aligned}$$

Continuing in this way gives the values of g in Table 7.3.

Table 7.2

Interval	Area	Total change in $g = \int_a^b g'(x)dx$
0–1	1/2	$1/2$
1–2	1	1
2–3	1	1
3–4	1/2	$1/2$
4–5	1/2	$-1/2$
5–6	1/2	$-1/2$

Table 7.3

x	$g(x)$
0	0
1	1/2
2	3/2
3	5/2
4	3
5	5/2
6	2

Notice: the graph of g will be a straight line from 1 to 3 because g' is horizontal there. Furthermore, the tangent line will be horizontal at $x = 4$, $x = 0$ and $x = 6$. The maximum is at $(4, 3)$. See Figure 7.7.

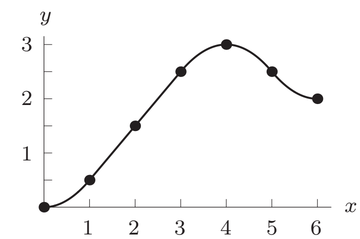

Figure 7.7

3. Since $F(0) = 0$, $F(b) = \int_0^b f(t)\,dt$. For each b we determine $F(b)$ graphically as follows:
$F(0) = 0$
$F(1) = F(0) +$ Area of 1×1 rectangle $= 0 + 1 = 1$
$F(2) = F(1) +$ Area of triangle $(\frac{1}{2} \cdot 1 \cdot 1) = 1 + 0.5 = 1.5$
$F(3) = F(2) +$ Negative of area of triangle $= 1.5 - 0.5 = 1$
$F(4) = F(3) +$ Negative of area of rectangle $= 1 - 1 = 0$
$F(5) = F(4) +$ Negative of area of rectangle $= 0 - 1 = -1$
$F(6) = F(5) +$ Negative of area of triangle $= -1 - 0.5 = -1.5$
The graph of $F(t)$, for $0 \le t \le 6$, is shown in Figure 7.8.

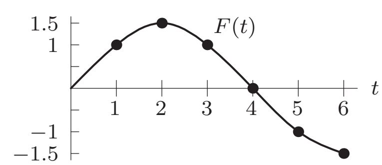

Figure 7.8

4. **(a)** The value of the integral is negative since the area below the x-axis is greater than the area above the x-axis. We count boxes: The area below the x-axis includes approximately 11.5 boxes and each box has area $(2)(1) = 2$, so

$$\int_0^5 f(x)dx \approx -23.$$

The area above the x-axis includes approximately 2 boxes, each of area 2, so

$$\int_5^7 f(x)dx \approx 4.$$

So we have

$$\int_0^7 f(x)dx = \int_0^5 f(x)dx + \int_5^7 f(x)dx \approx -23 + 4 = -19.$$

(b) By the Fundamental Theorem of Calculus, we have

$$F(7) - F(0) = \int_0^7 f(x)dx$$

so,

$$F(7) = F(0) + \int_0^7 f(x)dx = 25 + (-19) = 6.$$

5. See Figure 7.9.

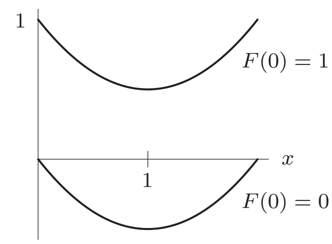

Figure 7.9

6. See Figure 7.10

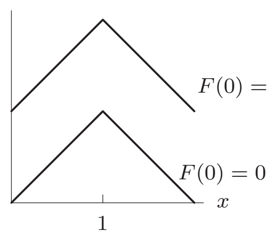

Figure 7.10

7. See Figure 7.11.

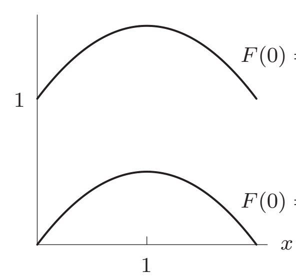

Figure 7.11

8. See Figure 7.12

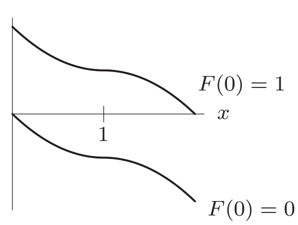

Figure 7.12

9. **(a)** The function $f(x)$ is increasing when $f'(x)$ is positive, so $f(x)$ is increasing for $x < -2$ or $x > 2$.
The function $f(x)$ is decreasing when $f'(x)$ is negative, so $f(x)$ is decreasing for $-2 < x < 2$.
Since $f(x)$ is increasing to the left of $x = -2$, decreasing between $x = -2$ and $x = 2$, and increasing to the right of $x = 2$, the function $f(x)$ has a local maximum at $x = -2$ and a local minimum at $x = 2$.

(b) See Figure 7.13.

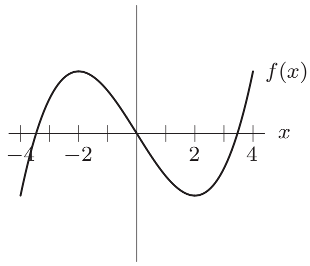

Figure 7.13

10. **(a)** The function $f(x)$ is increasing when $f'(x)$ is positive, so $f(x)$ is increasing for $-1 < x < 3$ or $x > 3$.
The function $f(x)$ is decreasing when $f'(x)$ is negative, so $f(x)$ is decreasing for $x < -1$.
Since $f(x)$ is decreasing to the left of $x = -1$ and increasing to the right of $x = -1$, the function has a local minimum at $x = -1$. Since $f(x)$ is increasing on both sides of $x = 3$, it has neither a local maximum nor a local minimum at that point.

(b) See Figure 7.14.

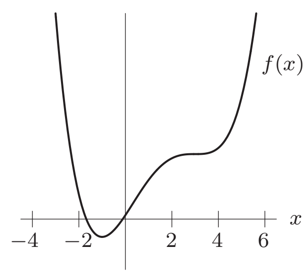

Figure 7.14

11. **(a)** $f(x)$ is increasing when $f'(x)$ is positive. $f'(x)$ is positive when $2 < x < 5$. So $f(x)$ is increasing when $2 < x < 5$.
$f(x)$ is decreasing when $f'(x)$ is negative. $f'(x)$ is negative when $x < 2$ or $x > 5$. So $f(x)$ is decreasing when $x < 2$ or $x > 5$.
A function has a local minimum at a point x when its derivative is zero at that point, and when it decreases immediately before x and increases immediately after x. $f'(2) = 0$, f decreases to the left of 2, and f increases immediately after 2, therefore $f(x)$ has a local minimum at $x = 2$.
A function has a local maximum at a point x when its derivative is zero at that point, and when it increases immediately before x and decreases immediately after x. $f'(5) = 0$, f increases before 5, and f decreases after 5. Therefore $f(x)$ has a local maximum at $x = 5$.

(b) Since we do not know any areas or vertical values, we can only sketch a rough graph. We start with the minimum and the maximum, then connect the graph between them. The graph could be more or less steep and further above or below the x-axis. See Figure 7.15.

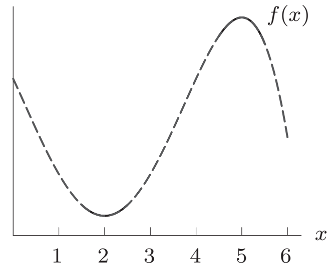

Figure 7.15

12. **(a)** The function f is increasing where f' is positive, so f is increasing for $x < -1$ or $x > 1$.
The function f is decreasing where f' is negative, so f is decreasing for $-1 < x < 1$.
The function f has critical points at $x = -1, 0, 1$. The point $x = -1$ is a local maximum (because f is increasing to the left of $x = -1$ and decreasing to the right of $x = -1$). The point $x = 1$ is a local minimum (because f decreases to the left of $x = 1$ and increases to the right). The point $x = 0$ is neither a local maximum nor a local minimum, since $f(x)$ is decreasing on both sides.
(b) See Figure 7.16.

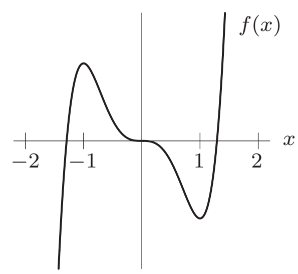

Figure 7.16

13. Since the rate at which the leaf grows is proportional to the rate of photosynthesis, the slope of the size graph is proportional to the given graph. Thus, if $S(t)$ is the size of the leaf and $p(t)$ is the rate of photosynthesis

$$S'(t) = kp(t) \qquad \text{for some positive } k.$$

We plot the antiderivative of $p(t)$ to get the graph of $S(t)$ in Figure 7.17. (Since no scale is given on the vertical axis, we can imagine $k = 1$.) The size of the leaf may be represented by its area, or perhaps by its weight.

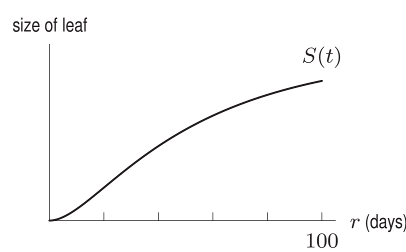

Figure 7.17

14. For every number b, the Fundamental Theorem tells us that

$$\int_0^b F'(x)\,dx = F(b) - F(0) = F(b) - 0 = F(b).$$

Therefore, the values of $F(1)$, $F(2)$, $F(3)$, and $F(4)$ are values of definite integrals. The definite integral is equal to the area of the regions under the graph above the x-axis minus the area of the regions below the x-axis above the graph. Let A_1, A_2, A_3, A_4 be the areas shown in Figure 7.18. The region between $x = 0$ and $x = 1$ lies above the x-axis, so $F(1)$ is positive, and we have

$$F(1) = \int_0^1 F'(x)\,dx = A_1.$$

The region between $x = 0$ and $x = 2$ also lies entirely above the x-axis, so $F(2)$ is positive, and we have

$$F(2) = \int_0^2 F'(x)\,dx = A_1 + A_2.$$

We see that $F(2) > F(1)$. The region between $x = 0$ and $x = 3$ includes parts above and below the x-axis. We have

$$F(3) = \int_0^3 F'(x)\,dx = (A_1 + A_2) - A_3.$$

Since the area A_3 is approximately the same as the area A_2, we have $F(3) \approx F(1)$. Finally, we see that

$$F(4) = \int_0^4 F'(x)\,dx = (A_1 + A_2) - (A_3 + A_4).$$

Since the area $A_1 + A_2$ appears to be larger than the area $A_3 + A_4$, we see that $F(4)$ is positive, but smaller than the others.

The largest value is $F(2)$ and the smallest value is $F(4)$. None of the numbers is negative.

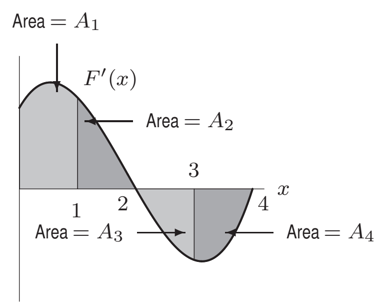

Figure 7.18

15. **(a)** The total volume emptied must increase with time and cannot decrease. The smooth graph (I) that is always increasing is therefore the volume emptied from the bladder. The jagged graph (II) that increases then decreases to zero is the flow rate.

(b) The total change in volume is the integral of the flow rate. Thus, the graph giving total change (I) shows an antiderivative of the rate of change in graph (II).

16. We can start by finding four points on the graph of $F(x)$. The first one is given: $F(2) = 3$. By the Fundamental Theorem of Calculus, $F(6) = F(2) + \int_2^6 F'(x)dx$. The value of this integral is -7 (the area is 7, but the graph lies below the x-axis), so $F(6) = 3 - 7 = -4$. Similarly, $F(0) = F(2) - 2 = 1$, and $F(8) = F(6) + 4 = 0$. We sketch a graph of $F(x)$ by connecting these points, as shown in Figure 7.19.

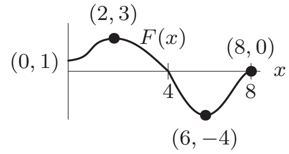

Figure 7.19

17. We see that

F decreases when $x < 1.5$ or $x > 4.67$, because F' is negative there.
F increases when $1.5 < x < 4.67$, because F' is positive there.

So

F has a local minimum at $x = 1.5$.
F has a local maximum at $x = 4.67$.

We have $F(0) = 14$. Since F' is negative between 0 and 1.5, the Fundamental Theorem of Calculus gives us

$$F(1.5) - F(0) = \int_0^{1.5} F'(x)\,dx = -34$$
$$F(1.5) = 14 - 34 = -20.$$

Similarly

$$F(4.67) = F(1.5) + \int_{1.5}^{4.67} F'(x)\,dx = -20 + 25 = 5.$$
$$F(6) = F(4.67) + \int_{4.67}^{6} F'(x)\,dx = 5 - 5 = 0.$$

A graph of F is in Figure 7.20. The local maximum is $(4.67, 5)$ and the local minimum is $(1.5, -20)$.

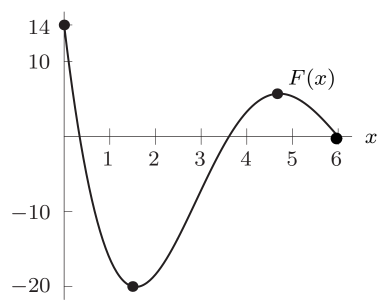

Figure 7.20

18. The areas given enable us to calculate the changes in the function F as we move along the t-axis. Areas above the axis count positively and areas below the axis count negatively. We know that $F(0) = 3$, so

$$F(2) - F(0) = \int_0^2 F'(t)\,dt = \begin{array}{c}\text{Area under } F' \\ 0 \le t \le 2\end{array} = 5$$

Thus,

$$F(2) = F(0) + 5 = 3 + 5 = 8.$$

Similarly,

$$F(5) - F(2) = \int_2^5 F'(t)\,dt = -16$$
$$F(5) = F(2) - 16 = 8 - 16 = -8$$

and

$$F(6) = F(5) + \int_5^6 F'(t)\,dt = -8 + 10 = 2.$$

A graph is shown in Figure 7.21.

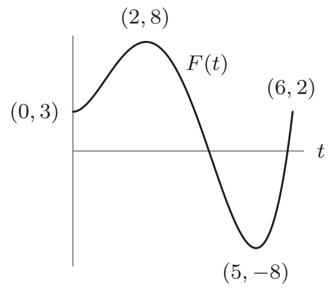

Figure 7.21

19. The critical points are at $(0, 5)$, $(2, 21)$, $(4, 13)$, and $(5, 15)$. A graph is given in Figure 7.22.

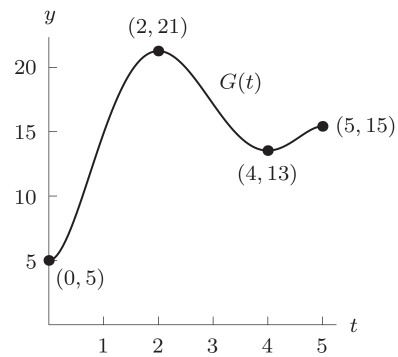

Figure 7.22

20. **(a)** Critical points of $F(x)$ are the zeros of f: $x = 1$ and $x = 3$.
(b) $F(x)$ has a local minimum at $x = 1$ and a local maximum at $x = 3$.
(c) See Figure 7.23.

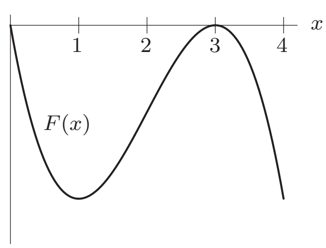

Figure 7.23

Notice that the graph could also be above or below the x-axis at $x = 3$.

21. **(a)** Critical points of $F(x)$ are $x = -1$, $x = 1$ and $x = 3$.
(b) $F(x)$ has a local minimum at $x = -1$, a local maximum at $x = 1$, and a local minimum at $x = 3$.
(c) See Figure 7.24.

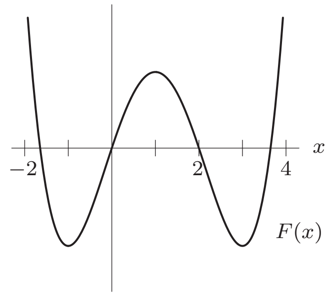

Figure 7.24

22. By the Fundamental Theorem,

$$f(1) - f(0) = \int_0^1 f'(x)\,dx,$$

Since $f'(x)$ is negative for $0 \le x \le 1$, this integral must be negative and so $f(1) < f(0)$.

23. First rewrite each of the quantities in terms of f', since we have the graph of f'. If A_1 and A_2 are the positive areas shown in Figure 7.25:

$$f(3) - f(2) = \int_2^3 f'(t)\,dt = -A_1$$

$$f(4) - f(3) = \int_3^4 f'(t)\,dt = -A_2$$

$$\frac{f(4) - f(2)}{2} = \frac{1}{2}\int_2^4 f'(t)\,dt = -\frac{A_1 + A_2}{2}$$

Since Area $A_1 >$ Area A_2,

$$A_2 < \frac{A_1 + A_2}{2} < A_1$$

so

$$-A_1 < -\frac{A_1 + A_2}{2} < -A_2$$

and therefore

$$f(3) - f(2) < \frac{f(4) - f(2)}{2} < f(4) - f(3).$$

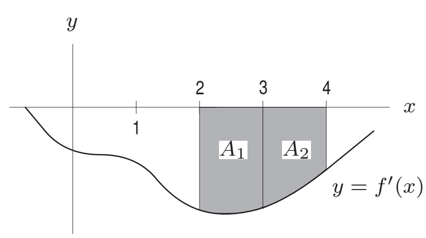

Figure 7.25

24. See Figure 7.26.

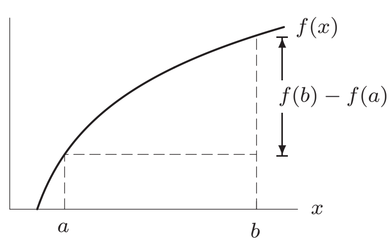

Figure 7.26

25. See Figure 7.27.

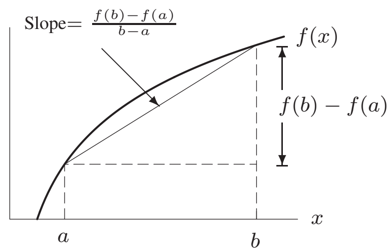

Figure 7.27

26. See Figure 7.28.

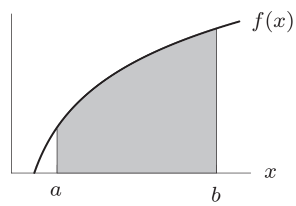

Figure 7.28

27. See Figure 7.29. Note that we are using the interpretation of the definite integral as the length of the interval times the average value of the function on that interval, which we developed in Section 6.1.

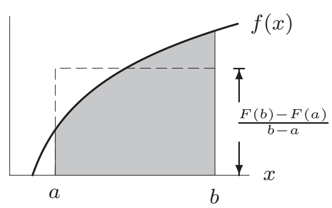

Figure 7.29

Solutions for Chapter 7 Review

1. $\frac{2}{3}t^3 + \frac{3}{4}t^4 + \frac{4}{5}t^5$

2. $t^3 + \dfrac{7t^2}{2} + t.$

3. $2x^3 - 4x^2 + 3x.$

4. $F(z) = e^z + 3z + C$

5. $P(r) = \pi r^2 + C$

6. $P(y) = \ln|y| + y^2/2 + y + C$

7. $-\dfrac{1}{t}$

8. $-\cos t$

9. Antiderivative $G(x) = \frac{x^4}{4} + x^3 + \frac{3x^2}{2} + x + C = \frac{(x+1)^4}{4} + C$

10. We use substitution with $w = 2x + 1$ and $dw = 2\,dx$. Then

$$\int f(x)\,dx = \int (2x+1)^3\,dx = \int w^3 \frac{1}{2}\,dw = \frac{w^4}{2 \cdot 4} + C = \frac{(2x+1)^4}{8} + C.$$

11. $2t^2 + 7t + C$

12. $\frac{3x^2}{2} + C$

13. $\frac{x^4}{4} - \frac{x^2}{2} + C.$

14. $\frac{-1}{0.05}e^{-0.05t} + C = -20e^{-0.05t} + C.$

15. $-\frac{5}{t} - \frac{3}{t^2} + C$

16. Since $f(x) = \frac{x+1}{x} = 1 + \frac{1}{x}$, the indefinite integral is $x + \ln|x| + C$

17. $4t^2 + 3t + C.$

18. $p + \ln|p| + C$

19. $2x^4 + \ln|x| + C.$

20. $5\sin x + 3\cos x + C$

21. $2\ln|x| - \pi\cos x + C$

22. $2x^2 + 2e^x + C$

23. Since $F'(x) = 12x^2 + 1$, we use $F(x) = 4x^3 + x$. By the Fundamental Theorem, we have

$$\int_0^2 (12x^2+1)dx = (4x^3+x)\Big|_0^2 = (4 \cdot 2^3 + 2) - (4 \cdot 0^3 + 0) = 34 - 0 = 34.$$

24. $\int_{-3}^{-1} \frac{2}{r^3}\,dr = -r^{-2}\Big|_{-3}^{-1} = -1 + \frac{1}{9} = -8/9 \approx -0.889.$

25. $\int_0^1 \sin\theta\,d\theta = -\cos\theta\Big|_0^1 = 1 - \cos 1 \approx 0.460.$

26. If $f(x) = 1/x$, then $F(x) = \ln|x|$ (since $\frac{d}{dx}\ln|x| = \frac{1}{x}$). By the Fundamental Theorem, we have

$$\int_1^2 \frac{1}{x}\,dx = \ln|x|\Big|_1^2 = \ln 2 - \ln 1 = \ln 2.$$

27. Since $F'(x) = \frac{1}{x^2} = x^{-2}$, we use $F(x) = \frac{x^{-1}}{-1} = -\frac{1}{x}$. By the Fundamental Theorem, we have

$$\int_1^2 \frac{1}{x^2}dx = \left(-\frac{1}{x}\right)\Big|_1^2 = -\frac{1}{2} - \left(-\frac{1}{1}\right) = -\frac{1}{2} + 1 = \frac{1}{2}.$$

28. $\int_0^2 \left(\frac{x^3}{3} + 2x\right)\,dx = \left(\frac{x^4}{12} + x^2\right)\Big|_0^2 = \frac{4}{3} + 4 = 16/3 \approx 5.333.$

29. $f(x) = 2x$, so $F(x) = x^2 + C$. $F(0) = 0$ implies that $0^2 + C = 0$, so $C = 0$. Thus $F(x) = x^2$ is the only possibility.

30. $f(x) = \frac{1}{4}x$, so $F(x) = \frac{x^2}{8} + C$. $F(0) = 0$ implies that $\frac{1}{8} \cdot 0^2 + C = 0$, so $C = 0$. Thus $F(x) = x^2/8$ is the only possibility.

31. We use the substitution $w = x^2 + 1$, $dw = 2xdx$.

$$\int \frac{2x}{\sqrt{x^2+1}}dx = \int w^{-1/2}dw = 2w^{1/2} + C = 2\sqrt{x^2+1} + C.$$

Check: $\frac{d}{dx}(2\sqrt{x^2+1} + C) = \frac{2x}{\sqrt{x^2+1}}$.

32. We use the substitution $w = x^2 + 1$, $dw = 2xdx$.

$$\int 2x(x^2+1)^5 dx = \int w^5 dw = \frac{w^6}{6} + C = \frac{1}{6}(x^2+1)^6 + C.$$

Check: $\frac{d}{dx}(\frac{1}{6}(x^2+1)^6 + C) = 2x(x^2+1)^5$.

33. We use the substitution $w = -x^2$, $dw = -2x\,dx$.

$$\int xe^{-x^2}\,dx = -\frac{1}{2}\int e^{-x^2}(-2x\,dx) = -\frac{1}{2}\int e^w\,dw$$
$$= -\frac{1}{2}e^w + C = -\frac{1}{2}e^{-x^2} + C.$$

Check: $\frac{d}{dx}(-\frac{1}{2}e^{-x^2} + C) = (-2x)(-\frac{1}{2}e^{-x^2}) = xe^{-x^2}$.

34. We use the substitution $w = x^4 + 1$, $dw = 4x^3dx$.

$$\int \frac{4x^3}{x^4+1}dx = \int \frac{1}{w}dw = \ln|w| + C = \ln(x^4+1) + C.$$

Check: $\frac{d}{dx}(\ln(x^4+1) + C) = \frac{4x^3}{x^4+1}$.

35. We use the substitution $w = 3x + 1$, $dw = 3dx$.

$$\int \frac{1}{(3x+1)^2}dx = \frac{1}{3}\int \frac{1}{w^2}dw = \frac{1}{3}\int w^{-2}dw = \frac{1}{3}\frac{w^{-1}}{-1} + C = -\frac{1}{3(3x+1)} + C.$$

36. We use the substitution $w = x^2 + 4$, $dw = 2xdx$.

$$\int \frac{x}{\sqrt{x^2+4}}dx = \frac{1}{2}\int w^{-1/2}dw = \frac{1}{2}\frac{w^{1/2}}{1/2} + C = \sqrt{x^2+4} + C.$$

37. We use the substitution $w = 5x - 7$, $dw = 5dx$.

$$\int (5x-7)^{10}dx = \frac{1}{5}\int w^{10}dw = \frac{1}{5}\frac{w^{11}}{11} + C = \frac{1}{55}(5x-7)^{11} + C.$$

38. We use the substitution $w = -0.2t$, $dw = -0.2dt$.

$$\int 100e^{-0.2t}dt = \frac{100}{-0.2}\int e^w dw = -500e^w + C = -500e^{-0.2t} + C.$$

39. We use the substitution $w = x^2 + 1$, $dw = 2xdx$.

$$\int x\sqrt{x^2+1}dx = \frac{1}{2}\int w^{1/2}dw = \frac{1}{3}w^{3/2} + C = \frac{1}{3}(x^2+1)^{3/2} + C.$$

Check: $\frac{d}{dx}\left(\frac{1}{3}(x^2+1)^{3/2} + C\right) = \frac{1}{3}\cdot\frac{3}{2}(x^2+1)^{1/2}\cdot 2x = x\sqrt{x^2+1}$.

40. We use the substitution $w = x^2$, $dw = 2xdx$.

$$\int x\sin(x^2)dx = \frac{1}{2}\int \sin w\,dw = -\frac{1}{2}\cos w + C = -\frac{1}{2}\cos(x^2) + C.$$

Check: $\frac{d}{dx}\left(-\frac{1}{2}\cos(x^2) + C\right) = -\frac{1}{2}(-\sin(x^2))\cdot 2x = x\sin(x^2)$.

41. We use the substitution $w = t^2$, $dw = 2t\,dt$.

$$\int t\cos(t^2)dt = \frac{1}{2}\int \cos(w)dw = \frac{1}{2}\sin(w) + C = \frac{1}{2}\sin(t^2) + C.$$

Check: $\frac{d}{dt}(\frac{1}{2}\sin(t^2) + C) = \frac{1}{2}\cos(t^2)(2t) = t\cos(t^2)$.

42. To find the area under the graph of $f(x) = xe^{x^2}$, we need to evaluate the definite integral

$$\int_0^2 xe^{x^2}\,dx.$$

This is done in Example 4, Section 7.2, using the substitution $w = x^2$, the result being

$$\int_0^2 xe^{x^2}\,dx = \frac{1}{2}(e^4 - 1).$$

43. Since $f(x) = 1/(x+1)$ is positive on the interval $x = 0$ to $x = 2$, we have

$$\text{Area} = \int_0^2 \frac{1}{x+1}dx = \ln(x+1)\Big|_0^2 = \ln 3 - \ln 1 = \ln 3.$$

The area is $\ln 3 \approx 1.0986$.

44. If $f(x) = \dfrac{1}{x+1}$, the average value of f on the interval $0 \le x \le 2$ is defined to be

$$\frac{1}{2-0}\int_0^2 f(x)\,dx = \frac{1}{2}\int_0^2 \frac{dx}{x+1}.$$

We'll integrate by substitution. We let $w = x+1$ and $dw = dx$, and we have

$$\int_{x=0}^{x=2} \frac{dx}{x+1} = \int_{w=1}^{w=3} \frac{dw}{w} = \ln w\Big|_1^3 = \ln 3 - \ln 1 = \ln 3.$$

Thus, the average value of $f(x)$ on $0 \le x \le 2$ is $\frac{1}{2}\ln 3 \approx 0.5493$. See Figure 7.30.

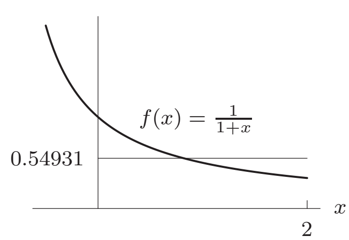

Figure 7.30

45. **(a)** Since r gives the rate of energy use, between 2005 and 2010 (where $t = 0$ and $t = 5$), we have

$$\text{Total energy used} = \int_0^5 462e^{0.019t}\,dt \text{ quadrillion BTUs.}$$

(b) The Fundamental Theorem of Calculus states that

$$\int_a^b f(t)\,dt = F(b) - F(a)$$

provided that $F'(t) = f(t)$. To apply this theorem, we need to find $F(t)$ such that $F'(t) = 462e^{0.019t}$; we take

$$F(t) = \frac{462}{0.019}e^{0.019t} = 24{,}316e^{0.019t}.$$

Thus,

$$\begin{aligned}\text{Total energy used} &= \int_0^5 462e^{0.019t}\,dt = F(5) - F(0)\\ &= 24{,}316e^{0.019t}\Big|_0^5\\ &= 24{,}316(e^{0.095} - e^0) = 2423 \text{ quadrillion BTUs.}\end{aligned}$$

Approximately 2423 quadrillion BTUs of energy were consumed between 2005 and 2010.

46. Since dP/dt is negative for $t < 3$ and positive for $t > 3$, we know that P is decreasing for $t < 3$ and increasing for $t > 3$. Between each two integer values, the magnitude of the change is equal to the area between the graph dP/dt and the t-axis. For example, between $t = 0$ and $t = 1$, we see that the change in P is -1. Since $P = 2$ at $t = 0$, we must have $P = 1$ at $t = 1$. The other values are found similarly, and are shown in Table 7.4.

Table 7.4

t	1	2	3	4	5
P	1	0	$-1/2$	0	1

47. See Figure 7.31.

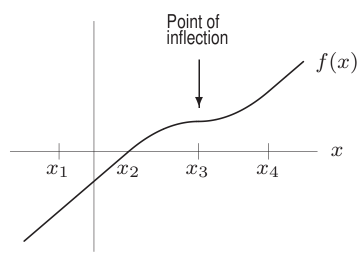

Figure 7.31

48. See Figure 7.32.

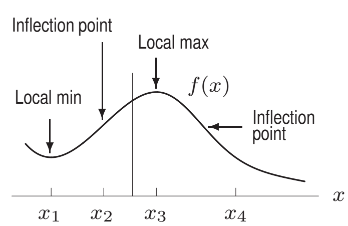

Figure 7.32

49. First notice that F will be decreasing on the interval $0 < x < 1$ and on the interval $3 < x < 4$ and will be increasing on the interval $1 < x < 3$. The areas tell us how much the function increases or decreases. By the Fundamental Theorem, we have

$$F(1) = F(0) + \int_0^1 F'(x)dx = 5 + (-6) = -1.$$
$$F(3) = F(1) + \int_1^3 F'(x)dx = -1 + 8 = 7.$$
$$F(4) = F(3) + \int_3^4 F'(x)dx = 7 + (-2) = 5.$$

50. **(a)** If $w = t/2$, then $dw = (1/2)dt$. When $t = 0$, $w = 0$; when $t = 4$, $w = 2$. Thus,

$$\int_0^4 g(t/2)\,dt = \int_0^2 g(w)\,2dw = 2\int_0^2 g(w)\,dw = 2 \cdot 5 = 10.$$

(b) If $w = 2 - t$, then $dw = -dt$. When $t = 0$, $w = 2$; when $t = 2$, $w = 0$. Thus,

$$\int_0^2 g(2-t)\,dt = \int_2^0 g(w)\,(-dw) = +\int_0^2 g(w)\,dw = 5.$$

51. **(a)** We sketch $f(x) = xe^{-x}$; see Figure 7.33. The shaded area to the right of the y-axis represents the integral $\int_0^\infty xe^{-x}\,dx$.

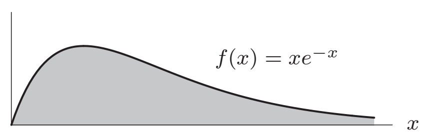

Figure 7.33

(b) Using a calculator or computer, we obtain

$$\int_0^5 xe^{-x}\,dx = 0.9596 \qquad \int_0^{10} xe^{-x}\,dx = 0.9995 \qquad \int_0^{20} xe^{-x}\,dx = 0.99999996.$$

(c) The answers to part (b) suggest that the integral converges to 1.

52. **(a)** A calculator or computer gives

$$\int_1^{100} \frac{1}{\sqrt{x}}\,dx = 18 \qquad \int_1^{1000} \frac{1}{\sqrt{x}}\,dx = 61.2 \qquad \int_1^{10000} \frac{1}{\sqrt{x}}\,dx = 198.$$

These values do not seem to be converging.

(b) An antiderivative of $F'(x) = \dfrac{1}{\sqrt{x}}$ is $F(x) = 2\sqrt{x}$ $\left(\text{since } \dfrac{d}{dx}(2\sqrt{x}) = \dfrac{1}{\sqrt{x}}\right)$. So, by the Fundamental Theorem, we have

$$\int_1^b \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x}\Big|_1^b = 2\sqrt{b} - 2\sqrt{1} = 2\sqrt{b} - 2.$$

(c) The limit of $2\sqrt{b} - 2$ as $b \to \infty$ does not exist, as $\sqrt{b}$ grows without bound. Therefore

$$\lim_{b\to\infty} \int_1^b \frac{1}{\sqrt{x}}\,dx = \lim_{b\to\infty} (2\sqrt{b} - 2) \quad \text{does not exist.}$$

So the improper integral $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ does not converge.

53. The original dose equals the quantity of drug eliminated. The quantity of drug eliminated is the definite integral of the rate. Thus, letting $t \to \infty$, we have the improper integral

$$\text{Total quantity of drug eliminated } = \int_0^\infty 50(e^{-0.1t} - e^{-0.2t})\, dt.$$

Using the fact that $\int e^{kt} dt = \frac{1}{k}e^{kt} + C$, we have

$$\begin{aligned}\text{Total quantity } &= \lim_{b\to\infty} 50\left(-\frac{1}{0.1}e^{-0.1t} + \frac{1}{0.2}e^{-0.2t}\right)\Big|_0^b \\ &= \lim_{b\to\infty} 50(-10e^{-0.1b} + 5e^{-0.2b} - (-10e^0 + 5e^0))\end{aligned}$$

Since $e^{-0.1b} \to 0$ and $e^{-0.2b} \to 0$ as $b \to \infty$, we have

$$\text{Total quantity } = 50(10 - 5) = 250 \text{ mg}.$$

54. Integration by parts with $u = x$, $v' = \cos x$ gives

$$\int x \cos x\, dx = x \sin x - \int \sin x\, dx + C = x \sin x + \cos x + C.$$

55. We integrate by parts, with $u = y$, $v' = \sin y$. We have $u' = 1$, $v = -\cos y$, and

$$\int y \sin y\, dy = -y \cos y - \int (-\cos y)\, dy = -y \cos y + \sin y + C.$$

Check:

$$\frac{d}{dy}(-y \cos y + \sin y + C) = -\cos y + y \sin y + \cos y = y \sin y.$$

56. Remember that $\ln(x^2) = 2 \ln x$. Therefore,

$$\int \ln(x^2)\, dx = 2 \int \ln x\, dx = 2x \ln x - 2x + C.$$

Check:

$$\frac{d}{dx}(2x \ln x - 2x + C) = 2 \ln x + \frac{2x}{x} - 2 = 2 \ln x = \ln(x^2).$$

CHECK YOUR UNDERSTANDING

1. True. We see that the derivative of $t^3/3 + 5$ is t^2.
2. False. When we add one to the exponent -2, we get -1. The function $-x^{-1}$ is an antiderivative of x^{-2}.
3. False. Antiderivatives of e^{3x} are of the form $(1/3)e^{3x} + C$.
4. True. This is a correct integral statement.
5. True. We know

$$\int z^{-1/2}\, dz = \frac{z^{1/2}}{1/2} + C = 2\sqrt{z} + C.$$

6. False. We know that $\int e^x\, dx = e^x + C$.
7. False. The derivative of $\ln |t|$ is $1/t$ so the correct integral statement is $\int (1/t)\, dt = \ln |t| + C$.
8. True, since the derivative of 2^x is $(\ln 2)2^x$.
9. True.
10. True. We know that all antiderivatives differ only by a constant.
11. False, since $dw = (3q^2 + 6q - 1)\, dq$ cannot be substituted.

12. True, since $dw = (1/x)\,dx$.

13. False. We have $dw = 2x\,dx$. Since the integral $\int e^{x^2}\,dx$ does not include an $x\,dx$ to be sustituted for dw, this integral cannot be evaluated using this substitution.

14. True, since $dw = 2x\,dx$.

15. False. This is almost true, but is off by a minus sign since $dw = -1\,ds$.

16. True. Since $dw = 2t\,dt$, we have

$$\int \frac{t}{\sqrt{t^2+1}}\,dt = \frac{1}{2}\int \frac{1}{\sqrt{w}}\,dw = \int \frac{1}{2\sqrt{w}}\,dw.$$

17. True, since $dw = (e^x - e^{-x})\,dx$.

18. False. The substitution $w = q^3 + 5$ would give the integral $\int (1/3)w^{10}\,dw$.

19. True, since $dw = \cos\alpha\,d\alpha$.

20. False. This is almost true, but is off by a minus sign, since $dw = -\sin x\,dx$.

21. False. We need to substitute the endpoints into an antiderivative of $1/x$.

22. True, since $\ln x$ is an antiderivative of $1/x$.

23. True, since x^2 is an antiderivative of $2x$.

24. False. We need to first find an antiderivative of $3x^2$.

25. False. For a definite integral, we need to substitute the endpoints into the antiderivative.

26. True. An antiderivative is e^t and we substitute the limits of integration and subtract.

27. False. When we make the substitution $w = x^2$, we must also substitute for the limits of integration. Since $w = 5^2 = 25$ when $x = 5$ and $w = 0$ when $x = 0$, the result of the substitution is $\int_0^{25} e^w\,dw$.

28. True, since $dw = (1/x)\,dx$ and $w = 1$ when $x = e$ and $w = 0$ when $x = 1$.

29. False. The two definite integrals represent two different quantities.

30. True. The function $y = e^{-kx}$ is positive, so the integral represents the area under the curve between $x = 1$ and $x = 2$ and so is positive.

31. True.

32. True. If a function is concave up, its second derivative is positive which implies that its derivative is increasing.

33. True. This is the Fundamental Theorem of Calculus.

34. False. The limits of integration on the integral need to be from 0 to 3 to make this a true statement.

35. True. Since f' is positive on the interval 3 to 4, the function is increasing on that interval.

36. False. Since f' is negative on the interval 1 to 2, the function is decreasing on that interval.

37. True. Since f' is positive on the interval 2 to 3, the function is increasing on that interval.

38. True. Since f' is negative on the interval 5 to 6, the function is decreasing on that interval.

39. False. Since f' is negative on the interval 0 to 1, the function is decreasing on that interval.

40. True. The area below the curve of f' between $x = 1$ and $x = 2$ is similar in size to the area above the curve between $x = 2$ and $x = 3$. Between $x = 1$ and $x = 3$, the function f increases approximately the same amount that it decreases, so $f(1) \approx f(3)$.

41. False. The integral has to be $\int u\,dv$ when u and dv are substituted. In this case, we should have $u = x^2$ and $dv = e^x\,dx$.

42. True.

43. False. We integrate dv to find v. We see $v = \int e^{3x}\,dx = (1/3)e^{3x}$.

44. False. We integrate dv to find v and $1/x$ is not the antiderivative of $\ln x$. It is the derivative of $\ln x$. In this case, the assignment of parts is wrong. We should try $u = \ln x$ and $dv = x^2\,dx$.

45. False. This integral is more appropriately evaluated using the method of substitution.

46. True.

47. True.

48. False. This integral is more appropriately evaluated using the method of substitution.

49. False. This integral is more appropriately evaluated using the method of substitution.

50. True. We use $u = \ln x$ and $dv = x^3\, dx$.

PROJECTS FOR CHAPTER SEVEN

1. (a) Suppose $Q(t)$ is the amount of water in the reservoir at time t. Then

$$Q'(t) = \begin{array}{c}\text{Rate at which water}\\ \text{in reservoir is changing}\end{array} = \begin{array}{c}\text{Inflow}\\ \text{rate}\end{array} - \begin{array}{c}\text{Outflow}\\ \text{rate}\end{array}$$

Thus the amount of water in the reservoir is increasing when the inflow curve is above the outflow, and decreasing when it is below. This means that $Q(t)$ is a maximum where the curves cross in July 2007 (as shown in Figure 7.34), and $Q(t)$ is decreasing fastest when the outflow is farthest above the inflow curve, which occurs about October 2007 (see Figure 7.34).

To estimate values of $Q(t)$, we use the Fundamental Theorem which says that the change in the total quantity of water in the reservoir is given by

$$Q(t) - Q(\text{Jan 2007}) = \int_{\text{Jan 07}}^{t} (\text{inflow rate} - \text{outflow rate})\, dt$$

or

$$Q(t) = Q(\text{Jan 2007}) + \int_{\text{Jan 07}}^{t} (\text{Inflow rate} - \text{Outflow rate})\, dt.$$

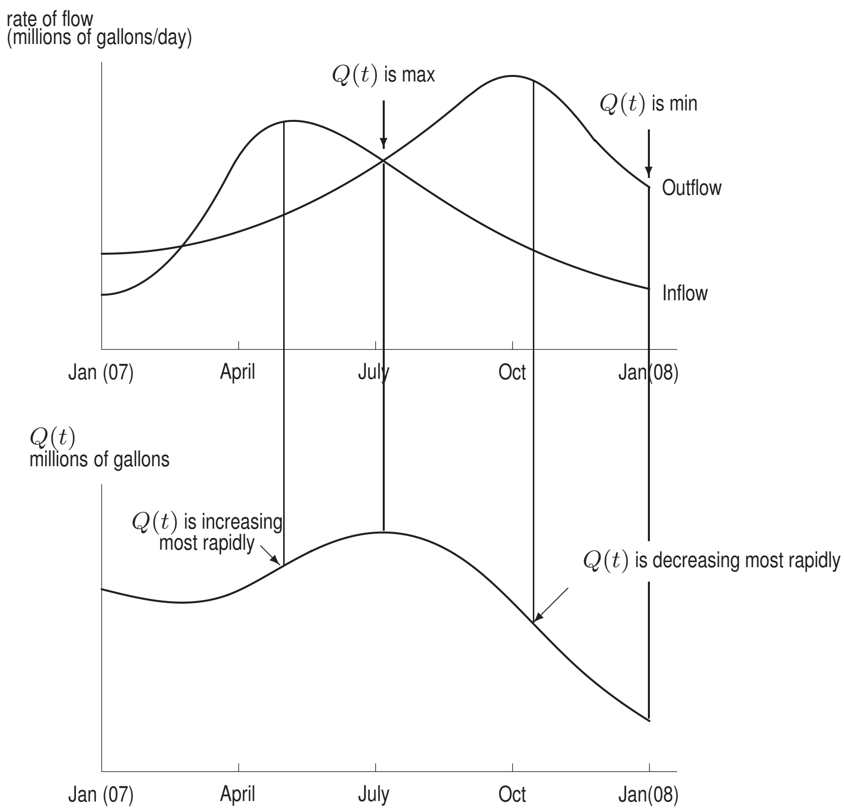

Figure 7.34

(b) See Figure 7.34. Maximum in July 2007. Minimum in Jan 2008.

(c) See Figure 7.34. Increasing fastest in May 2007. Decreasing fastest in Oct 2007.

(d) In order for the water to be the same as Jan 2007 the total amount of water which has flowed into the reservoir minus the total amount of water which has flowed out of the reservoir must be 0. Referring to Figure 7.35, we have

$$\int_{\text{Jan 07}}^{\text{July 08}} (\text{Inflow} - \text{Outflow})\, dt = -A_1 + A_2 - A_3 + A_4 = 0$$

giving $A_1 + A_3 = A_2 + A_4$

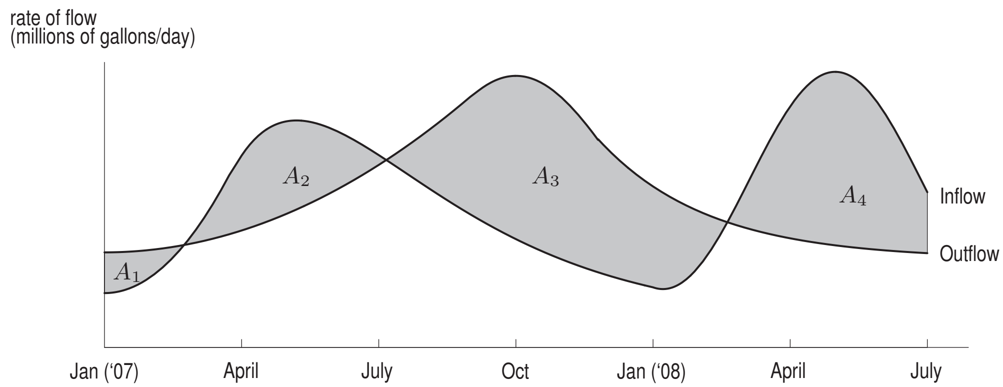

Figure 7.35

Solutions to Practice Problems on Integration

1. $\dfrac{q^3}{3} + \dfrac{5q^2}{2} + 2q + C$

2. $\displaystyle\int (u^4 + 5)\, du = \frac{u^5}{5} + 5u + C$

3. $\dfrac{x^3}{3} + x + C.$

4. Since $\dfrac{d}{dx}(e^{-3t}) = -3e^{-3t}$, we have $\displaystyle\int e^{-3t}\, dt = -\frac{1}{3}e^{-3t} + C.$

5. $4x^{3/2} + C$

6. $\displaystyle\int (ax^2 + b)\, dx = a \cdot \frac{x^3}{3} + bx + C$

7. $\dfrac{x^4}{4} + 2x^2 + 8x + C.$

8. $\displaystyle\int 100e^{-0.5t}\, dt = 100\left(\frac{1}{-0.5}e^{-0.5t}\right) + C = -200e^{-0.5t} + C$

9. $$\int (w^4 - 12w^3 + 6w^2 - 10)\, dw = \frac{w^5}{5} - 12 \cdot \frac{w^4}{4} + 6 \cdot \frac{w^3}{3} - 10 \cdot w + C$$
$$= \frac{w^5}{5} - 3w^4 + 2w^3 - 10w + C$$

10. $\displaystyle\int \left(\frac{4}{x} + 5x^{-2}\right) dx = 4\ln|x| + \frac{5x^{-1}}{-1} + C = 4\ln|x| - \frac{5}{x} + C$

11. $\displaystyle\int q^{-1/2}\, dq = \frac{q^{1/2}}{1/2} + C = 2q^{1/2} + C$

12. $\int 3\sin\theta\,d\theta = -3\cos\theta + C$

13. $\int (p^2 + \frac{5}{p})\,dp = \frac{p^3}{3} + 5\ln|p| + C$

14. $\int P_0 e^{kt}\,dt = P_0\left(\frac{1}{k}e^{kt}\right) + C = \frac{P_0}{k}e^{kt} + C$

15. $$\int (q^3 + 8q + 15)\,dq = \frac{q^4}{4} + 8\cdot\frac{q^2}{2} + 15q + C$$
$$= \frac{q^4}{4} + 4q^2 + 15q + C$$

16. $\int 1000e^{0.075t}\,dt = 1000\left(\frac{1}{0.075}e^{0.075t}\right) + C = 13333e^{0.075t} + C$

17. $\int (5\sin x + 3\cos x)\,dx = -5\cos x + 3\sin x + C$

18. $\int (10 + 5\sin x)\,dx = 10x - 5\cos x + C$

19. $\int \frac{5}{w}\,dw = 5\ln|w| + C$

20. $\int \pi r^2 h\,dr = \pi h\left(\frac{r^3}{3}\right) + C = \frac{\pi}{3}hr^3 + C$

21. $\int (q + q^{-3})\,dq = \frac{q^2}{2} + \frac{q^{-2}}{-2} + C = \frac{q^2}{2} - \frac{1}{2q^2} + C$

22. $\int 15p^2q^4\,dp = 15\left(\frac{p^3}{3}\right)q^4 + C = 5p^3q^4 + C$

23. $\int 15p^2q^4\,dq = 15p^2\left(\frac{q^5}{5}\right) + C = 3p^2q^5 + C$

24. $$\int (3x^2 + 6e^{2x})\,dx = 3\cdot\frac{x^3}{3} + 6\cdot\frac{e^{2x}}{2} + C$$
$$= x^3 + 3e^{2x} + C$$

25. $\int 5e^{2q}\,dq = 5\cdot\frac{1}{2}e^{2q} + C = 2.5e^{2q} + C$

26. $\int \left(p^3 + \frac{1}{p}\right)\,dp = \frac{p^4}{4} + \ln|p| + C$

27. $\int (Ax^3 + Bx)\,dx = \frac{Ax^4}{4} + \frac{Bx^2}{2} + C$

28. $\int (6x^{1/2} + 15)\,dx = 6\cdot\frac{x^{3/2}}{3/2} + 15x + C = 4x^{3/2} + 15x + C$

29. $\int (x^2 + 8 + e^x)\,dx = \frac{x^3}{3} + 8x + e^x + C$

30. $-150e^{-0.2t} + C$

31. $\frac{t^3}{3} - 3t^2 + 5t + C.$

32. $\int \left(a\left(\frac{1}{x}\right) + bx^{-2}\right)\,dx = a\ln|x| + b\frac{x^{-1}}{-1} + C = a\ln|x| - \frac{b}{x} + C$

33. $\int (Aq + B)dq = \frac{Aq^2}{2} + Bq + C$

34. $\int (6x^{-1/2} + 8x^{1/2})dx = 6\frac{x^{1/2}}{1/2} + 8\frac{x^{3/2}}{3/2} = 12\sqrt{x} + \frac{16}{3}x^{3/2} + C$

35. $\int (e^{2t}+5)dt = \frac{1}{2}e^{2t} + 5t + C$

36. $\int \sin(3x)dx = -\frac{1}{3}\cos(3x) + C$

37. $\int 12\cos(4x)dx = 3\sin(4x) + C$

38. We use the substitution $w = y+2$, $dw = dy$:

$$\int \frac{1}{y+2}dy = \int \frac{1}{w}dw = \ln|w| + C = \ln|y+2| + C.$$

39. We use the substitution $w = y^2+5$, $dw = 2y\,dy$.

$$\begin{aligned}\int y(y^2+5)^8\,dy &= \frac{1}{2}\int (y^2+5)^8(2y\,dy)\\ &= \frac{1}{2}\int w^8\,dw = \frac{1}{2}\frac{w^9}{9} + C\\ &= \frac{1}{18}(y^2+5)^9 + C.\end{aligned}$$

Check: $\frac{d}{dy}(\frac{1}{18}(y^2+5)^9 + C) = \frac{1}{18}[9(y^2+5)^8(2y)] = y(y^2+5)^8$.

40. $\frac{1}{4}\sin(4x) + C$

41. $\int A\sin(Bt)dt = -\frac{A}{B}\cos(Bt) + C$

42. We use the substitution $w = 3x+1$, $dw = 3dx$:

$$\int \sqrt{3x+1}dx = \frac{1}{3}\int w^{1/2}dw = \frac{1}{3}\frac{w^{3/2}}{3/2} + C = \frac{2}{9}(3x+1)^{3/2} + C.$$

43. We use the substitution $w = 2+e^x$, $dw = e^x\,dx$.

$$\int \frac{e^x}{2+e^x}\,dx = \int \frac{dw}{w} = \ln|w| + C = \ln(2+e^x) + C.$$

(We can drop the absolute value signs since $2+e^x \geq 0$ for all x.)
Check: $\frac{d}{dx}[\ln(2+e^x) + C] = \frac{1}{2+e^x}\cdot e^x = \frac{e^x}{2+e^x}$.

44. We use the substitution $w = \sin 5\theta$, $dw = 5\cos 5\theta\,d\theta$.

$$\int \sin^6 5\theta \cos 5\theta\,d\theta = \frac{1}{5}\int w^6\,dw = \frac{1}{5}(\frac{w^7}{7}) + C = \frac{1}{35}\sin^7 5\theta + C.$$

Check: $\frac{d}{d\theta}(\frac{1}{35}\sin^7 5\theta + C) = \frac{1}{35}[7\sin^6 5\theta](5\cos 5\theta) = \sin^6 5\theta\cos 5\theta$.
Note that we could also use Problem 23 to solve this problem, substituting $w = 5\theta$ and $dw = 5\,d\theta$ to get:

$$\begin{aligned}\int \sin^6 5\theta\cos 5\theta\,d\theta &= \frac{1}{5}\int \sin^6 w\cos w\,dw\\ &= \frac{1}{5}(\frac{\sin^7 w}{7}) + C = \frac{1}{35}\sin^7 5\theta + C.\end{aligned}$$

45. We use the substitution $w = 1+\sin x$, $dw = \cos x dx$:

$$\int \frac{\cos x}{\sqrt{1+\sin x}}dx = \int w^{-1/2}dw = \frac{w^{1/2}}{1/2} + C = 2\sqrt{1+\sin x} + C.$$

46. Integration by parts with $u = \ln x$, $v' = x$ gives

$$\int x \ln x \, dx = \frac{x^2}{2} \ln x - \int \frac{1}{2} x \, dx = \frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 + C.$$

47.

$$\begin{aligned} \int x e^x \, dx &= x e^x - \int e^x \, dx \qquad \text{(let } x = u, e^x = v', e^x = v\text{)} \\ &= x e^x - e^x + C, \end{aligned}$$

where C is a constant.

48.

$$\begin{aligned} \int_0^{10} z e^{-z} \, dz &= \left[-z e^{-z}\right]\Big|_0^{10} - \int_0^{10} -e^{-z} \, dz \qquad \text{(let } z = u, e^{-z} = v', -e^{-z} = v\text{)} \\ &= -10e^{-10} - \left[e^{-z}\right]\Big|_0^{10} \\ &= -10e^{-10} - e^{-10} + 1 \\ &= -11e^{-10} + 1. \end{aligned}$$