CHAPTER NINE

Solutions for Section 9.1

1. We make a table by calculating values for $C = f(d, m)$ for each value of d and m. Such a table is shown in Table 9.1

Table 9.1

		d			
		1	2	3	4
m	100	55	95	135	175
	200	70	110	150	190
	300	85	125	165	205
	400	100	140	180	220

2. **(a)** $f(3, 200) = 40(3) + 0.15(200) = 150$. Renting a car for three days and driving it 200 miles will cost \$150.
(b) $f(3, m)$ is the value of C for various values of m with d fixed at 3. In other words, it is the cost of renting a car for three days and driving it various numbers of miles. A plot of $C = f(3, m)$ against m is shown in Figure 9.1.

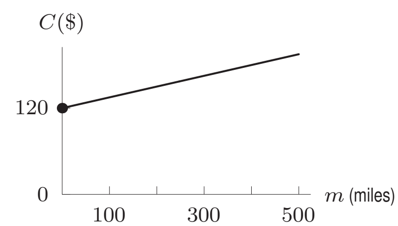


Figure 9.1

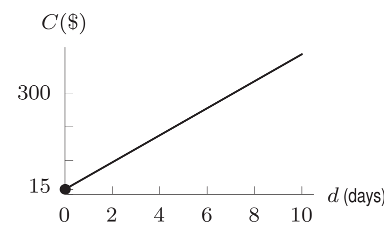


Figure 9.2

(c) $f(d, 100)$ is the value of C for various values of d with m fixed at 100. In other words, it is the cost of renting a car for various lengths of time and driving it 100 miles. A graph of $C = f(d, 100)$ against d is shown in Figure 9.2.

3. Beef consumption by households making \$20,000/year is given by Row 1 of Table 9.2 on page 352 of the text.

Table 9.2

p	3.00	3.50	4.00	4.50
$f(20, p)$	2.65	2.59	2.51	2.43

For households making \$20,000/year, beef consumption decreases as price goes up.
Beef consumption by households making \$100,000/year is given by Row 5 of Table 9.2.

Table 9.3

p	3.00	3.50	4.00	4.50
$f(100, p)$	5.79	5.77	5.60	5.53

For households making \$100,000/year, beef consumption also decreases as price goes up.
Beef consumption by households when the price of beef is \$3.00/lb is given by Column 1 of Table 9.2.

Table 9.4

I	20	40	60	80	100
$f(I, 3.00)$	2.65	4.14	5.11	5.35	5.79

When the price of beef is \$3.00/lb, beef consumption increases as income increases.
Beef consumption by households when the price of beef is \$4.00/lb is given by Column 3 of Table 9.2.

Table 9.5

I	20	40	60	80	100
$f(I, 4.00)$	2.51	3.94	4.97	5.19	5.60

When the price of beef is \$4.00/lb, beef consumption increases as income increases.

4. If the price of beef is held constant, beef consumption for households with various incomes can be read from a fixed column in Table 9.2 on page 352 of the text. For example, the column corresponding to $p = 3.00$ gives the function $h(I) = f(I, 3.00)$; it tells you how much beef a household with income I will buy at \$3.00/lb. Looking at the column from the top down, you can see that it is an increasing function of I. This is true in every column. This says that at any fixed price for beef, consumption goes up as household income goes up—which makes sense. Thus, f is an increasing function of I for each value of p.

5. The amount of money spent on beef equals the product of the unit price p and the quantity C of beef consumed:

$$M = pC = pf(I, p).$$

Thus, we multiply each entry in Table 9.2 on page 352 of the text by the price at the top of the column. This yields Table 9.6.

Table 9.6 *Amount of money spent on beef (\$/household/week)*

		Price			
		3.00	3.50	4.00	4.50
Income	20	7.95	9.07	10.04	10.94
	40	12.42	14.18	15.76	17.46
	60	15.33	17.50	19.88	21.78
	80	16.05	18.52	20.76	22.82
	100	17.37	20.20	22.40	24.89

6. Table 9.7 gives the amount M spent on beef per household per week. Thus, the amount the household spent on beef in a year is $52M$. Since the household's annual income is I thousand dollars, the proportion of income spent on beef is

$$P = \frac{52M}{1000I} = 0.052\frac{M}{I}.$$

Thus, we need to take each entry in Table 9.7, divide it by the income at the left, and multiply by 0.052. Table 9.8 shows the results.

Table 9.7 *Money spent on beef (\$/household/week)*

		Price of Beef (\$)			
		3.00	3.50	4.00	4.50
Income (\$1,000)	20	7.95	9.07	10.04	10.94
	40	12.42	14.18	15.76	17.46
	60	15.33	17.50	19.88	21.78
	80	16.05	18.52	20.76	22.82
	100	17.37	20.20	22.40	24.89

Table 9.8 *Proportion of annual income spent on beef*

		Price of Beef (\$)			
		3.00	3.50	4.00	4.50
Income (\$1,000)	20	0.021	0.024	0.026	0.028
	40	0.016	0.018	0.020	0.023
	60	0.013	0.015	0.017	0.019
	80	0.010	0.012	0.013	0.015
	100	0.009	0.011	0.012	0.013

7. In the answer to Problem 6 we saw that

$$P = 0.052\frac{M}{I},$$

and in the answer to Problem 5 we saw that

$$M = pf(I, p).$$

Putting the expression for M into the expression for P, gives:

$$P = 0.052\frac{pf(I, p)}{I}.$$

8. We have $M = f(B, t) = B(1.05)^t$.

Figure 9.3 gives the graphs of f as a function of t for B fixed at 10, 20, and 30. For each fixed B, the function $f(B, t)$ is an increasing function of t. The larger the fixed value of B, the larger the value of $f(B, t)$.

Figure 9.4 gives the graphs of f as a function of B for t fixed at 0, 5, and 10. For each fixed t, $f(B, t)$ is an increasing (and in fact linear) function of B. The larger t is, the larger the slope of the line.

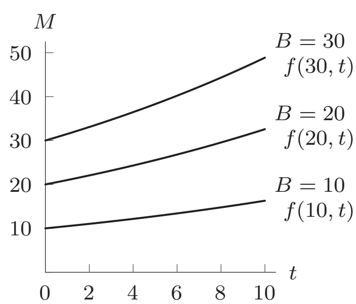


Figure 9.3

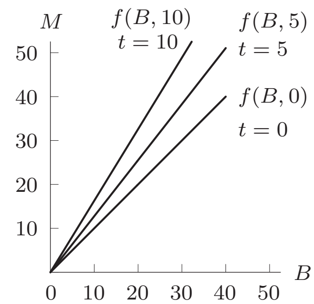


Figure 9.4

9. Asking if f is an increasing or decreasing function of p is the same as asking how does f vary as we vary p, when we hold a fixed. Intuitively, we know that as we increase the price p, total sales of the product will go down. Thus, f is a decreasing function of p. Similarly, if we increase a, the amount spent on advertising, we can expect f to increase and therefore f is an increasing function of a.

10. **(a)** Decreasing, because, other things being equal, we expect the sales of cars to drop if gas prices increase.
(b) Decreasing, because, other things being equal, we expect car sales to drop if car prices increase.

11. **(a)** It feels like 81°F.
(b) At 30% relative humidity, 90°F feels like 90°F.
(c) By finding the temperature which has heat index 105°F for each humidity level, we get Table 9.9:

Table 9.9 *Estimates of danger temperatures*

Relative humidity(%)	0	10	20	30	40	50	60
Temperature($^\circ$F)	117	110	105	101	97	94	92

(d) With a high humidity your body cannot cool itself as well by sweating. With a low humidity your body is capable of cooling itself to below the actual temperature. Therefore a high humidity feels hotter and a low humidity feels cooler.

12.

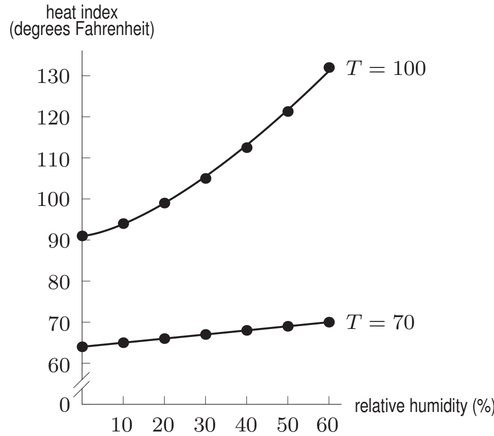


Figure 9.5

Both graphs are increasing because at any fixed temperature the air feels hotter as the humidity increases. The fact that the graph for $T = 100$ increases more rapidly with humidity than the graph for $T = 70$ tells us that when it is hot (100°F), high humidity has more effect on how we feel than at lower temperatures (70°F).

13. We expect P to be an increasing function of A and r. (If you borrow more, your payments go up; if the interest rates go up, your payments go up.) However, P is a decreasing function of t. (If you spread out your payments over more years, you pay less each month.)

14. **(a)** The daily fuel cost is calculated:

$$\text{Cost} = \text{Price per gallon} \times \text{Number of gallons}.$$

Table 9.10 *Cost vs. Price & Gallons*

		Price per gallon (dollars)						
		1.00	1.05	1.10	1.15	1.20	1.25	1.30
Number of gallons	5	5.00	5.25	5.50	5.75	6.00	6.25	6.50
	6	6.00	6.30	6.60	6.90	7.20	7.50	7.80
	7	7.00	7.35	7.70	8.05	8.40	8.75	9.10
	8	8.00	8.40	8.80	9.20	9.60	10.00	10.40
	9	9.00	9.45	9.90	10.35	10.80	11.25	11.70
	10	10.00	10.50	11.00	11.50	12.00	12.50	13.00
	11	11.00	11.55	12.10	12.65	13.20	13.75	14.30
	12	12.00	12.60	13.20	13.80	14.40	15.00	15.60

Note: Table entries may vary depending on the price increase interval chosen.

(b) If the daily travel distance and the price of gas are given, then:

$$\text{Number of gallons} = (\text{Distance in miles}) / (30 \text{ miles per gallon})$$
$$\text{Cost} = \text{Price per gallon} \times \text{Number of gallons}$$

Table 9.11 *Cost vs. Price & Distance*

	Price per gallon (dollars)						
Distance (miles)	1.00	1.05	1.10	1.15	1.20	1.25	1.30
100	3.33	3.50	3.67	3.83	4.00	4.17	4.33
150	5.00	5.25	5.50	5.75	6.00	6.25	6.50
200	6.67	7.00	7.33	7.67	8.00	8.33	8.67
250	8.33	8.75	9.17	9.58	10.00	10.42	10.83
300	10.00	10.50	11.00	11.50	12.00	12.50	13.00
350	11.67	12.25	12.83	13.42	14.00	14.58	15.17
400	13.33	14.00	14.67	15.33	16.00	16.67	17.33
450	15.00	15.75	16.50	17.25	18.00	18.75	19.50

Note: Table entries may vary depending on the price increase interval chosen.

15. At any fixed time t, as we get farther from the volcano, the fallout decreases exponentially. At a fixed distance, d, at the start of the eruption, there is no fallout; as time passes, the fallout increases. This models the behavior of an actual volcano, as we expect the fallout to increase with time but diminish with distance. See Figure 9.6.

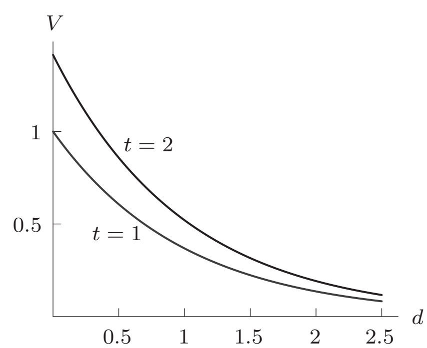


Figure 9.6

16. At any fixed distance, d, as time passes, the fallout increases, although increasing more slowly as time passes. Also, the fallout decreases with distance. This answer is as we would expect: the fallout increases with time but decreases with distance from the volcano. See Figure 9.7.

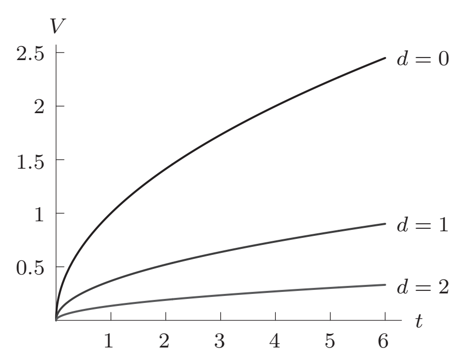


Figure 9.7

17. **(a)** We wish to find $H(20, 0.3)$. The plot of $H(T, 0.3)$ is shown on the given graph. To find $H(20, 0.3)$, we can just find the value of $H(T, 0.3)$ where $T = 20$. Looking at the graph, we see that $H(20, 0.3) \approx 260$ cal/m^3. It takes about 260 calories to clear one cubic meter of air at 20° C with 0.3 grams of water in it.

(b) Proceeding in an analogous manner we can estimate H for the other combinations of T and w; these are shown in Table 9.12:

Table 9.12

T (°C) \ w (gm/m^3)	0.1	0.2	0.3	0.4
0	150	290	425	590
10	110	240	330	450
20	100	180	260	350
30	70	150	220	300
40	65	140	200	270

Solutions for Section 9.2

1. We can see that as we move horizontally to the right, we are increasing x but not changing y. As we take such a path at $y = 2$, we cross decreasing contour lines, starting at the contour line 6 at $x = 1$ to the contour line 1 at around $x = 5.7$. This trend holds true for all of horizontal paths. Thus, z is a decreasing function of x. Similarly, as we move up along a vertical line, we cross increasing contour lines and thus z is an increasing function of y.

2. Looking at the contour diagram, we can see that $Q(x, y)$ is an increasing function of x and a decreasing function of y. It stands to reason that as the price of orange juice goes up, the demand for orange juice will go down. Thus, the demand is a decreasing function of the price of orange juice and thus the y-axis corresponds to the price of orange juice. Also, as the price of apple juice goes up, so will the demand for orange juice and therefore the demand for orange juice is an increasing function of the price of apple juice. Thus, the price of apple juice corresponds to the x-axis.

3. The temperature is decreasing away from the window, suggesting that heat is flowing in from the window. As time goes by the temperature at each point in the room increases. This could be caused by opening the window of an air conditioned room at $t = 0$ thus letting heat from the hot summer day outside raise the temperature inside.

4. To find the hills, we look for areas of the map with concentric contour arcs that increase as we move closer to the center of the area. We see three hills: a height 9 hill centered near $(0, 2)$, a height 7 hill with a peak at $(3, 7)$, and a height 7 hill near $(7, 1.5)$. The highest hill is just the one with the highest contour line, in this case the hill near $(0, 2)$.

We can expect the river to look like a valley, which is represented on a contour diagram as roughly parallel lines. We see such a pattern running from the bottom left at around $(2, 0)$ to the upper right at $(8, 8)$. To see where the river is flowing, we have to see which end of the river is at a higher altitude. At the bottom left of the map, the river has altitude between 3 and 4 and at the upper right, it reaches an altitude of 0. Thus, since rivers flow from areas of high altitude to areas of low altitude, the river flows toward the upper right—or northeast—section of the map.

5. The contour for $C = 50$ is given by
$$40d + 0.15m = 50.$$
This is the equation of a line with intercepts $d = 50/40 = 1.25$ and $m = 50/0.15 \approx 333$. (See Figure 9.8.) The contour for $C = 100$ is given by
$$40d + 0.15m = 100.$$
This is the equation of a parallel line with intercepts $d = 100/40 = 2.5$ and $m = 100/0.15 \approx 667$. The contours for $C = 150$ and $C = 200$ are parallel lines drawn similarly. (See Figure 9.8.)

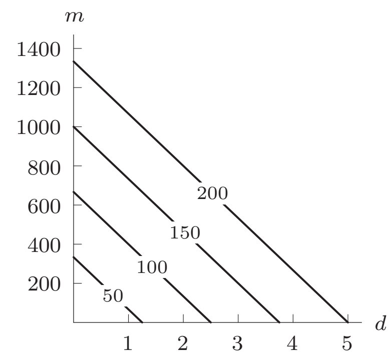


Figure 9.8: A contour diagram for $C = 40d + 0.15m$

6. **(a)** 80-90°F
(b) 60-72°F
(c) 60-100°F

7.

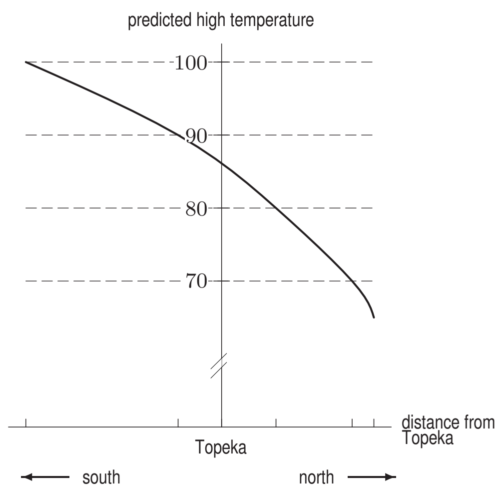


Figure 9.9

8.

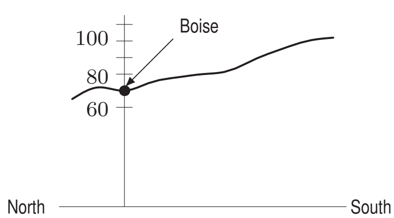


Figure 9.10

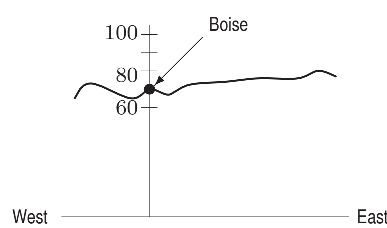


Figure 9.11

9. If we let x represent the high temperature and y represent the low temperature, then the contours should be highest for large x values and small y values. The contour values are greatest in the lower right corner of Quadrant I. One possible contour diagram is shown in Figure 9.12. Other answers are also possible.

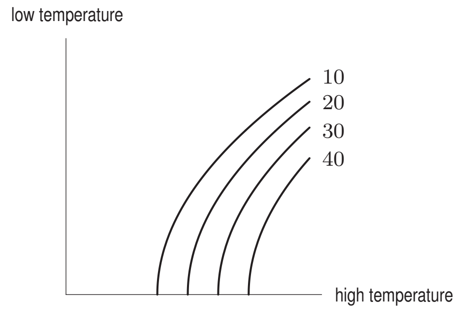


Figure 9.12

10. **(a)** The profit is given by the following:

$$\pi = (\text{Revenue from } q_1) + (\text{ Revenue from } q_2) - \text{Cost}.$$

Measuring π in thousands, we obtain:

$$\pi = 3q_1 + 12q_2 - 4.$$

(b) A contour diagram of π is in Figure 9.13. Note that the units of π are in thousands.

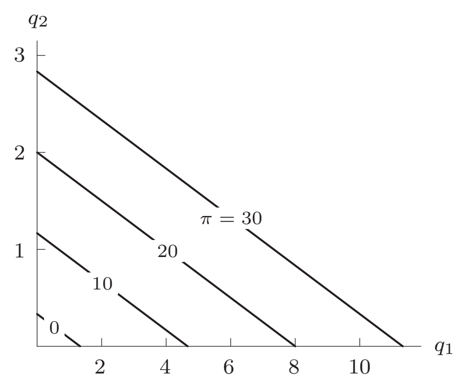


Figure 9.13

11. The contours of $z = y - \sin x$ are of the form $y = \sin x + c$ for a constant c. They are sinusoidal graphs shifted vertically by the value of z on the contour line. The contours are equally spaced vertically for equally spaced z values. See Figure 9.14.

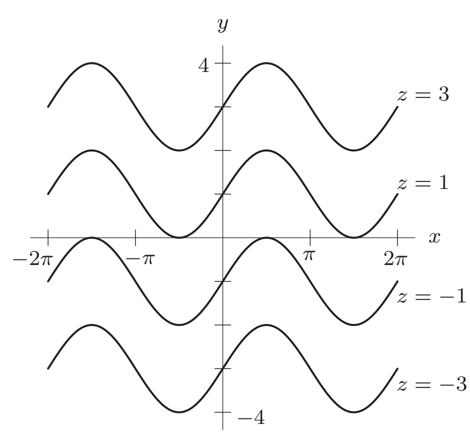


Figure 9.14

12. **(a)** (i) As money increases, with love fixed, your happiness goes up, reaches a maximum and then goes back down. Evidently, there is such a thing as too much money.

(ii) On the other hand, as love increases, with money fixed, your happiness keeps going up.

(b) A cross-section with love fixed will show your happiness as money increases; the curve goes up to a maximum then back down, as in Figure 9.15. The higher cross-section, showing more overall happiness, corresponds to a larger amount of love, because as love increases so does happiness. Figure 9.16 shows two cross-sections with money fixed. Happiness increases as love increases. We cannot say, however, which cross-section corresponds to a larger fixed amount of money, because as money increases happiness can either increase or decrease.

Figure 9.15: Cross-sections with love fixed

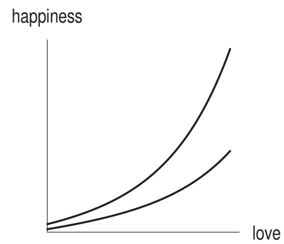


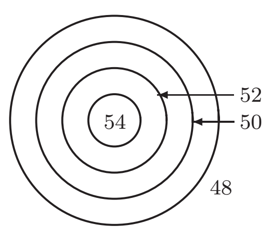
Figure 9.16: Cross-sections with money fixed

13. We first investigate the behavior of f with t fixed. We choose a value for t and move horizontally across the diagram looking at how the values on the contours change. For $t = 1$ hour, as we move from the left at $x = 0$ to the right at $x = 5$, we cross contours of 0.1, 0.2, 0.3, 0.4, and 0.5. The concentration of the drug is increasing as the initial dose, x, increases. (This is what we saw in Figure 9.2 in Section 9.1.) For each choice of f with t fixed (each horizontal line), we see that the contour values are increasing as we move from left to right, showing that, at any given time, the concentration of the drug increases as the size of the dose increases.

The values of f with x fixed are read vertically. For $x = 4$, as we move up from $t = 0$ to $t = 5$, we see that the contours go up from 0.1 to 0.2 to 0.3, and then back down from 0.3 to 0.2 to 0.1. The maximum value of C is very close to 0.4 and is reached at about $t = 1$ hour. The concentration increases quickly at first (the contour lines are closer together), reaches its maximum, and then decreases slowly (the contour lines are farther apart.) All of the contours of f with x fixed are similar to this, although the maximum value varies. For the contour at $x = 3$, we see that the maximum value is slightly below 0.2. If the dose of the drug is 1 mg, the concentration of the drug in the bloodstream never gets as high as 0.1, since we cross no contours on the cross-section of f for $x = 1$.

14. **(a)** The TMS map of an eye of constant curvature will have only one color, with no contour lines dividing the map.

(b) The contour lines are circles, because the cross-section is the same in every direction. The largest curvature is in the center. See picture below.



15. The contour where $f(x, y) = x + y = c$, or $y = -x + c$, is the graph of the straight line with slope -1 as shown in Figure 9.17. Note that we have plotted the contours for $c = -3, -2, -1, 0, 1, 2, 3$. The contours are evenly spaced.

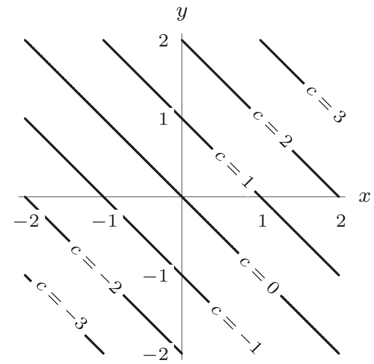


Figure 9.17

16. The contour where $f(x, y) = 3x + 3y = c$ or $y = -x + c/3$ is the graph of the straight line of slope -1 as shown in Figure 9.18. Note that we have plotted the contours for $c = -9, -6, -3, 0, 3, 6, 9$. The contours are evenly spaced.

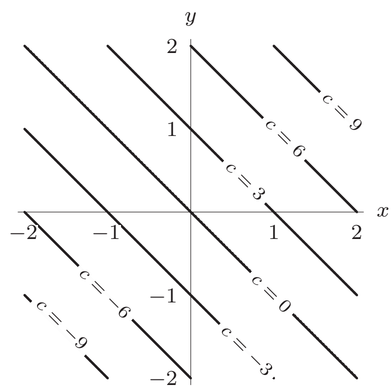


Figure 9.18

17. The contour where $f(x, y) = x + y + 1 = c$ or $y = -x + c - 1$ is the graph of the straight line of slope -1 as shown in Figure 9.19. Note that we have plotted the contours for $c = -2, -1, 0, 1, 2, 3, 4$. The contours are evenly spaced.

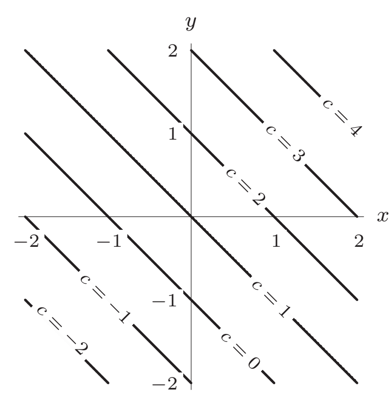


Figure 9.19

18. The contour where $f(x,y) = 2x - y = c$ is the graph of the straight line $y = 2x - c$ of slope 2 as shown in Figure 9.20. Note that we have plotted the contours for $c = -3, -2, -1, 0, 1, 2, 3$. The contours are evenly spaced.

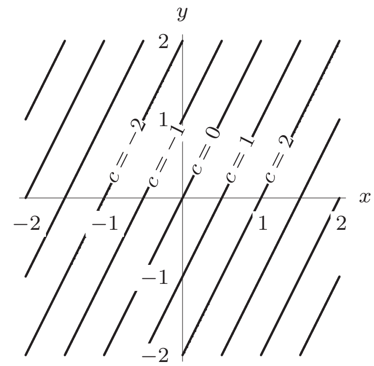


Figure 9.20

19. The contour where $f(x,y) = -x - y = c$ or $y = -x - c$ is the graph of the straight line of slope -1 as shown in Figure 9.21. Note that we have plotted contours for $c = -3, -2, -1, 0, 1, 2, 3$. The contours are evenly spaced.

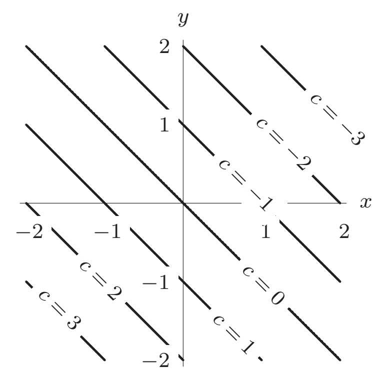


Figure 9.21

20. The contour where $f(x,y) = y - x^2 = c$ is the graph of the parabola $y = x^2 + c$, with vertex $(0, c)$ and symmetric about the y-axis, shown in Figure 9.22. Note that we have plotted the contours for $c = -2, -1, 0, 1$. The contours become more closely packed as we move farther from the y-axis.

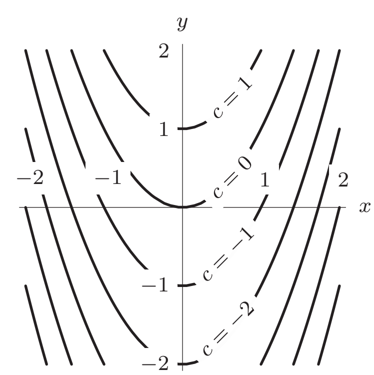


Figure 9.22

21. **(a)** *False*. The values on the level curves are decreasing as you go northward in Canada.

(b) *True*. In general, the contour levels are increasing from peninsulas to mainland. This is true for all three examples. For instance, the density on the Baja peninsula is mostly below 180 whereas on the mainland nearby it is 200 and up to 280.

(c) *False*. It is below 100, since the values on the level curves are decreasing as you go southward in Florida and Miami is south of the 100 level curve.

(d) *True*. Pick the point P where the level curves are the closest together, and pick the direction in which the values on the level curves are increasing fastest. See the arrow marked in Figure 9.23.

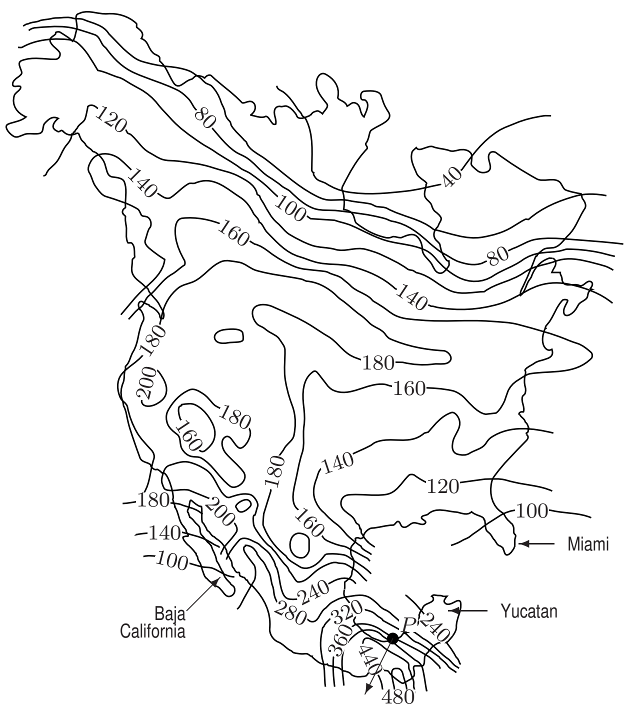


Figure 9.23

22. One possible answer follows.

(a) If the middle line is a highway in the city, there may be many people living around it since it provides mass transit nearby that nearly everyone needs and uses. Thus, the population density would be greatest near the highway, and less dense farther from the highway, as is shown in diagram (I). In an alternative solution, highways may be considered inherently very noisy and dirty. In this scenario, the people in a certain community may purposely live away from the highway. Then the population density would follow the pattern in diagram (III), or in an extreme case, even diagram (II).

(b) If the middle line is a sewage canal, there will be no one living within it, and very few people living close to it. This is represented in diagram (II).

(c) If the middle line is a railroad line in the city, then one scenario is that very few people would enjoy living nearby a railroad yard, with all of the noise and difficulty in crossing it. Then the population density would be smallest near the middle line, and greatest farther from the railroad, as in diagram (III). In an alternative solution, a community might well depend upon the railroad for transit, news or supplies, in which case there would be a denser population nearer the railroad than farther from it, as in diagram (I).

23. **(a)** We have $N = 300$ and $V = 200$ so production

$$P = 2(300)^{0.6}(200)^{0.4} = 510.17 \text{ thousand pages per day.}$$

(b) If the labor force, N, is doubled to 600 workers and the value of the equipment, V, remains at 200, we have

$$P = 2(600)^{0.6}(200)^{0.4} = 773.27.$$

The production P has increased from 510.17 to 773.27, an increase of 263.10 thousand pages per day.

(c) If the value of the equipment, V, is doubled to 400 and the labor force, N, remains at 300 workers, we have

$$P = 2(300)^{0.6}(400)^{0.4} = 673.17.$$

The production P has increased from 510.17 to 673.17, an increase of 163.00 thousand pages per day. We see that doubling the work force has a greater effect on production than doubling the value of the equipment. We might have expected this, since the exponent for N is larger than the exponent for V.

(d) What happens if we double both N and V? Then

$$P = 2(600)^{0.6}(400)^{0.4} = 1020.34 \text{ thousand pages per day.}$$

This is exactly double our original production level of 510.17. In this case, doubling both N and V doubled P.

24. (a) The contour lines are much closer together on path A, so path A is steeper.

(b) If you are on path A and turn around to look at the countryside, you find hills to your left and right, obscuring the view. But the ground falls away on either side of path B, so you are likely to get a much better view of the countryside from path B.

(c) There is more likely to be a stream alongside path A, because water follows the direction of steepest descent.

25. (a) In this company success only increases when money increases, so success will remain constant along the work axis. However, as money increases so does success, which is shown in Graph (III).

(b) As both work and money increase, success never increases, which corresponds to Graph (II).

(c) If the money does not matter, then regardless of how much the money increases success will be constant along the money axis. However, success increases as work increases. This is best represented in Graph (I).

(d) This company's success increases as both money and work increase, which is demonstrated in Graph (IV).

26. (a) To find cardiac output o when pressure $p = 4$ mm Hg and time $t = 0$ hours (i.e. when the patient goes into shock) we go to the coordinate $(0, 4)$ on the graph; we see that this point falls at where $o = 12$ L/min, so initial cardiac output for $p = 4$ mm Hg is 12 L/min. Three hours later, $t = 3$ and p still $= 4$. Going to $(3, 4)$ on the graph, we find that we are somewhat to the right of where $o = 8$. Since output is decreasing as we move to the right, output at $(3, 4)$ will be somewhat less than 8. Estimating, 3 hours after shock with pressure of 4 mm Hg, $o \approx 7.5$ L/min. Since $o = 12$ at $t = 0$, we wish to find the time corresponding to $o = 12/2 = 6$ and p still $= 4$. Going along the line of $o = 6$, we find that this intersects $p = 4$ at $t \approx 4.2$. So about 4.2 hours have elapsed when cardiac output halves.

(b) Looking at the graph, we see that output increases as we move up a vertical line.

(c) Looking at the graph, we see that output decreases as we move to the right along a horizontal line.

(d) If $p = 3$, the cardiac output decreases rather slowly between $t = 0$ and $t = 2$, the first two hours. We can see this because there is a large gap between initial output, which is approximately 9 L/min, and $o = 8$. In these two hours, output only falls by 1 L/min, a fairly gradual change. Between $t = 2$ and $t = 4$, output falls from 8 L/min to 6 L/min. In the last hour, output falls all the way from 6 L/min to 0 L/min, corresponding to death. So the decrease in output accelerates rapidly. This information might be very useful to physicians treating shock. It tells them that in the first two hours the patient's condition will not worsen very much; in the next two hours, it will deteriorate at a faster rate. After four hours, the patient may well die unless something is done to maintain the patient's cardiac output.

27. (a) The $P = 30$ contour crosses the horizontal line GFR $= 50$ at about $t = 4$. In such a patient, it takes about 4 hours to excrete 30% of the dose.

(b) When GFR $= 60$ and $t = 5$, we are approximately on the 40 contour. Forty percent of the dose has been excreted after 5 hours.

(c) The contour lines of P are approximately horizontal lines for $t \geq 12$. If we look across the horizontal line for any fixed GFR, we see that the value of P changes very little for $t \geq 12$. This means that the percent excreted changes very little then.

(d) If we fix GFR and increase t, we see that the values of P increase. The percent excreted is an increasing function of time. This makes sense because as time goes by, more of the drug will pass through the patient's system.

(e) If we fix time and increase GFR along a vertical line, the values of P increase. The percent excreted is an increasing function of GFR. We see that in sicker patients with lower GFRs, less of the dose is excreted. This is why physicians are careful not to administer frequent doses of antibiotics to patients with kidney disease.

Solutions for Section 9.3

1. **(a)** Positive.
   (b) Negative.
   (c) Positive.
   (d) Zero.
2. $f_x(5,2)$, because the contour lines in the positive x direction are closer together at $(5,2)$ than at $(3,1)$.
3. We estimate $\partial I/\partial H$ and $\partial I/\partial T$ by using difference quotients. We have

$$\frac{\partial I}{\partial H} \approx \frac{f(H+\Delta H,T)-f(H,T)}{\Delta H} \quad \text{and} \quad \frac{\partial I}{\partial T} \approx \frac{f(H,T+\Delta T)-f(H,T)}{\Delta T}$$

Choosing $\Delta H = 10$ and reading the values from Table 9.4 on page 358 in the text we get

$$\left.\frac{\partial I}{\partial H}\right|_{(10,100)} \approx \frac{f(10+10,100)-f(10,100)}{10} = \frac{99-95}{10} = 0.4.$$

Similarly, choosing $\Delta T = 5$ we get

$$\left.\frac{\partial I}{\partial T}\right|_{(10,100)} \approx \frac{f(10,100+5)-f(10,100)}{5} = \frac{100-95}{5} = 1.$$

The fact that $\left.\frac{\partial I}{\partial H}\right|_{(10,100)} \approx 0.4$ means that the rate of change of the heat index per unit increase in humidity is about 0.4. This means that the heat index increases by approximately 0.4°F for every percentage point increase in humidity. This rate is positive, because as the humidity increases, the heat index increases.

The partial derivative $\partial I/\partial T$ gives the rate of increase of heat index with respect to temperature. It is positive because the heat index increases as temperature increases. Knowing that $\left.\frac{\partial I}{\partial T}\right|_{(10,100)} \approx 1$ tells us that as the temperature increases by 1°F, the temperature you feel increases by 1°F also. In other words, at this humidity and temperature, the changes in temperature that you feel are approximately equal to the actual changes in temperature.

The fact that $\frac{\partial I}{\partial T} > \frac{\partial I}{\partial H}$ at $(10,100)$ tells us that a unit increase in temperature has a greater effect on the heat index in Tucson than a unit increase in humidity.

4. We estimate $\partial I/\partial H$ and $\partial I/\partial T$ by using difference quotients. We have

$$\frac{\partial I}{\partial H} \approx \frac{f(H+\Delta H,T)-f(H,T)}{\Delta H} \quad \text{and} \quad \frac{\partial I}{\partial T} \approx \frac{f(H,T+\Delta T)-f(H,T)}{\Delta T}$$

Choosing $\Delta H = 10$ and reading the values from Table 9.4 on page 358 in the text we have

$$\left.\frac{\partial I}{\partial H}\right|_{(50,80)} \approx \frac{f(50+10,80)-f(50,80)}{10} = \frac{82-81}{10} = 0.1.$$

Similarly, choosing $\Delta T = 5$ we get

$$\left.\frac{\partial I}{\partial T}\right|_{(50,80)} \approx \frac{f(50,80+5)-f(50,80)}{5} = \frac{88-81}{5} = 1.4.$$

The fact that $\left.\frac{\partial I}{\partial H}\right|_{(50,80)} \approx 0.1$ means that the average rate of change of the heat index per unit increase in humidity is about 0.1°F. This means that the heat index increases by approximately 0.1 for every percentage point increase in humidity. This rate is positive, because as the humidity increases, the heat index increases.

The partial derivative $\partial I/\partial T$ gives the rate of increase of heat index with respect to temperature. It is positive because the heat index increases as temperature increases. Knowing that $\left.\frac{\partial I}{\partial T}\right|_{(50,80)} \approx 1.4$ tells us that as the temperature increases by 1°F, the temperature you feel increases by 1.4°F. Thus, at this humidity and temperature , the changes in temperature you feel are even larger then the actual change.

The fact that $\partial I/\partial T > \partial I/\partial H$ at (50,80) tells the residents of Boston that a unit increase in temperature has a greater effect on the heat index than a unit increase in humidity.

5. (a) We expect the demand for coffee to decrease as the price of coffee increases (assuming the price of tea is fixed.) Thus we expect f_c to be negative. We expect people to switch to coffee as the price of tea increases (assuming the price of coffee is fixed), so that the demand for coffee will increase. We expect f_t to be positive.

(b) The statement $f(3,2) = 780$ tells us that if coffee costs \$3 per pound and tea costs \$2 per pound, we can expect 780 pounds of coffee to sell each week. The statement $f_c(3,2) = -60$ tells us that, if the price of coffee then goes up \$1 and the price of tea stays the same, the demand for coffee will go down by about 60 pounds. The statement $20 = f_t(3,2)$ tells us that if the price of tea goes up \$1 and the price of coffee stays the same, the demand for coffee will go up by about 20 pounds.

6. The partial derivative, $\partial Q/\partial b$ is the rate of change of the quantity of beef purchased with respect to the price of beef, when the price of chicken stays constant. If the price of beef increases and the price of chicken stays the same, we expect consumers to buy less beef and more chicken. Thus when b increases, we expect Q to decrease, so $\partial Q/\partial b < 0$.

On the other hand, $\partial Q/\partial c$ is the rate of change of the quantity of beef purchased with respect to the price of chicken, when the price of beef stays constant. An increase in the price of chicken is likely to cause consumers to buy less chicken and more beef. Thus when c increases, we expect Q to increase, so $\partial Q/\partial c > 0$.

7. (a) The units of $\partial c/\partial x$ are units of concentration/distance. (For example, (gm/cm^3)/cm.) The practical interpretation of $\partial c/\partial x$ is the rate of change of concentration with distance as you move down the blood vessel at a fixed time. We expect $\partial c/\partial x < 0$ because the further away you get from the point of injection, the less of the drug you would expect to find (at a fixed time).

(b) The units of $\partial c/\partial t$ are units of concentration/time. (For example, (gm/cm^3)/sec.) The practical interpretation of $\partial c/\partial t$ is the rate of change of concentration with time, as you look at a particular point in the blood vessel. We would expect the concentration to first increase (as the drug reaches the point) and then decrease as the drug dies away. Thus, we expect $\partial c/\partial t > 0$ for small t and $\partial c/\partial t < 0$ for large t.

8. (a) We see that f_w is positive since B increases as w increases, when s is held constant. We see that f_s is also positive since B increases as s increases, when w is held constant. This means that a heavier person will burn more calories per minute while roller-blading than a lighter person, and the number of calories burned per minute goes up as the speed goes up.

(b) We have

$$f_w(160,10) \approx \frac{\Delta B}{\Delta w} = \frac{f(180,10) - f(160,10)}{180-160} = \frac{10.2-9.2}{180-160} = 0.05.$$

For a 160 lb person roller-blading at a speed of 10 mph, the number of calories burned per minute increases by about 0.05 if the person's weight is 1 pound more.

To find f_s, we have

$$f_s(160,10) \approx \frac{\Delta B}{\Delta s} = \frac{f(160,11) - f(160,10)}{11-10} = \frac{10.8-9.2}{11-10} = 1.6.$$

For a 160 lb person roller-blading at a speed of 10 mph, the number of calories burned per minute increases by about 1.6 as the speed of the roller-blader increases by 1 mph.

9. We know $z_x(1,0)$ is the rate of change of z in the x-direction at $(1,0)$.Therefore

$$z_x(1,0) \approx \frac{\Delta z}{\Delta x} \approx \frac{1}{0.5} = 2, \quad \text{so } z_x(1,0) \approx 2.$$

We know $z_x(0,1)$ is the rate of change of z in the x-direction at the point $(0,1)$. Since we move along the contour, the change in z

$$z_x(0,1) \approx \frac{\Delta z}{\Delta x} \approx \frac{0}{\Delta x} = 0.$$

We know $z_y(0,1)$ is the rate of change of z in the y-direction at the point $(0,1)$ so

$$z_y(0,1) \approx \frac{\Delta z}{\Delta y} \approx \frac{1}{0.1} = 10.$$

10. (a) This means you must pay a mortgage payment of \$1090.08/month if you have borrowed a total of \$92,000 at an interest rate of 14%, on a 30-year mortgage.

(b) This means that the rate of change of the monthly payment with respect to the interest rate is \$72.82; i.e., your monthly payment will go up by approximately \$72.82 for one percentage point increase in the interest rate for the \$92,000 borrowed under a 30-year mortgage.

(c) It should be *positive*, because the monthly payments will increase if the total amount borrowed is increased.

(d) It should be *negative*, because as you increase the number of years in which to pay the mortgage, you should have to pay less each month.

11. **(a)** We expect f_p to be negative because if the price of the product increases, the sales usually decrease.

(b) If the price of the product is \$8 per unit and if \$12000 has been spent on advertising, sales increase by approximately 150 units if an additional \$1000 is spent on advertising.

12. Estimate $\partial P/\partial r$ and $\partial P/\partial L$ by using difference quotients and reading values of P from the graph:

$$\left.\frac{\partial P}{\partial r}\right|_{(8,5000)} \approx \frac{P(15,5000) - P(8,5000)}{15-8} = \frac{120-100}{7} = 2.9,$$

and

$$\left.\frac{\partial P}{\partial L}\right|_{(8,5000)} \approx \frac{P(8,4000) - P(8,5000)}{4000-5000} = \frac{80-100}{-1000} = 0.02.$$

$P_r(8,5000) \approx 2.9$ means that at an interest rate of 8% and a loan amount of \$5000 the monthly payment increases by approximately \$2.90 for every one percent increase of the interest rate. $P_L(8,5000) \approx 0.02$ means the monthly payment increases by approximately \$0.02 for every \$1 increase in the loan amount at an 8% rate and a loan amount of \$5000.

13. **(a)** We know that we can approximate f_R at a point (R,T) by $f_R(R,T) \approx \frac{f(R+\Delta R,T)-f(R,T)}{\Delta R}$. Approximating with $\Delta R = 3$ at the point $(15,76)$ gives:

$$f_R(15,76) \approx \frac{f(15+3,76) - f(15,76)}{3} = \frac{f(18,76) - f(15,76)}{3}$$

Looking at the graph, we see that $f(15,76) \approx 100$ and $f(18,76) \approx 110$, so

$$f_R(15,76) \approx \frac{110-100}{3} \approx 3.3\% \text{ per in.}$$

Each additional inch of rainfall increases corn yield by about 3.3% from the present production level.

(b) We estimate f_T similarly; we approximate with $\Delta T = 2$.

$$f_T(15,76) \approx \frac{f(15,76+2) - f(15,76)}{2} = \frac{f(15,78) - f(15,76)}{2}$$

Looking at the graph, we see that $f(15,76) \approx 100$ and $f(15,78) \approx 90$, so

$$f_T(15,76) \approx \frac{90-100}{2} = -5\% \text{ per }^\circ\text{F}.$$

For every Fahrenheit degree that the temperature increases, corn yield falls by 5% from the current production level.

14. Reading values of H from the graph gives Table 9.13. In order to compute $H_T(T,w)$ at $T = 30$, it is useful to have values of $H(T,w)$ for $T = 40^\circ\ C$. The column corresponding to $w = 0.4$ is not used in this problem.

Table 9.13 *Estimated values of $H(T,w)$ (in calories/meter^3)*

T (°C) \ w (gm/m^3)	0.1	0.2	0.3	0.4
10	110	240	330	450
20	100	180	260	350
30	70	150	220	300
40	65	140	200	270

Table 9.14 *Estimated values of $H_T(T,w)$ (in calories/meter^3)*

T (°C) \ w (gm/m^3)	0.1	0.2	0.3
10	−1.0	−6.0	−7.0
20	−3.0	−3.0	−4.0
30	−0.5	−1.0	−2.0

The estimates for $H_T(T, w)$ in Table 9.14 are now computed using the formula

$$H_T(T, w) \approx \frac{H(T + 10, w) - H(T, w)}{10}.$$

Practically, the values $H_T(T, W)$ show how much less heat (in calories per cubic meter of fog) is needed to clear the fog if the temperature is increased by 10° C, with w fixed.

15. The sign of $\partial f/\partial P_1$ tells you whether f (the number of people who ride the bus) increases or decreases when P_1 is increased. Since P_1 is the price of taking the bus, as it increases, f should decrease. This is because fewer people will be willing to pay the higher price, and more people will choose to ride the train. On the other hand, the sign of $\dfrac{\partial f}{\partial P_2}$ tells you the change in f as P_2 increases. Since P_2 is the cost of riding the train, as it increases, f should increase. This is because fewer people will be willing to pay the higher fares for the train, and more people will choose to ride the bus.

Therefore, $\dfrac{\partial f}{\partial P_1} < 0$ and $\dfrac{\partial f}{\partial P_2} > 0$.

16. $\dfrac{\partial q}{\partial I} > 0$ because, other things being constant, as people get richer, more beer will be bought.

$\dfrac{\partial q}{\partial p_1} < 0$ because, other things being constant, if the price of beer rises, less beer will be bought.

$\dfrac{\partial q}{\partial p_2} > 0$ because, other things being constant, if the price of other goods rises, but the price of beer does not, more beer will be bought.

17. **(a)** The table gives $f(200, 400) = 150{,}000$. This means that sales of 200 full-price tickets and 400 discount tickets generate \$150,000 in revenue.

(b) The notation $f_x(200, 400)$ represents the rate of change of f as we fix y at 400 and increase x from 200. What happens to the revenue as we look along the row $y = 400$ in the table? Revenue increases, so $f_x(200, 400)$ is positive. The notation $f_y(200, 400)$ represents the rate of change of f as we fix x at 200 and increase y from 400. What happens to the revenue as we look down the column $x = 200$ in the table? Revenue increases, so $f_y(200, 400)$ is positive.

Both partial derivatives are positive. This makes sense since revenue increases if more of either type of ticket is sold.

(c) To estimate $f_x(200, 400)$, we calculate $\Delta R/\Delta x$ as x increases from 200 to 300 while y is fixed at 400. We have

$$f_x(200, 400) \approx \frac{\Delta R}{\Delta x} = \frac{185{,}000 - 150{,}000}{300 - 200} = 350 \text{ dollars/ticket.}$$

The partial derivative of f with respect to x is 350 dollars per full-price ticket. This means that the price of a full-price ticket is \$350.

To estimate $f_y(200, 400)$, we calculate $\Delta R/\Delta y$ as y increases from 400 and 600 while x is fixed at 200. We have

$$f_y(200, 400) \approx \frac{\Delta R}{\Delta y} = \frac{190{,}000 - 150{,}000}{600 - 400} = 200 \text{ dollars/ticket.}$$

The partial derivative of f with respect to y is 200 dollars per discount ticket. This means that the price of a discount ticket is \$200.

18. **(a)** The partial derivative $f_x = 350$ tells us that R increases by \$350 as x increases by 1. Thus, $f(201, 400) = f(200, 400) + 350 = 150{,}000 + 350 = 150{,}350$.

(b) The partial derivative $f_y = 200$ tells us that R increases by \$200 as y increases by 1. Since y is increasing by 5, we have $f(200, 405) = f(200, 400) + 5(200) = 150{,}000 + 1{,}000 = 151{,}000$.

(c) Here x is increasing by 3 and y is increasing by 6. We have $f(203, 406) = f(200, 400) + 3(350) + 6(200) = 150{,}000 + 1{,}050 + 1{,}200 = 152{,}250$.

In this problem, the partial derivatives gave exact results, but in general they only give an estimate of the changes in the function.

19. We are asked to find $f(26, 15)$. The closest to this point in the data given in the table is $f(24, 15)$, so we will approximate based on this value. First we need to calculate the partial derivative f_t. Using $\Delta t = -2$, we have

$$f(24, 15) \approx \frac{f(22, 15) - f(24, 15)}{-2} = \frac{58 - 36}{-2} = -11.$$

Given this, we can use $\Delta f \approx \Delta t f_t + \Delta c f_c$ to obtain

$$f(26, 15) \approx f(24, 15) + 2 \cdot f_t(24, 15) + 0 \cdot f_c(26, 15)$$

$$\approx 36 - 2 \cdot 11 + 0$$
$$= 14\%.$$

20. We can use the formula $\Delta f \approx \Delta x f_x + \Delta y f_y$. Applying this formula in order to estimate $f(105, 21)$ from the known value of $f(100, 20)$ gives

$$\begin{aligned} f(105,21) &\approx f(100,20) + (105-100)f_x(100,20) + (21-20)f_y(100,20) \\ &= 2750 + 5 \cdot 4 + 1 \cdot 7 \\ &= 2777. \end{aligned}$$

21. We can use the formula $\Delta f \approx \Delta r f_r + \Delta s f_s$. Applying this formula in order to estimate $f(52, 108)$ from the known value of $f(50, 100)$ gives

$$\begin{aligned} f(52,108) &\approx f(50,100) + (52-50)f_r(50,100) + (108-100)f_s(50,100) \\ &= 5.67 + 2(0.60) + 8(-0.15) \\ &= 5.67. \end{aligned}$$

22. **(a)** $\dfrac{\partial p}{\partial c} = f_c(c, s) =$ rate of change in blood pressure as cardiac output increases while systemic vascular resistance remains constant.

(b) Suppose that $p = kcs$. Note that c (cardiac output), a volume, s (SVR), a resistance, and p, a pressure, must all be positive. Thus k must be positive, and our level curves should be confined to the first quadrant. Several level curves are shown in Figure 9.24. Each level curve represents a different blood pressure level. Each point on a given curve is a combination of cardiac output and SVR that results in the blood pressure associated with that curve.

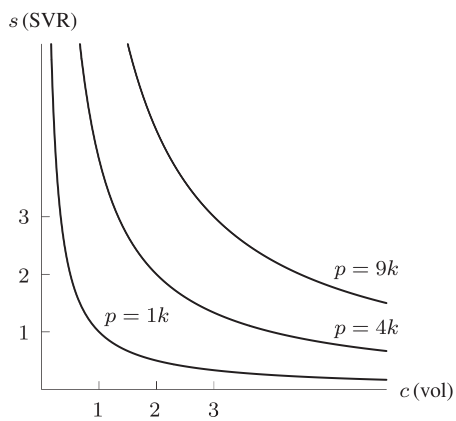


Figure 9.24

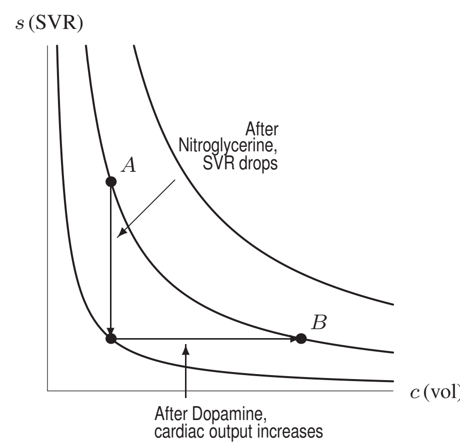


Figure 9.25

(c) Point B in Figure 9.25 shows that if the two doses are correct, the changes in pressure will cancel. The patient's cardiac output will have increased and his SVR will have decreased, but his blood pressure won't have changed.

(d) At point F in Figure 9.26, the patient's blood pressure is normalized, but his/her cardiac output has dropped and his SVR is up.

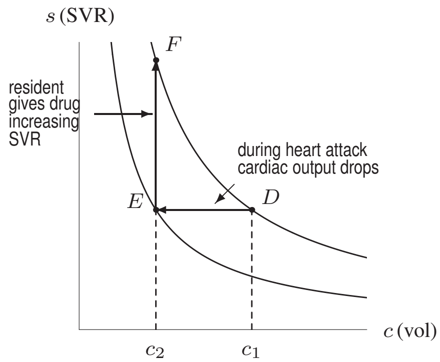


Figure 9.26

Note: c_1 and c_2 are the cardiac outputs before and after the heart attack, respectively.

23. **(a)** Since $f_x > 0$, the values on the contours increase as you move to the right. Since $f_y > 0$, the values on the contours increase as you move upward. See Figure 9.27.

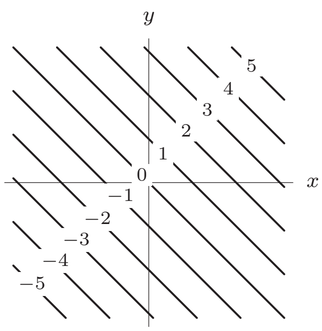


Figure 9.27: $f_x > 0$ and $f_y > 0$

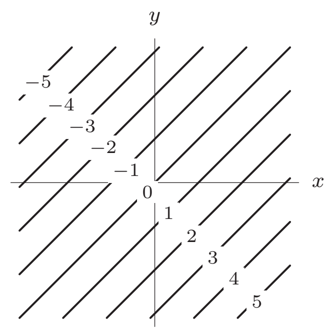


Figure 9.28: $f_x > 0$ and $f_y < 0$

(b) Since $f_x > 0$, the values on the contours increase as you move to the right. Since $f_y < 0$, the values on the contours decrease as you move upward. See Figure 9.28.

(c) Since $f_x < 0$, the values on the contours decrease as you move to the right. Since $f_y > 0$, the values on the contours increase as you move upward. See Figure 9.29.

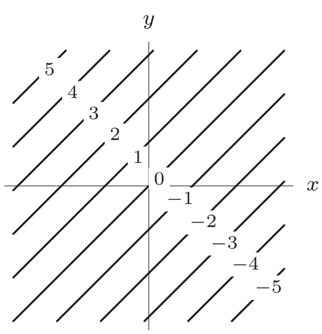


Figure 9.29: $f_x < 0$ and $f_y > 0$

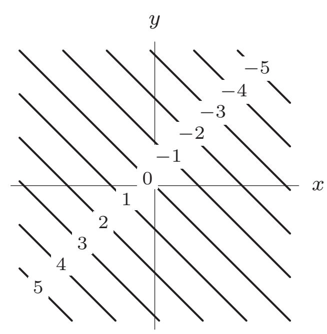


Figure 9.30: $f_x < 0$ and $f_y < 0$

(d) Since $f_x < 0$, the values on the contours decrease as you move to the right. Since $f_y < 0$, the values on the contours decrease as you move upward. See Figure 9.30.

24. The fact that $f_x(P) > 0$ tells us that the values of the function on the contours increase as we move to the right in Figure 9.6 past the point P. Thus, the values of the function on the contours

(a) Decrease as we move upward past P. Thus $f_y(P) < 0$.

(b) Decrease as we move upward past Q (since Q and P are on the same contour line.) Thus $f_y(Q) < 0$.

(c) Decrease as we move to the right past Q. Thus $f_x(Q) < 0$

Solutions for Section 9.4

1. $f_x(x, y) = 4x + 0 = 4x$
$f_y(x, y) = 0 + 6y = 6y$

2. $f_x(x, y) = 2 \cdot 100xy = 200xy$
$f_y(x, y) = 100x^2 \cdot 1 = 100x^2$

3. $f_x(x, y) = 2x + 2y$, $f_y(x, y) = 2x + 3y^2$.

4. $z_x = 2xy + 10x^4y$

5. $f_u(u, v) = 2u + 5v + 0 = 2u + 5v$
$f_v(u, v) = 0 + 5u + 2v = 5u + 2v$

6. $\dfrac{\partial z}{\partial x} = 2xe^y$

7. $\dfrac{\partial Q}{\partial p} = 5a^2 - 9ap^2$

8. $f_t(t, a) = 3 \cdot 5a^2t^2 = 15a^2t^2$.

9. $f_x(x, y) = 10xy^3 + 8y^2 - 6x$ and $f_y(x, y) = 15x^2y^2 + 16xy$.

10. $f_x(x, y) = 20xe^{3y}$, $f_y(x, y) = 30x^2e^{3y}$.

11. $\dfrac{\partial P}{\partial r} = 100te^{rt}$

12. $\dfrac{\partial A}{\partial h} = \dfrac{1}{2}(a + b)$

13. $\dfrac{\partial}{\partial m}\left(\dfrac{1}{2}mv^2\right) = \dfrac{1}{2}v^2$

14.

$$\begin{aligned} f(1, 2) &= (1)^3 + 3(2)^2 = 13 \\ f_x(x, y) &= 3x^2 + 0 \Rightarrow f_x(1, 2) = 3(1)^2 = 3 \\ f_y(x, y) &= 0 + 6y \Rightarrow f_y(1, 2) = 6(2) = 12 \end{aligned}$$

15.

$$\begin{aligned} f(3, 1) &= 5(3)(1)^2 = 15 \\ f_u(u, v) &= 5v^2 \Rightarrow f_u(3, 1) = 5(1)^2 = 5 \\ f_v(u, v) &= 10uv \Rightarrow f_v(3, 1) = 10(3)(1) = 30 \end{aligned}$$

16. **(a)** From contour diagram,

$$\begin{aligned} f_x(2, 1) &\approx \frac{f(2.3, 1) - f(2, 1)}{2.3 - 2} \\ &= \frac{6 - 5}{0.3} = 3.3, \\ f_y(2, 1) &\approx \frac{f(2, 1.4) - f(2, 1)}{1.4 - 1} \\ &= \frac{6 - 5}{0.4} = 2.5. \end{aligned}$$

(b) A table of values for f is given in Table 9.15.

Table 9.15

		y		
		0.9	1.0	1.1
x	1.9	4.42	4.61	4.82
	2.0	4.81	5.00	5.21
	2.1	5.22	5.41	5.62

From Table 9.15 we estimate $f_x(2,1)$ and $f_y(2,1)$ using difference quotients:

$$f_x(2,1) \approx \frac{5.41 - 5.00}{2.1 - 2} = 4.1$$
$$f_y(2,1) \approx \frac{5.21 - 5.00}{1.1 - 1} = 2.1.$$

We obtain better estimates by finer data in the table.

(c) $f_x(x,y) = 2x$, $f_y(x,y) = 2y$. So the true values are $f_x(2,1) = 4$, $f_y(2,1) = 2$.

17. To calculate $\partial B/\partial t$, we hold P and r constant and differentiate B with respect to t:

$$\frac{\partial B}{\partial t} = \frac{\partial}{\partial t}(Pe^{rt}) = Pre^{rt}.$$

In financial terms, $\partial B/\partial t$ represents the change in the amount of money in the bank as one unit of time passes by.

To calculate $\partial B/\partial r$, we hold t and P constant and differentiate with respect to r:

$$\frac{\partial B}{\partial r} = \frac{\partial}{\partial r}(Pe^{rt}) = Pte^{rt}.$$

In financial terms, $\partial B/\partial r$ represents the rate of change in the amount of money in the back as the interest rate changes.

To calculate $\partial B/\partial P$, we hold t and r constant and differentiate B with respect to P:

$$\frac{\partial B}{\partial P} = \frac{\partial}{\partial P}(Pe^{rt}) = e^{rt}.$$

In financial terms, $\partial B/\partial P$ represents the rate of change in the amount of money in the bank at time t as you change the amount of money that was initially deposited.

18. Differentiating with respect to d gives

$$\frac{\partial C}{\partial d} = 40 + 0 = \$40 \text{ per day},$$

so the cost of renting a car an additional day is \$40.
Differentiating with respect to m gives

$$\frac{\partial C}{\partial m} = 0 + 0.15 = \$0.15 \text{ per mile},$$

so the cost of driving the car an additional mile is \$0.15.

19. N is the number of workers, so $N = 80$. V is the value of equipment in units of \$25,000, so
$V = (\text{value of equipment})/\$25{,}000 = 30$.

$$f(80, 30) = 5(80)^{0.75}(30)^{0.25} = 313 \text{ tons.}$$

Using 80 workers and \$750,000 worth of equipment produces about 313 tons of output.

$$f_N(N,V) = 0.75 \cdot 5N^{-0.25}V^{0.25}, \qquad \text{so } f_N(80,30) = 0.75 \cdot 5(80)^{-0.25}(30)^{0.25} = 2.9 \text{ tons/worker.}$$

When the company is using 80 workers and \$750,000 worth of equipment, each additional worker would add about 2.9 tons to total output.

$$f_V(N,V) = 0.25 \cdot 5N^{0.75}V^{-0.75}, \qquad \text{so } f_V(80,30) = 0.25 \cdot 5(80)^{0.75}(30)^{-0.75} = 2.6 \text{ tons/\$25,000.}$$

When the company is using 80 workers and \$750,000 worth of equipment, each additional \$25,000 of equipment would add about 2.6 tons to total output.

20. $f_x = 2xy$ and $f_y = x^2$, so $f_{xx} = 2y$, $f_{xy} = 2x$, $f_{yy} = 0$ and $f_{yx} = 2x$.

21. $f_x = 2x + 2y$ and $f_y = 2x + 2y$, so $f_{xx} = 2$, $f_{xy} = 2$, $f_{yy} = 2$ and $f_{yx} = 2$.

22. $f_x = e^y$ and $f_y = xe^y$, so $f_{xx} = 0$, $f_{xy} = e^y$, $f_{yy} = xe^y$ and $f_{yx} = e^y$.

23. $f_x = 2/y$ and $f_y = -2x/y^2$, so $f_{xx} = 0$, $f_{xy} = -2/y^2$, $f_{yy} = 4x/y^3$ and $f_{yx} = -2/y^2$.

24. $f_x = 2xy^2$ and $f_y = 2x^2y$, so $f_{xx} = 2y^2$, $f_{xy} = 4xy$, $f_{yy} = 2x^2$ and $f_{yx} = 4xy$.

25. $f_x = ye^{xy}$ and $f_y = xe^{xy}$, so $f_{xx} = y^2e^{xy}$, $f_{xy} = xye^{xy}+e^{xy} = (xy+1)e^{xy}$, $f_{yy} = x^2e^{xy}$ and $f_{yx} = xye^{xy}+e^{xy} = (xy+1)e^{xy}$.

26. $Q_{p_1} = 10p_1p_2^{-1}$ and $Q_{p_2} = -5p_1^2p_2^{-2}$, so $Q_{p_1p_1} = 10p_2^{-1}$, $Q_{p_2p_2} = 10p_1^2p_2^{-3}$ and $Q_{p_1p_2} = Q_{p_2p_1} = -10p_1p_2^{-2}$.

27. $V_r = 2\pi rh$ and $V_h = \pi r^2$, so $V_{rr} = 2\pi h$, $V_{hh} = 0$ and $V_{rh} = V_{hr} = 2\pi r$.

28. $P_K = 2L^2$ and $P_L = 4KL$, so $P_{KK} = 0$, $P_{LL} = 4K$ and $P_{KL} = P_{LK} = 4L$.

29. $B_x = 5e^{-2t}$ and $B_t = -10xe^{-2t}$, so $B_{xx} = 0$, $B_{tt} = 20xe^{-2t}$ and $B_{xt} = B_{tx} = -10e^{-2t}$.

30. $f_x = -8xt$ and $f_t = 3t^2 - 4x^2$, so $f_{xx} = -8t$, $f_{tt} = 6t$ and $f_{xt} = f_{tx} = -8x$.

31. $f_r = 100te^{rt}$ and $f_t = 100re^{rt}$, so $f_{rr} = 100t^2e^{rt}$, $f_{rt} = f_{tr} = 100tre^{rt} + 100e^{rt} = 100(rt + 1)e^{rt}$ and $f_{tt} = 100r^2e^{rt}$.

32. Since $f_x(x, y) = 4x^3y^2 - 3y^4$, we could have

$$f(x, y) = x^4y^2 - 3xy^4.$$

In that case,

$$f_y(x, y) = \frac{\partial}{\partial y}(x^4y^2 - 3xy^4) = 2x^4y - 12xy^3$$

as expected. More generally, we could have $f(x, y) = x^4y^2 - 3xy^4 + C$, where C is any constant.

33. We compute the partial derivatives:

$$\frac{\partial Q}{\partial K} = b\alpha K^{\alpha-1}L^{1-\alpha} \quad \text{so} \quad K\frac{\partial Q}{\partial K} = b\alpha K^{\alpha}L^{1-\alpha}$$
$$\frac{\partial Q}{\partial L} = b(1-\alpha)K^{\alpha}L^{-\alpha} \quad \text{so} \quad L\frac{\partial Q}{\partial L} = b(1-\alpha)K^{\alpha}L^{1-\alpha}$$

Adding these two results, we have:

$$K\frac{\partial Q}{\partial K} + L\frac{\partial Q}{\partial L} = b(\alpha + 1 - \alpha)K^{\alpha}L^{1-\alpha} = Q.$$

34. **(a)** We have

$$\frac{\partial M}{\partial B} = \frac{c+1}{c+r}.$$

(b) Positive, because $c > 0$ and $r > 0$.

(c) If people and banks hold constant the ratio, c, of cash to checking deposits and the fraction, r, of checking account deposits held as cash, then increasing the amount of cash in an economy increases its money supply.

35. **(a)** Since $M = (c + 1)(c + r)^{-1}B$ we have

$$\frac{\partial M}{\partial r} = -(c+1)(c+r)^{-2}B = -\frac{c+1}{(c+r)^2}B.$$

(b) Negative, because $c > 0$, $r > 0$ and $B > 0$.

(c) Suppose that people hold a constant ratio of cash to checking account balances, and that the amount of cash in an economy is constant. If banks start to keep a greater fraction of their deposits in cash, then the money supply decreases.

36. **(a)** The quotient rule gives

$$\frac{\partial M}{\partial c} = \frac{(c+r)1 - (c+1)1}{(c+r)^2}B = -\frac{1-r}{(c+r)^2}B.$$

(b) Negative sign, because $c > 0$, $0 < r < 1$, and $B > 0$.

(c) Suppose that banks hold a constant ratio of cash to checking account balances, and that the amount of cash in an economy is constant. If people start to keep a greater fraction of their money in cash and less in checking accounts, then the money supply decreases.

Solutions for Section 9.5

1. We can identify local extreme points on contour plots because these points will be the centers of a series of concentric circles that close around them or will lie on the edges of the plot. Looking at the graph, we see that $(0,0)$, $(13,30)$, $(37,18)$, and $(32,34)$ appear to be such points. Since the points near $(0,0)$ increase in functional value as they close around $(0,0)$, $f(0,0)$ will be somewhat more than its nearest contour. So $f(0,0) \approx 12.5$. Similarly, since the contours near $(0,0)$ are less in functional value than $f(0,0)$, $f(0,0)$ is a local maximum. Applying analogous arguments to the other extreme points, we see that $f(13,30) \approx 4.5$ and is a local minimum; $f(37,18) \approx 2.5$ and is a local minimum; and $f(32,34) \approx 10.5$ and is a local maximum. Since none of the local maxima are greater in value than $f(0,0) \approx 12.5$, $f(0,0)$ is a global maximum. Since none of the local minima are lower in value than $f(37,18) \approx 2.5$, $f(37,18)$ is a global minimum.

2. We can identify local extreme points on a contour diagram because these points will either be the centers of a series of concentric circles that close around them, or will lie on the edges of the diagram. Looking at the graph, we see that $(2,10)$, $(6,4)$, $(6.5,16)$ and $(9,10)$ appear to be such points. Since the points near $(2,10)$ decrease in functional value as they close around $(2,10)$, $f(2,10)$ will be somewhat less than its nearest contour. So $f(2,10) \approx 0.5$. Similarly, since the contours near $(2,10)$ are greater in functional value than $f(2,10)$, $f(2,10)$ is a local minimum. Applying analogous arguments to the point $(6,4)$, we see that $f(6,4) \approx 9.5$ and is a local maximum. The contour values are increasing as we approach $(6.5,16)$ along any path, so $f(6.5,16) \approx 10$ is a local maximum and $(9,10)$ is a local minimum.

Since none of the local minima are less in value than $f(2,10) \approx 0.5$, $f(2,10)$ is a global minimum. Since none of the local maxima are greater in value than $f(6.5,16) \approx 10$, $f(6.5,16)$ is a global maximum.

3. The maximum value, which is about 11, occurs at $(5.1,4.9)$. The minimum value, which is about -1, occurs at $(1,3.9)$.

4. The maximum value, which is slightly above 30, say 30.5, occurs approximately at the origin. The minimum value, which is about 20.5, occurs at $(2.5,5)$.

5. The maxima occur at about $(\pi/2,0)$ and $(\pi/2,2\pi)$. The minimum occurs at $(\pi/2,\pi)$. The maximum value is about 1, the minimum value is about -1.

6. At a critical point $f_x = 2x+4 = 0$ and $f_y = 2y = 0$, so $(-2,0)$ is the only critical point. Since $f_{xx} = 2 > 0$ and $f_{xx}f_{yy} - f_{xy}^2 = 4 > 0$, the point $(-2,0)$ is a local minimum.

7. At a critical point $f_x = 2x+y = 0$ and $f_y = x+3 = 0$, so $(-3,6)$ is the only critical point. Since $f_{xx}f_{yy} - f_{xy}^2 = -1 < 0$, the point $(-3,6)$ is neither a local maximum nor a local minimum.

8. At a critical point $f_x = 2x+6 = 0$ and $f_y = 2y-10 = 0$, so $(-3,5)$ is the only critical point. Since $f_{xx} = 2 > 0$ and $f_{xx}f_{yy} - f_{xy}^2 = 4 > 0$, the point $(-3,5)$ is a local minimum.

9. At a critical point $f_x = -3y+6 = 0$ and $f_y = 3y^2 - 3x = 0$, so $(4,2)$ is the only critical point. Since $f_{xx}f_{yy} - f_{xy}^2 = -9 < 0$, the point $(4,2)$ is neither a local maximum nor a local minimum.

10. To find the critical points, we solve $f_x = 0$ and $f_y = 0$ for x and y. Solving

$$\begin{aligned} f_x &= 2x - 2y = 0, \\ f_y &= -2x + 6y - 8 = 0. \end{aligned}$$

We see from the first equation that $x = y$. Substituting this into the second equation shows that $y = 2$. The only critical point is $(2,2)$.

We have

$$D = (f_{xx})(f_{yy}) - (f_{xy})^2 = (2)(6) - (-2)^2 = 8.$$

Since $D > 0$ and $f_{xx} = 2 > 0$, the function f has a local minimum at the point $(2,2)$.

11. To find the critical points, we solve $f_x = 0$ and $f_y = 0$ for x and y. Solving

$$\begin{aligned} f_x &= 3x^2 - 3 = 0 \\ f_y &= 3y^2 - 3 = 0 \end{aligned}$$

shows that $x = \pm 1$ and $y = \pm 1$. There are four critical points: $(1,1)$, $(-1,1)$, $(1,-1)$ and $(-1,-1)$.

We have

$$D = (f_{xx})(f_{yy}) - (f_{xy})^2 = (6x)(6y) - (0)^2 = 36xy.$$

At the points $(1,-1)$ and $(-1,1)$, we have $D = -36 < 0$, so f has neither a local maximum nor a local minimum at $(1,-1)$ and $(-1,1)$. At $(1,1)$, we have $D = 36 > 0$ and $f_{xx} = 6 > 0$, so f has a local minimum at $(1,1)$. At $(-1,-1)$, we have $D = 36 > 0$ and $f_{xx} = -6 < 0$, so f has a local maximum at $(-1,-1)$.

12. To find the critical points, we solve $f_x = 0$ and $f_y = 0$ for x and y. Solving

$$f_x = 3x^2 - 6x = 0$$
$$f_y = 2y + 10 = 0$$

shows that $x = 0$ or $x = 2$ and $y = -5$. There are two critical points: $(0, -5)$ and $(2, -5)$.

We have

$$D = (f_{xx})(f_{yy}) - (f_{xy})^2 = (6x - 6)(2) - (0)^2 = 12x - 12.$$

When $x = 0$, we have $D = -12 < 0$, so f has neither a local maximum nor a local minimum at $(0, -5)$. When $x = 2$, we have $D = 12 > 0$ and $f_{xx} = 6 > 0$, so f has a local minimum at $(2, -5)$.

13. To find the critical points, we solve $f_x = 0$ and $f_y = 0$ for x and y. Solving

$$f_x = 3x^2 - 3 = 0$$
$$f_y = 3y^2 - 12y = 0$$

shows that $x = -1$ or $x = 1$ and $y = 0$ or $y = 4$. There are four critical points: $(-1, 0)$, $(1, 0)$, $(-1, 4)$, and $(1, 4)$.

We have

$$D = (f_{xx})(f_{yy}) - (f_{xy})^2 = (6x)(6y - 12) - (0)^2 = (6x)(6y - 12).$$

At critical point $(-1, 0)$, we have $D > 0$ and $f_{xx} < 0$, so f has a local maximum at $(-1, 0)$.
At critical point $(1, 0)$, we have $D < 0$, so f has neither a local maximum nor a local minimum at $(1, 0)$.
At critical point $(-1, 4)$, we have $D < 0$, so f has neither a local maximum nor a local minimum at $(-1, 4)$.
At critical point $(1, 4)$, we have $D > 0$ and $f_{xx} > 0$, so f has a local minimum at $(1, 4)$.

14. To find the critical points, we solve $f_x = 0$ and $f_y = 0$ for x and y. Solving

$$f_x = 3x^2 - 6x = 0$$
$$f_y = 3y^2 - 3 = 0$$

shows that $x = 0$ or $x = 2$ and $y = -1$ or $y = 1$. There are four critical points: $(0, -1)$, $(0, 1)$, $(2, -1)$, and $(2, 1)$.

We have

$$D = (f_{xx})(f_{yy}) - (f_{xy})^2 = (6x - 6)(6y) - (0)^2 = (6x - 6)(6y).$$

At the point $(0, -1)$, we have $D > 0$ and $f_{xx} < 0$, so f has a local maximum.
At the point $(0, 1)$, we have $D < 0$, so f has neither a local maximum nor a local minimum.
At the point $(2, -1)$, we have $D < 0$, so f has neither a local maximum nor a local minimum.
At the point $(2, 1)$, we have $D > 0$ and $f_{xx} > 0$, so f has a local minimum.

15. The partial derivatives are

$$f_x = -6x - 4 + 2y \quad \text{and} \quad f_y = 2x - 10y + 48.$$

Set $f_x = 0$ and $f_y = 0$ to find the critical point, thus

$$2y - 6x = 4 \quad \text{and} \quad 10y - 2x = 48.$$

Now, solve these equations simultaneously to obtain $x = 1$ and $y = 5$.

Since $f_{xx} = -6$, $f_{yy} = -10$ and $f_{xy} = 2$ for all (x, y), at $(1, 5)$ the discriminant $D = (-6)(-10) - (2)^2 = 56 > 0$ and $f_{xx} < 0$. Thus $f(x, y)$ has a local maximum value at $(1, 5)$.

16. Mississippi lies entirely within a region designated as 80s so we expect both the maximum and minimum daily high temperatures within the state to be in the 80s. The southwestern-most corner of the state is close to a region designated as 90s, so we would expect the temperature here to be in the high 80s, say 87-88. The northern-most portion of the state is located near the center of the 80s region. We might expect the high temperature there to be between 83-87.

Alabama also lies completely within a region designated as 80s so both the high and low daily high temperatures within the state are in the 80s. The southeastern tip of the state is close to a 90s region so we would expect the temperature here to be about 88-89 degrees. The northern-most part of the state is near the center of the 80s region so the temperature there is 83-87 degrees.

Pennsylvania is also in the 80s region, but it is touched by the boundary line between the 80s and a 70s region. Thus we expect the low daily high temperature to occur there and be about 80 degrees. The state is also touched by a boundary line of a 90s region so the high will occur there and be 89-90 degrees.

New York is split by a boundary between an 80s and a 70s region, so the northern portion of the state is likely to be about 74-76 while the southern portion is likely to be in the low 80s, maybe 81-84 or so.

California contains many different zones. The northern coastal areas will probably have the daily high as low as 65-68, although without another contour on that side, it is difficult to judge how quickly the temperature is dropping off to the west. The tip of Southern California is in a 100s region, so there we expect the daily high to be 100-101.

Arizona will have a low daily high around 85-87 in the northwest corner and a high in the 100s, perhaps 102-107 in its southern regions.

Massachusetts will probably have a high daily high around 81-84 and a low daily high of 70.

17. At a local maximum value of f,

$$\frac{\partial f}{\partial x} = -2x - B = 0.$$

We are told that this is satisfied by $x = -2$. So $-2(-2) - B = 0$ and $B = 4$. In addition,

$$\frac{\partial f}{\partial y} = -2y - C = 0$$

and we know this holds for $y = 1$, so $-2(1) - C = 0$, giving $C = -2$. We are also told that the value of f is 15 at the point $(-2, 1)$, so

$$15 = f(-2,1) = A - ((-2)^2 + 4(-2) + 1^2 - 2(1)) = A - (-5), \text{ so } A = 10.$$

Now we check that these values of A, B, and C give $f(x, y)$ a local maximum at the point $(-2, 1)$. Since

$$f_{xx}(-2,1) = -2,$$

$$f_{yy}(-2,1) = -2$$

and

$$f_{xy}(-2,1) = 0,$$

we have that $f_{xx}(-2,1)f_{yy}(-2,1) - f_{xy}^2(-2,1) = (-2)(-2) - 0 > 0$ and $f_{xx}(-2,1) < 0$. Thus, f has a local maximum value 15 at $(-2, 1)$.

18. We calculate the partial derivatives and set them to zero.

$$\frac{\partial\,(\text{range})}{\partial t} = -10t - 6h + 400 = 0$$

$$\frac{\partial\,(\text{range})}{\partial h} = -6t - 6h + 300 = 0.$$

$$10t + 6h = 400$$

$$6t + 6h = 300$$

solving we obtain

$$4t = 100$$

so

$$t = 25$$

Solving for h, we obtain $6h = 150$, yielding $h = 25$. Since the range is quadratic in h and t, the second derivative test tells us this is a local and global maximum. So the optimal conditions are $h = 25\%$ humidity and $t = 25^\circ$C.

19. **(a)** This tells us that an increase in the price of either product causes a decrease in the quantity demanded of both products. An example of products with this relationship is tennis rackets and tennis balls. An increase in the price of either product is likely to lead to a decrease in the quantity demanded of both products as they are used together. In economics, it is rare for the quantity demanded of a product to increase if its price increases, so for q_1, the coefficient of p_1 is negative as expected. The coefficient of p_2 in the expression could be either negative or positive. In this case, it is negative showing that the two products are complementary in use. If it were positive, however, it would indicate that the two products are competitive in use, for example Coke and Pepsi.

(b) The revenue from the first product would be $q_1p_1 = 150p_1 - 2p_1^2 - p_1p_2$, and the revenue from the second product would be $q_2p_2 = 200p_2 - p_1p_2 - 3p_2^2$. The total sales revenue of both products, R, would be

$$R(p_1, p_2) = 150p_1 + 200p_2 - 2p_1p_2 - 2p_1^2 - 3p_2^2.$$

Note that R is a function of p_1 and p_2. To find the critical points of R, we solve

$$\frac{\partial R}{\partial p_1} = \frac{\partial R}{\partial p_2} = 0.$$

This gives
$$\frac{\partial R}{\partial p_1} = 150 - 2p_2 - 4p_1 = 0$$
and
$$\frac{\partial R}{\partial p_2} = 200 - 2p_1 - 6p_2 = 0$$
Solving simultaneously, we have $p_1 = 25$ and $p_2 = 25$. Therefore the point $(25, 25)$ is a critical point for R. Further,
$$\frac{\partial^2 R}{\partial p_1^2} = -4, \frac{\partial^2 R}{\partial p_2^2} = -6, \frac{\partial^2 R}{\partial p_1 \partial p_2} = -2,$$
so
$$\frac{\partial^2 R}{\partial^2 P_1^2}\frac{\partial^2 R}{\partial^2 P_2^2} - \left(\frac{\partial^2 R}{\partial P_1 \partial P_2}\right)^2 = (-4)(-6) - (-2)^2 = 20.$$
Since $D > 0$ and $\partial^2 R/\partial p_1^2 < 0$, this critical point is a local maximum. Since R is quadratic in p_1 and p_2, this is a global maximum. Therefore the maximum possible revenue is
$$\begin{aligned} R &= 150(25) + 200(25) - 2(25)(25) - 2(25)^2 - 3(25)^2 \\ &= (6)(25)^2 + 8(25)^2 - 7(25)^2 \\ &= 4375. \end{aligned}$$

This is obtained when $p_1 = p_2 = 25$. Note that at these prices, $q_1 = 75$ units, and $q_2 = 100$ units.

20. **(a)** The revenue $R = p_1q_1 + p_2q_2$. Profit $= P = R - C = p_1q_1 + p_2q_2 - 2q_1^2 - 2q_2^2 - 10$.
$$\begin{aligned} \frac{\partial P}{\partial q_1} &= p_1 - 4q_1 = 0 \quad \text{gives } q_1 = \frac{p_1}{4} \\ \frac{\partial P}{\partial q_2} &= p_2 - 4q_2 = 0 \quad \text{gives } q_2 = \frac{p_2}{4} \end{aligned}$$

Since $\frac{\partial^2 P}{\partial q_1^2} = -4$, $\frac{\partial^2 P}{\partial q_2^2} = -4$ and $\frac{\partial^2 P}{\partial q_1 \partial q_2} = 0$, at $(p_1/4, p_2/4)$ we have that the discriminant, $D = (-4)(-4) > 0$ and $\frac{\partial^2 P}{\partial q_1^2} < 0$, thus P has a local maximum value at $(q_1, q_2) = (p_1/4, p_2/4)$. Since P is quadratic in q_1 and q_2, this is a global maximum. So the maximum profit is
$$P = \frac{p_1^2}{4} + \frac{p_2^2}{4} - 2\frac{p_1^2}{16} - 2\frac{p_2^2}{16} - 10 = \frac{p_1^2}{8} + \frac{p_2^2}{8} - 10.$$

(b) The rate of change of the maximum profit as p_1 increases is
$$\frac{\partial}{\partial p_1}(\max\ P) = \frac{2p_1}{8} = \frac{p_1}{4}.$$

21. The total revenue is
$$R = pq = (60 - 0.04q)q = 60q - 0.04q^2,$$
and as $q = q_1 + q_2$, this gives
$$R = 60q_1 + 60q_2 - 0.04q_1^2 - 0.08q_1q_2 - 0.04q_2^2.$$

Therefore, the profit is
$$\begin{aligned} P(q_1, q_2) &= R - C_1 - C_2 \\ &= -13.7 + 60q_1 + 60q_2 - 0.07q_1^2 - 0.08q_2^2 - 0.08q_1q_2. \end{aligned}$$

At a local maximum point, we have:
$$\begin{aligned} \frac{\partial P}{\partial q_1} &= 60 - 0.14q_1 - 0.08q_2 = 0, \\ \frac{\partial P}{\partial q_2} &= 60 - 0.16q_2 - 0.08q_1 = 0. \end{aligned}$$

Solving these equations, we find that

$$q_1 = 300 \quad \text{and} \quad q_2 = 225.$$

To see whether or not we have found a local maximum, we compute the second-order partial derivatives:

$$\frac{\partial^2 P}{\partial q_1^2} = -0.14, \quad \frac{\partial^2 P}{\partial q_2^2} = -0.16, \quad \frac{\partial^2 P}{\partial q_1 \partial q_2} = -0.08.$$

Therefore,

$$D = \frac{\partial^2 P}{\partial q_1^2}\frac{\partial^2 P}{\partial q_2^2} - \frac{\partial^2 P}{\partial q_1 \partial q_2} = (-0.14)(-0.16) - (-0.08)^2 = 0.016,$$

and so we have found a local maximum point. The graph of $P(q_1, q_2)$ has the shape of an upside down paraboloid since P is quadratic in q_1 and q_2, hence $(300, 225)$ is a global maximum point.

Solutions for Section 9.6

1. We wish to optimize $f(x,y) = x^2 + 4xy$ subject to the constraint $g(x,y) = x + y = 100$. To do this we must solve the following system of equations:

$$\begin{aligned} f_x(x,y) &= \lambda g_x(x,y), \qquad \text{so } 2x + 4y = \lambda \\ f_y(x,y) &= \lambda g_y(x,y), \qquad \text{so } 4x = \lambda \\ g(x,y) &= 100, \qquad \text{so } x + y = 100 \end{aligned}$$

Solving these equations produces:

$$x \approx 66.7 \quad y \approx 33.3 \quad \lambda \approx 266.8$$

corresponding to optimal $f(x,y) \approx (66.7)^2 + 4(66.7)(33.3) \approx 13{,}333$.

2. We wish to optimize $f(x,y) = xy$ subject to the constraint $g(x,y) = 5x + 2y = 100$. To do this we must solve the following system of equations:

$$\begin{aligned} f_x(x,y) &= \lambda g_x(x,y), \qquad \text{so } y = 5\lambda \\ f_y(x,y) &= \lambda g_y(x,y), \qquad \text{so } x = 2\lambda \\ g(x,y) &= 100, \qquad \text{so } 5x + 2y = 100 \end{aligned}$$

We substitute in the third equation to obtain $5(2\lambda) + 2(5\lambda) = 100$, so $\lambda = 5$. Thus,

$$x = 10 \quad y = 25 \quad \lambda = 5$$

corresponding to optimal $f(x,y) = (10)(25) = 250$.

3. We wish to optimize $f(x,y) = x^2 + 3y^2 + 100$ subject to the constraint $g(x,y) = 8x + 6y = 88$. To do this we must solve the following system of equations:

$$\begin{aligned} f_x(x,y) &= \lambda g_x(x,y), \qquad \text{so } x = 4\lambda \\ f_y(x,y) &= \lambda g_y(x,y), \qquad \text{so } y = \lambda \\ g(x,y) &= 88, \qquad \text{so } 8x + 6y = 88 \end{aligned}$$

Solving these equations produces:

$$x = 9.26 \quad y = 2.32 \quad \lambda = 2.32$$

corresponding to optimal $f(x,y) = (9.26)^2 + 3(2.32)^2 + 100 = 201.9$.

4. We wish to optimize $f(x,y) = 5xy$ subject to the constraint $g(x,y) = x + 3y = 24$. To do this we must solve the following system of equations:

$$\begin{aligned} f_x(x,y) &= \lambda g_x(x,y), \ \text{so } 5y = \lambda \\ f_y(x,y) &= \lambda g_y(x,y), \ \text{so } 5x = 3\lambda \\ g(x,y) &= 24, \ \text{so } x + 3y = 24 \end{aligned}$$

Solving these equations produces:

$$x = 12 \quad y = 4 \quad \lambda = 20$$

corresponding to optimal $f(x,y) = 5(12)(4) = 240$.

5. Our objective function is $f(x, y) = x + y$ and our equation of constraint is $g(x, y) = x^2 + y^2 = 1$. To optimize $f(x, y)$ with Lagrange multipliers, we solve the following system of equations

$$f_x(x, y) = \lambda g_x(x, y), \qquad \text{so } 1 = 2\lambda x$$
$$f_y(x, y) = \lambda g_y(x, y), \qquad \text{so } 1 = 2\lambda y$$
$$g(x, y) = 1, \qquad \text{so } x^2 + y^2 = 1$$

Solving for λ gives

$$\lambda = \frac{1}{2x} = \frac{1}{2y},$$

which tells us that $x = y$. Going back to our equation of constraint, we use the substitution $x = y$ to solve for y:

$$\begin{aligned} g(y, y) = y^2 + y^2 &= 1 \\ 2y^2 &= 1 \\ y^2 &= \frac{1}{2} \\ y &= \pm\sqrt{\frac{1}{2}} = \pm\frac{\sqrt{2}}{2}. \end{aligned}$$

Since $x = y$, our critical points are $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ and $(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$. Since the constraint is closed and bounded, maximum and minimum values of f subject to the constraint exist. Evaluating f at the critical points we find that the maximum value is $f(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) = \sqrt{2}$ and the minimum value is $f(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}) = -\sqrt{2}$.

6. The objective function is $f(x, y) = x^2 + y^2$ and the constraint equation is $g(x, y) = 4x - 2y = 15$, so $f_x = 2x$, $f_y = 2y$ and $g_x = 4$, $g_y = -2$. We have:

$$2x = 4\lambda,$$
$$2y = -2\lambda.$$

From the first equation we have $\lambda = x/2$, and from the second equation we have $\lambda = -y$. Setting these equal gives

$$y = -0.5x.$$

Substituting this into the constraint equation $4x - 2y = 15$ gives $x = 3$. The only critical point is $(3, -1.5)$.

We have $f(3, -1.5) = (3)^2 + (1.5)^2 = 11.25$. One way to determine if this point gives a maximum or minimum value or neither for the given constraint is to examine the contour diagram of f with the constraint sketched in, Figure 9.31. It appears that moving away from the point $P = (3, -1.5)$ in either direction along the constraint increases the value of f, so $(3, -1.5)$ is a point of minimum value.

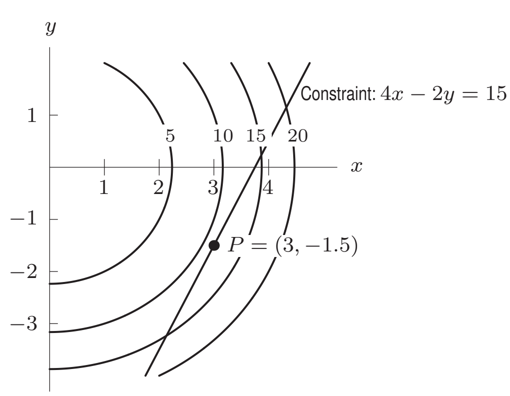


Figure 9.31

7. Our objective function is $f(x, y) = 3x - 2y$ and our equation of constraint is $g(x, y) = x^2 + 2y^2 = 44$. To optimize $f(x, y)$ with Lagrange multipliers we solve the following system of equations

$$\begin{aligned} f_x(x, y) &= \lambda g_x(x, y), \qquad \text{so } 3 = 2\lambda x \\ f_y(x, y) &= \lambda g_y(x, y), \qquad \text{so } -2 = 4\lambda y \\ g(x, y) &= 44, \qquad \text{so } x^2 + y^2 = 44 \end{aligned}$$

Solving for λ gives us

$$\lambda = \frac{3}{2x} = \frac{-2}{4y},$$

which we can use to find x in terms of y:

$$\begin{aligned} \frac{3}{2x} &= \frac{-2}{4y} \\ -4x &= 12y \\ x &= -3y. \end{aligned}$$

Using this relation in our equation of constraint, we can solve for y:

$$\begin{aligned} x^2 + 2y^2 &= 44 \\ (-3y)^2 + 2y^2 &= 44 \\ 9y^2 + 2y^2 &= 44 \\ 11y^2 &= 44 \\ y^2 &= 4 \\ y &= \pm 2. \end{aligned}$$

Thus, the critical points are $(-6, 2)$ and $(6, -2)$. Since the constraint is closed and bounded, maximum and minimum values of f subject to the constraint exist. Evaluating f at the critical points, we find that the maximum is $f(6, -2) = 18 + 4 = 22$ and the minimum value is $f(-6, 2) = -18 - 4 = -22$.

8. Our objective function is $f(x, y) = x^2 + y$ and our equation of constraint is $g(x, y) = x^2 - y^2 = 1$. We have

$$\begin{aligned} f_x &= 2x, \qquad f_y = 1 \\ g_x &= 2x, \qquad g_y = -2y. \end{aligned}$$

The Lagrange multiplier equations are

$$\begin{aligned} 2x &= \lambda 2x \\ 1 &= -\lambda 2y \end{aligned}$$

But x cannot be zero, since the constraint equation, $-y^2 = 1$, would then have no real solution for y. Thus, the first equation gives $\lambda = 1$. Then the second equation gives

$$\begin{aligned} -2y &= 1 \\ y &= -\frac{1}{2}. \end{aligned}$$

Substituting this into our equation of constraint we find

$$\begin{aligned} g(x, -\frac{1}{2}) = x^2 - \left(-\frac{1}{2}\right)^2 &= 1 \\ x^2 &= \frac{5}{4} \\ x &= \pm\frac{\sqrt{5}}{2}. \end{aligned}$$

So the critical points are $(\frac{\sqrt{5}}{2}, -\frac{1}{2})$ and $(-\frac{\sqrt{5}}{2}, -\frac{1}{2})$. Evaluating f at these points we find $f(\frac{\sqrt{5}}{2}, -\frac{1}{2}) = f(-\frac{\sqrt{5}}{2}, -\frac{1}{2}) = \frac{5}{4} - \frac{1}{2} = \frac{3}{4}$. This is the minimum value for $f(x, y)$ constrained to $g(x, y) = 1$. To see this, note that for $x^2 = y^2 + 1$, $f(x, y) = y^2 + 1 + y = (y + 1/2)^2 + 3/4 \geq 3/4$. Alternatively, see Figure 9.32. To see that f has no maximum on $g(x, y) = 1$, note that $f \to \infty$ as $x \to \infty$ and $y \to \infty$ on the part of the graph of $g(x, y) = 1$ in quadrant I.

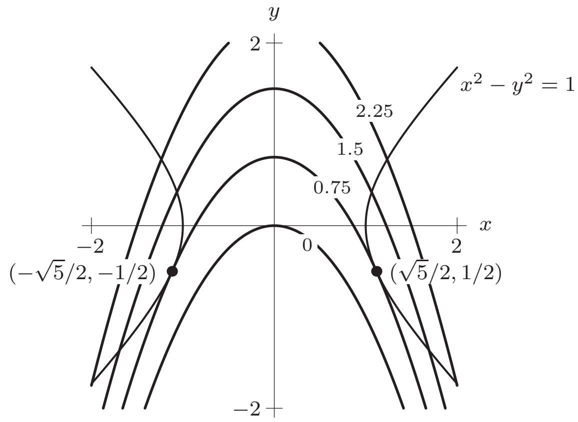


Figure 9.32: Graph of $x^2-y^2=1$

9. Our objective function is $f(x,y)=xy$ and our equation of constraint is $g(x,y)=4x^2+y^2=8$. Their partial derivatives are

$$f_x=y,\qquad f_y=x$$
$$g_x=8x,\qquad g_y=2y.$$

This gives

$$8x\lambda=y \quad\text{and}\quad 2y\lambda=x.$$

Multiplying, we get

$$8x^2\lambda=2y^2\lambda.$$

If $\lambda=0$, then $x=y=0$, which does not satisfy the constraint equation. So $\lambda\neq 0$ and we get

$$\begin{aligned} 2y^2&=8x^2\\ y^2&=4x^2\\ y&=\pm 2x. \end{aligned}$$

To find x, we substitute for y in our equation of constraint.

$$\begin{aligned} 4x^2+y^2&=8\\ 4x^2+4x^2&=8\\ x^2&=1\\ x&=\pm 1 \end{aligned}$$

So our critical points are $(1,2)$, $(1,-2)$, $(-1,2)$ and $(-1,-2)$. Evaluating $f(x,y)$ at the critical points, we have

$$f(1,2)=f(-1,-2)=2$$
$$f(1,-2)=f(1,-2)=-2.$$

Thus, the maximum value of f on $g(x,y)=8$ is 2, and the minimum value is -2.

10. The objective function is $f(x,y)=x^2+y^2$ and the equation of constraint is $g(x,y)=x^4+y^4=2$. Their partial derivatives are

$$f_x=2x,\qquad f_y=2y$$
$$g_x=4x^3,\qquad g_y=4y^3.$$

We have

$$2x=4\lambda x^3,$$
$$2y=4\lambda y^3.$$

Now if $x = 0$, the first equation is true for any value of λ. In particular, we can choose λ which satisfies the second equation. Similarly, $y = 0$ is solution.

Assuming both $x \neq 0$ and $y \neq 0$, we can divide to solve for λ and find

$$\begin{aligned}\lambda = \frac{2x}{4x^3} &= \frac{2y}{4y^3}\\ \frac{1}{2x^2} &= \frac{1}{2y^2}\\ y^2 &= x^2\\ y &= \pm x.\end{aligned}$$

Going back to our equation of constraint, we find

$$\begin{aligned}g(0,y) &= 0^4 + y^4 = 2, \quad \text{so } y = \pm\sqrt[4]{2}\\ g(x,0) &= x^4 + 0^4 = 2, \quad \text{so } x = \pm\sqrt[4]{2}\\ g(x,\pm x) &= x^4 + (\pm x)^4 = 2, \quad \text{so } x = \pm 1.\end{aligned}$$

Thus, the critical points are $(0, \pm\sqrt[4]{2})$, $(\pm\sqrt[4]{2}, 0)$, $(1, \pm 1)$ and $(-1, \pm 1)$. Evaluating f at the critical points, we find

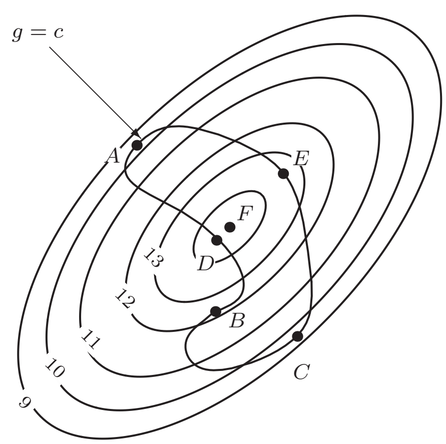
$$\begin{aligned}f(1,1) = f(1,-1) = f(-1,1) = f(-1,-1) = 2,\\ f(0,\sqrt[4]{2}) = f(0,-\sqrt[4]{2}) = f(\sqrt[4]{2},0) = f(-\sqrt[4]{2},0) = \sqrt{2}.\end{aligned}$$

Thus, the minimum value of $f(x,y)$ on $g(x,y) = 2$ is $\sqrt{2}$ and the maximum value is 2.

11. To maximize $f(x,y)$ subject to $g(x,y) = c$, we wish to find the highest possible value of $f(x,y)$ where (x,y) is on the line corresponding to our constraint. From our constraint, we see that $f = 400$ is the highest curve that intersects the constraint line. We can verify this either by noting that all higher contours lie above the constraint $g(x,y) = c$ or by noting that the contour $f = 400$ is tangent to the constraint. The point of intersection, from looking at the graph, occurs at approximately $x = 6, y = 6$, giving us maximal $f(x,y) = 400$.

12. **(a)** Point F since the value of f is greatest at this point.

(b) At points A, B, C, D, E, the level curve of f and the constraint curve are parallel. Point D has the greatest f value of these points.



13. **(a)** Objective function: $C = 127x_1 + 92x_2$.

(b) Constraint: $x_1^{0.6}x_2^{0.4} = 500$.

14. We want to minimize cost $C = 100L + 200K$ subject to $Q = 900L^{1/2}K^{2/3} = 36000$. We solve the system of equations:

$$\begin{aligned}C_L &= \lambda Q_L, \quad &&\text{so } 100 = \lambda 450L^{-1/2}K^{2/3}\\ C_K &= \lambda Q_K, \quad &&\text{so } 200 = \lambda 600L^{1/2}K^{-1/3}\\ Q &= 36000, \quad &&\text{so } 900L^{1/2}K^{2/3} = 36000\end{aligned}$$

Since $\lambda \neq 0$ this gives

$$450L^{-1/2}K^{2/3} = 300L^{1/2}K^{-1/3}.$$

Solving, we get $L = (3/2)K$. Substituting into $Q = 36{,}000$ gives

$$900\left(\frac{3}{2}K\right)^{1/2}K^{2/3} = 36{,}000.$$

Solving yields $K = \left[40 \cdot \left(\frac{2}{3}\right)^{1/2}\right]^{6/7} \approx 19.85$, so $L \approx \frac{3}{2}(19.85) = 29.78$. We can thus calculate cost using $K = 20$ and $L = 30$ which gives $C = \$7,000$.

15. **(a)** Because of budget constraints, we are limited in the size of the labor force and the amount of total equipment. This constraint is described in the formula

$$300L + 100K = 15{,}000$$

We can let L range from 0 to about 50. Each choice of L determines a choice of K. We have, as a result, the Table 9.16.

Table 9.16

L	5	10	15	20	25	30	35	40	45	50
K	135	120	105	90	75	60	45	30	15	0
P	48	82	111	135	156	172	184	189	181	0

(b) We wish to maximize P subject to that cost C satisfies $C = 300L + 100K = \$15{,}000$. This is accomplished by solving the following system of equations:

$$\begin{aligned} P_L &= \lambda C_L, &&\text{so } 4L^{-0.2}K^{0.2} = 300\lambda \\ P_K &= \lambda C_K, &&\text{so } L^{0.8}K^{-0.8} = 100\lambda \\ C &= 15{,}000, &&\text{so } 300L + 100K = 15{,}000 \end{aligned}$$

We can solve these equations by dividing the first by the second and then substituting into the budget constraint. Doing so produces the solution $K = 30$, $L = 40$. So the optimal choices of labor and capital are $L = 40$, $K = 30$.

16. We want to minimize

$$C = f(q_1, q_2) = 2q_1^2 + q_1q_2 + q_2^2 + 500$$

subject to the constraint $q_1 + q_2 = 200$ or $g(q_1,\, q_2) = q_1 + q_2 = 200$.

We solve the system of equations:

$$\begin{aligned} C_{q_1} &= \lambda g_{q_1}, &&\text{so } 4q_1 + q_2 = \lambda \\ C_{q_2} &= \lambda g_{q_2}, &&\text{so } 2q_2 + q_1 = \lambda \\ g &= 200, &&\text{so } q_1 + q_2 = 200. \end{aligned}$$

Solving we get

$$4q_1 + q_2 = 2q_2 + q_1$$

so

$$3q_1 = q_2.$$

We want

$$q_1 + q_2 = 200$$
$$q_1 + 3q_1 = 4q_1 = 200.$$

Therefore

$$q_1 = 50 \text{ units}, \quad q_2 = 150 \text{ units}.$$

17. (a) To be producing the maximum quantity Q under the cost constraint given, the firm should be using K and L values given by

$$\begin{aligned}\frac{\partial Q}{\partial K} &= 0.6aK^{-0.4}L^{0.4} = 20\lambda \\ \frac{\partial Q}{\partial L} &= 0.4aK^{0.6}L^{-0.6} = 10\lambda \\ 20K + 10L &= 150.\end{aligned}$$

Hence $\dfrac{0.6aK^{-0.4}L^{0.4}}{0.4aK^{0.6}L^{-0.6}} = 1.5\dfrac{L}{K} = \dfrac{20\lambda}{10\lambda} = 2$, so $L = \dfrac{4}{3}K$. Substituting in $20K + 10L = 150$, we obtain $20K + 10\left(\dfrac{4}{3}\right)K = 150$. Then $K = \dfrac{9}{2}$ and $L = 6$, so capital should be reduced by $\dfrac{1}{2}$ unit, and labor should be increased by 1 unit.

(b) $\dfrac{\text{New production}}{\text{Old production}} = \dfrac{a4.5^{0.6}6^{0.4}}{a5^{0.6}5^{0.4}} \approx 1.01$, so tell the board of directors, "Reducing the quantity of capital by $1/2$ unit and increasing the quantity of labor by 1 unit will increase production by 1% while holding costs to \$150."

18. The value of λ tells us how the optimum value of f changes when we change the constraint. In particular, if the constraint increases by 1, the optimum value of f increases by λ.

(a) So, if we raise the quota by 1 product, cost rises by $\lambda = 15$. So $C = 1200 + 15 = 1215$.
(b) Similarly, if we lower the quota by 1, cost falls by $\lambda = 15$. So $C = 1200 - 15 = 1185$.

19. (a) The objective function is the one we are trying to optimize, so $P(x, y)$ is the objective function. The constraint equations tells us what condition must be satisfied; here, we must have costs equal to \$50,000, so $C(x, y) = 50{,}000$ is the constraint equation. The meaning of λ is how much the objective would increase if the budget available increased by 1 and we optimized again with the new budget. Here, this means how much optimal production would increase if the budget increased to \$50,001.

(b) Here, we are trying to minimize cost, so $C(x, y)$ is the objective function. Since we must satisfy $P(x, y) = 2000$, our constraint equation becomes $P(x, y) = 2000$. In this situation, λ represents the change in the minimal cost when production is increased by one unit to 2001.

20. (a) The objective function is the function that you want to maximize, that is, your grade as a function of the time spent on each project. Its units are points.
(b) The constraint is the condition that limits your options. The limitation is the amount of time set aside, 20 hours.
(c) The Lagrange multiplier tells you the rate at which the grade changes when the total time to work on the projects is increased. Its units are points per hour.
(d) Working 21 hours instead of 20 hours will improve your grade by approximately 5 points.

21. (a) The objective function is the function that is optimized. Since the problem refers to maximum production, the objective function is the production function $P(K, L)$.
(b) Production is maximized subject to a budget restriction, which is the constraint. The constraint equation is $C(K, L) = 600{,}000$.
(c) The Lagrange multiplier tells you the rate at which maximum production changes when the budget is increased. Its units are tons of steel per dollar of budget, or simply tons/dollar.
(d) Increasing the budget from \$600,000 to \$$(600{,}000+a)$ increases the maximum possible production from 2,500,000 tons to approximately $(2{,}500{,}000 + 3.17a)$ tons. Every extra dollar of budget increases maximal production by approximately $\lambda = 3.17$ tons.

22. (a) We wish to maximize q subject to the constraint that cost $C = 10W + 20K = 3000$. To optimize q according to this, we must solve the following system of equations:

$$\begin{aligned} q_W &= \lambda C_W, &\quad& \text{so } \frac{9}{2}W^{-\frac{1}{4}}K^{\frac{1}{4}} = \lambda 10 \\ q_K &= \lambda C_K, && \text{so } \frac{3}{2}W^{\frac{3}{4}}K^{-\frac{3}{4}} = \lambda 20 \\ C &= 3000, && \text{so } 10W + 20K = 3000 \end{aligned}$$

Dividing yields $K = \frac{1}{6}W$, so substituting into C gives

$$10W + 20\left(\frac{1}{6}W\right) = \frac{40}{3}W = 3000.$$

Thus $W = 225$ and $K = 37.5$. Substituting both answers to find λ gives

$$\lambda = \frac{\frac{9}{2}(225)^{-\frac{1}{4}}(37.5)^{\frac{1}{4}}}{10} = 0.2875.$$

We also find the optimum quantity produced, $q = 6(225)^{\frac{3}{4}}(37.5)^{\frac{1}{4}} = 862.57$.

(b) When the budget is increased by one dollar, we substitute the relation $K_1 = \frac{1}{6}W_1$ into $10W_1 + 20K_1 = 3001$ which gives $10W_1 + 20(\frac{1}{6}W_1) = \frac{40}{3}W_1 = 3001$. Solving yields $W_1 = 225.075$ and $K_1 = 37.513$, so $q_1 = 862.86 = q + 0.29$. Thus production has increased by $0.29 \approx \lambda$, the Lagrange Multiplier.

23. (a) The objective function is the function that you want to minimize, the quantity of fuel. Since we put x_1 liters in the first stage of the rocket and x_2 liters into the second, the objective function is $f(x_1, x_2) = x_1 + x_2$. Its units are liters.

(b) The constraint is the condition that limits your options. The limitation is that the fuel x_1 and x_2 used in the two stages must produce a given terminal velocity. If $g(x_1, x_2)$ denotes the terminal velocity (in meters/second) when using x_1 liters in the first stage and x_2 liters in the second stage and we want to attain velocity v_0 meters/sec, then the constraint is the equation $g(x_1, x_2) = v_0$.

(c) The Lagrange multiplier tells you the rate at which minimum fuel required changes when the terminal velocity is increased. Its units are liters of fuel per meters/second.

(d) It takes approximately 8 liters more fuel to reach terminal velocity 51 meters/second than to reach 50 meters/second.

24. (a) The curves are shown in Figure 9.33.

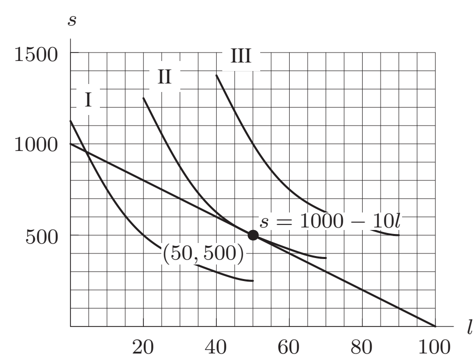


Figure 9.33

(b) The income equals \$10/hour times the number of hours of work:

$$s = 10(100 - l) = 1000 - 10l.$$

(c) The graph of this constraint is the straight line in Figure 9.33.

(d) For any given salary, curve III allows for the most leisure time, curve I the least. Similarly, for any amount of leisure time, curve III also has the greatest salary, and curve I the least. Thus, any point on curve III is preferable to any point on curve II, which is preferable to any point on curve I. We prefer to be on the outermost curve that our constraint allows. We want to choose the point on $s = 1000 - 10l$ which is on the most preferable curve. Since all the curves are concave up, this occurs at the point where $s = 1000 - 10l$ is *tangent* to curve II. So we choose $l = 50$, $s = 500$, and work 50 hours a week.

25. The constraint is

$$g(x_1, x_2) = x_1 + 3x_2 = 100.$$

We need to solve the equations

$$\frac{\partial U}{\partial x_1} = \lambda\frac{\partial g}{\partial x_1}, \qquad \frac{\partial U}{\partial x_2} = \lambda\frac{\partial g}{\partial x_2}, \qquad x_1 + 3x_2 = 100.$$

These equations are

$$\begin{aligned} 2x_2 + 3 &= \lambda \\ 2x_1 &= 3\lambda \\ x_1 + 3x_2 &= 100. \end{aligned}$$

The first equation gives

$$x_2 = \frac{\lambda - 3}{2}$$

and the second that

$$x_1 = \frac{3\lambda}{2}.$$

Substituting these into the third equation gives

$$\frac{3\lambda}{2} + 3\left(\frac{\lambda - 3}{2}\right) = 100,$$

therefore, $\lambda = 209/6$ so $x_1 = 209/4$ and $x_2 = 191/12$. The maximum utility is

$$U\left(\frac{209}{4}, \frac{191}{12}\right) = 2\frac{209}{4}\frac{191}{12} + 3\frac{209}{4} = 1820.04.$$

The approximate change in maximum utility due to a one unit increase in the consumer's disposable income is λ. For a \$6 increase in disposable income, the maximum utility increases by about 6λ which is 209.

Solutions for Chapter 9 Review

1. To see whether f is an increasing or decreasing function of x, we need to see how f varies as we increase x and hold y fixed. We note that each column of the table corresponds to a fixed value of y. Scanning down the $y = 2$ column, we can see that as x increases, the value of the function decreases from 114 when $x = 0$ down to 93 when $x = 80$. Thus, f may be decreasing. In order for f to actually be decreasing however, we have to make sure that f decreases for *every* column. In this case, we see that f indeed does decrease for every column. Thus, f is a decreasing function of x. Similarly, to see whether f is a decreasing function of y we need to look at the rows of the table. As we can see, f increases for every row as we increase y. Thus, f is an increasing function of y.

2. **(a)** We expect B to be an increasing function of all three variables.
(b) A deposit of \$1250 at a 1% annual interest rate leads to a balance of \$1276 after 25 months.

3. Contours are lines of the form $3x - 5y + 1 = c$ as shown in Figure 9.34. Note that for the regions of x and y given, the c values range from $-12 < c < 12$ and are evenly spaced.

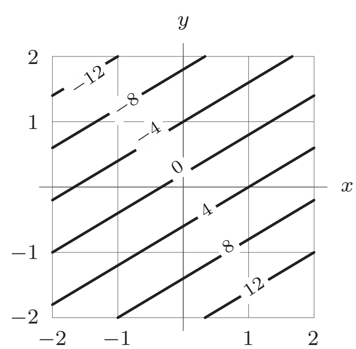


Figure 9.34

4. Contours are ellipses of the form $2x^2 + y^2 = c$ as shown in Figure 9.35. Note that for the ranges of x and y given, the range of c value is $1 \le c < 9$ and are closer together farther from the origin.

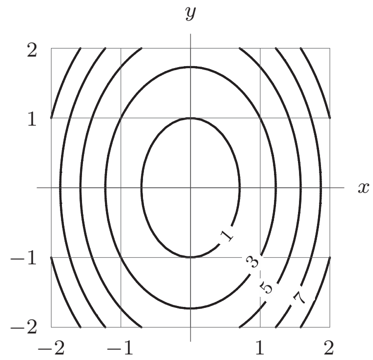


Figure 9.35

5. To draw a contour for a wind-chill of $W = 20$, we need a few combinations of temperature and wind velocity (T, v) such that $W(T, v) = 20$. Estimating from the table, some such points are $(24, 5)$ and $(33, 10)$. We can connect these points to get a contour for $W = 20$. Similarly, some points that have wind-chill of about $0^\circ F$ are $(5, 5)$, $(17.5, 10)$, $(23.5, 15)$, $(27, 20)$, and $(29, 25)$. By connecting these points we get the contour for $W = 0$. If we carry out this procedure for more values of W, we get a full contour diagram such as is shown in Figure 9.36:

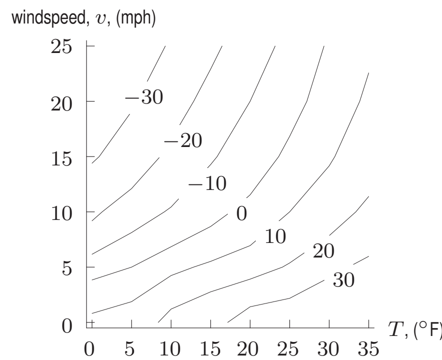


Figure 9.36

6. **(a)** According to Table 9.13 of the problem, it feels like -19°F.

(b) A wind of 20 mph, according to Table 9.13.

(c) About 17.5 mph. Since at a temperature of 25°F, when the wind increases from 15 mph to 20 mph, the temperature adjusted for wind-chill decreases from 13°F to 11°F, we can say that a 5 mph increase in wind speed causes an 2°F decrease in the temperature adjusted for wind-chill. Thus, each 2.5 mph increase in wind speed brings *about* a 1°F drop in the temperature adjusted for wind-chill. If the wind speed at 25°F increases from 15 mph to 17.5 mph, then the temperature you feel will be $13 - 1 = 12^\circ$F.

(d) Table 9.13 shows that with wind speed 20 mph the temperature will feel like 0°F when the air temperature is somewhere between 15°F and 20°F. When the air temperature drops 5°F from 20°F to 15°F, the temperature adjusted for wind-chill drops 6°F from 4°F to -2°F. We can say that for every 1°F decrease in air temperature there is *about* a $6/5 = 1.2^\circ$F drop in the temperature you feel. To drop the temperature you feel from 4°F to 0°F will take an air temperature drop of about $4/1.2 = 3.3^\circ$F from 20°F. With a wind of 20 mph, approximately $20 - 3.3 = 16.7^\circ$F would feel like 0°F."

7.

Table 9.17 *Temperature adjusted for wind-chill at* $20^\circ F$

Wind speed (mph)	5	10	15	20	25
Adjusted temperature (°F)	13	9	6	4	3

Table 9.18 *Temperature adjusted for wind-chill at* $0^\circ F$

Wind speed (mph)	5	10	15	20	25
Adjusted temperature (°F)	−11	−16	−19	−22	−24

8.

Table 9.19 *Temperature adjusted for wind-chill at 5 mph*

Temperature (°F)	35	30	25	20	15	10	5	0
Adjusted temperature (°F)	31	25	19	13	7	1	−5	−11

Table 9.20 *Temperature adjusted for wind-chill at 20 mph*

Temperature (°F)	35	30	25	20	15	10	5	0
Adjusted temperature (°F)	24	17	11	4	−2	−9	−15	−22

9. Using our economic intuition, we know that the total sales of a product should be an increasing function of the amount spent on advertising. From the graph, Q is a decreasing function of x and an increasing function of y. Thus, the y-axis corresponds to the amount spent on advertising and the x-axis corresponds to the price of the product.

10. One possible answer follows.

(a) If there is a city at the center of the diagram, then the population is very dense at the center, but progressively less dense as you move into the suburbs, further from the city center. This scenario corresponds to diagram (I) or (II). We pick (I) because it has the highest density, as we would expect in a city.

(b) If the center of the diagram is a lake, and is a very busy and thriving center, where lake front property is considered the most desirable, then the most dense area will be at lakeside, and decrease as you move further from the lake in the center, as in diagram (I) or (II). We pick (II) because we expect the population density at lake front to be less than that in the center of a city.

In an alternative solution, if the lake were in the middle of nowhere, the entire area would be very sparsely populated, and there would be slightly fewer people living on the actual lake shore, as in diagram (III).

(c) If the center of the diagram is a power plant and if the plant is not in a densely populated area, where people can and will choose not to live anywhere near it, the population density will then be very low nearby, increasing slightly further from the plant, as in diagram (III).

11. Many different answers are possible. Answers are in degrees Celsius.

(a) Minnesota in winter. See Figure 9.37.

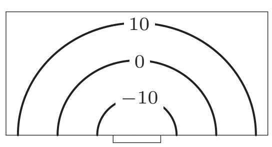


Figure 9.37

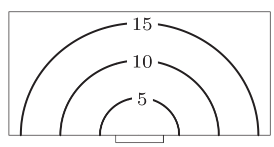


Figure 9.38

(b) San Francisco in winter. See Figure 9.38.

(c) Houston in summer. See Figure 9.39.

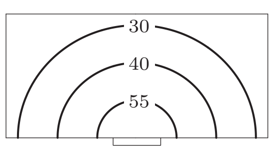


Figure 9.39

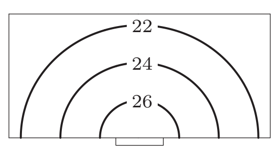


Figure 9.40

(d) Oregon in summer. See Figure 9.40.

12. (a) The point representing 8% and $6000 on the graph lies between the 120 and 140 contours. We estimate the monthly payment to be about $122.

(b) Since the interest rate has dropped, we will be able to borrow more money and still make a monthly payment of \$122. To find out how much we can afford to borrow, we find where the interest rate of 6% intersects the \$122 contour and read off the loan amount to which these values correspond. Since the \$122 contour is not shown, we estimate its position from the \$120 and \$140 contours. We find that we can borrow an amount of money that is more than \$6000 but less than \$6500. So we can borrow about \$350 more without increasing the monthly payment.

(c) The entries in the table will be the amount of loan at which each interest rate intersects the 122 contour. Using the \$122 contour from (b) we make table 9.21.

Table 9.21 *Amount borrowed at a monthly payment of \$122.*

Interest Rate (%)	0	1	2	3	4	5	6	7
Loan Amount (\$)	7400	7200	7000	6800	6650	6500	6350	6200
Interest rate (%)	8	9	10	11	12	13	14	15
Loan Amount (\$)	6000	5850	5700	5600	5500	5400	5300	5200

13. The values in Table (a) are not constant along rows or columns and therefore cannot be the lines shown in (I) or (IV). Also observe that as you move away from the origin, whose contour value is 0, the z-values on the contours increase. Thus, this table corresponds to diagram (II).

The values in Table (b) are also not constant along rows or columns. Since the contour values are decreasing as you move away from the origin, this table corresponds to diagram (III).

Table (c) shows that for each fixed value of x, we have constant contour value, suggesting a straight vertical line at each x-value, as in diagram (IV).

Table (d) also shows lines, however these are horizontal since for each fixed value of y we have constant contour values. Thus, this table matches diagram (I).

14. (A) In graph I, $L = 1, K = 1$ gives us $F = 1$, and $L = 3, K = 3$ gives us $F = 3$. So tripling all inputs in graph I triples output; graph I corresponds to statement (A).

(B) In graph II, $L = 1, K = 1$ gives us $F = 1$, and $L = 2.2, K = 2.2$ gives us $F = 1.5$. Extrapolating from this ratio, $L = 4, K = 4$ should gives us $F = 2$. So, quadrupling all inputs in graph II doubles output; graph II corresponds to statement (B).

(C) In graph III, $L = 1, K = 1$ gives us $F = 1$, and $L = 2, K = 2$ gives us $F = 2.8$. So, doubling the inputs in graph III almost tripled output; graph III corresponds to statement (C).

15. The values of z increase as we move in the direction of increasing x-values, so f_x is positive. The values of z decrease as we move in the direction of increasing y-values, so f_y is negative. We see in the contour diagram that $f(2, 1) = 10$. We estimate the partial derivatives:

$$f_x(2,1) \approx \frac{\Delta z}{\Delta x} = \frac{14-10}{4-2} = 2,$$
$$f_y(2,1) \approx \frac{\Delta z}{\Delta y} = \frac{6-10}{2-1} = -4.$$

16. **(a)** If you borrow \$8000 at an interest rate of 1% per month and pay it off in 24 months, your monthly payments are \$376.59.

(b) The increase in your monthly payments for borrowing an extra dollar under the same terms as in (a) is about 4.7 cents.

(c) If you borrow the same amount of money for the same time period as in (a), but if the interest rate increases by 1%, the increase in your monthly payments is about \$44.83.

17. For $f_w(10, 25)$ we get

$$f_w(10,25) \approx \frac{f(10+h,25) - f(10,25)}{h}.$$

Choosing $h = 5$ and reading values from Table 9.13 on page 390 of the text, we get

$$f_w(10,25) \approx \frac{f(15,25) - f(10,25)}{5} = \frac{13-15}{5} = -0.4^\circ\text{F/mph}$$

This means that when the wind speed is 10 mph and the true temperature is 25°F, as the wind speed increases from 10 mph by 1 mph we feel an approximately 0.4°F drop in temperature. This rate is negative because the temperature you feel drops as the wind speed increases.

18. Using a difference quotient with $h = 5$, we get

$$f_T(5,20) \approx \frac{f(5,20+5) - f(5,20)}{5} = \frac{19-13}{5} = 1.2^\circ\text{F}/^\circ\text{F}.$$

This means that when the wind speed is 5 mph and the true temperature is 20°F, the apparent temperature increases by approximately 1.2°F for every increase of 1°F in the true temperature. This rate is positive because the true temperature you feel increases as true temperature increases.

19. Since the average rate of change of the temperature adjusted for wind-chill is about -0.8 (drops by 0.8°F), with every 1 mph increase in wind speed from 5 mph to 10 mph, when the true temperature stays constant at 20°F, we know that

$$f_w(5,20) \approx -0.8.$$

20. **(a)** $\dfrac{\partial q_1}{\partial x}$ is the rate of change of the quantity of the first brand sold as its price increases. Since this brand competes with another brand, an increase in the price of the first brand should result in a decrease in the quantity sold of this same brand. Thus $\dfrac{\partial q_1}{\partial x} < 0$. Similarly, $\dfrac{\partial q_2}{\partial y} < 0$.

(b) Again we take into consideration the competition between the two brands. If the second brand were to increase its price, then more of the first brand should sell. Thus $\dfrac{\partial q_1}{\partial y} > 0$. Similarly, $\dfrac{\partial q_2}{\partial x} > 0$.

21. **(a)** An increase in the price of a new car will decrease the number of cars bought annually. Thus $\dfrac{\partial q_1}{\partial x} < 0$. Similarly, an increase in the price of gasoline will decrease the amount of gas sold, implying $\dfrac{\partial q_2}{\partial y} < 0$.

(b) Since the demands for a car and gas complement each other, an increase in the price of gasoline will decrease the total number of cars bought. Thus $\dfrac{\partial q_1}{\partial y} < 0$. Similarly, we may expect $\dfrac{\partial q_2}{\partial x} < 0$.

22. A graph with kilometers north fixed at 50 is in Figure 9.41.

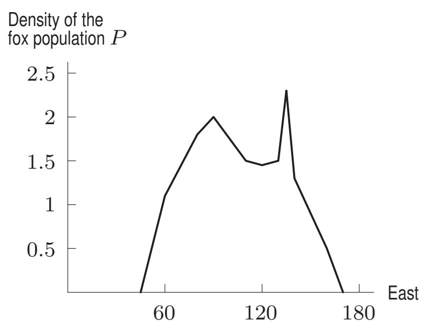


Figure 9.41

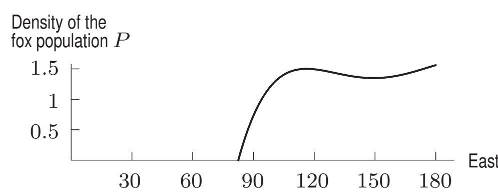


Figure 9.42

A graph with kilometers north fixed at 100 is in Figure 9.42.
A graph with kilometers east fixed at 60 is in Figure 9.43.

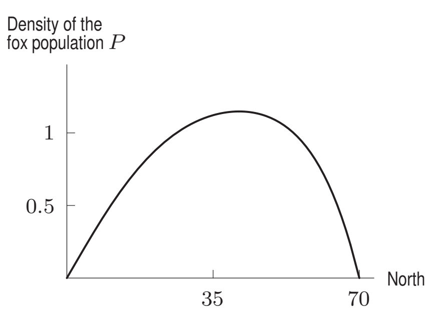


Figure 9.43

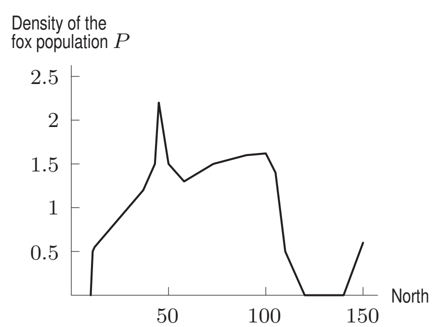


Figure 9.44

A graph with kilometers east fixed at 120 is in Figure 9.44.

23. Estimating from the contour diagram, using positive increments for Δx and Δy, we have, for point A,

$$\left.\frac{\partial n}{\partial x}\right|_{(A)} \approx \frac{1.5-1}{67-59} = \frac{1/2}{8} = \frac{1}{16} \approx 0.06 \quad \frac{\text{foxes/km}^2}{\text{km}}$$
$$\left.\frac{\partial n}{\partial y}\right|_{(A)} \approx \frac{0.5-1}{60-51} = -\frac{1/2}{9} = -\frac{1}{18} \approx -0.06 \quad \frac{\text{foxes/km}^2}{\text{km}}.$$

So, from point A the fox population density increases as we move eastward. The population density decreases as we move north from A.

At point B,

$$\left.\frac{\partial n}{\partial x}\right|_{(B)} \approx \frac{0.75-1}{135-115} = -\frac{1/4}{20} = -\frac{1}{80} \approx -0.01 \quad \frac{\text{foxes/km}^2}{\text{km}}$$
$$\left.\frac{\partial n}{\partial y}\right|_{(B)} \approx \frac{0.5-1}{120-110} = -\frac{1/2}{10} = -\frac{1}{20} \approx -0.05 \quad \frac{\text{foxes/km}^2}{\text{km}}.$$

So, fox population density decreases as we move both east and north of B. However, notice that the partial derivative $\partial n/\partial x$ at B is smaller in magnitude than the others. Indeed if we had taken a negative Δx we would have obtained an estimate of the opposite sign. This suggests that better estimates for$B$ are

$$\left.\frac{\partial n}{\partial x}\right|_{(B)} \approx 0 \quad \frac{\text{foxes/km}^2}{\text{km}}$$
$$\left.\frac{\partial n}{\partial y}\right|_{(B)} \approx -0.05 \quad \frac{\text{foxes/km}^2}{\text{km}}.$$

At point C,

$$\left.\frac{\partial n}{\partial x}\right|_{(C)} \approx \frac{2-1.5}{135-115} = \frac{1/2}{20} = \frac{1}{40} \approx 0.02 \quad \frac{\text{foxes/km}^2}{\text{km}}$$
$$\left.\frac{\partial n}{\partial y}\right|_{(C)} \approx \frac{2-1.5}{80-55} = \frac{1/2}{25} = \frac{1}{50} \approx 0.02 \quad \frac{\text{foxes/km}^2}{\text{km}}.$$

So, the fox population density increases as we move east and north of C. Again, if these estimates were made using negative values for Δx and Δy we would have had estimates of the opposite sign. Thus, better estimates are

$$\left.\frac{\partial n}{\partial x}\right|_{(C)} \approx 0 \quad \frac{\text{foxes/km}^2}{\text{km}}$$
$$\left.\frac{\partial n}{\partial y}\right|_{(C)} \approx 0 \quad \frac{\text{foxes/km}^2}{\text{km}}.$$

24. $f_x = 2x + y$, $f_y = 2y + x$.

25. $P_a = 2a - 2b^2$, $P_b = -4ab$.

26. $\dfrac{\partial Q}{\partial p_1} = 50p_2$, $\dfrac{\partial Q}{\partial p_2} = 50p_1 - 2p_2$.

27. $\dfrac{\partial f}{\partial x} = 5e^{-2t}$, $\dfrac{\partial f}{\partial t} = -10xe^{-2t}$.

28. $\dfrac{\partial P}{\partial K} = 7K^{-0.3}L^{0.3}$, $\dfrac{\partial P}{\partial L} = 3K^{0.7}L^{-0.7}$.

29. $f_x = \frac{x}{\sqrt{x^2+y^2}}$, $f_y = \frac{y}{\sqrt{x^2+y^2}}$.

30.

$$\begin{aligned} f(500, 1000) &= 16 + 1.2(500) + 1.5(1000) + 0.2(500)(1000) \\ &= \$102{,}116 \end{aligned}$$

The cost of producing 500 units of item 1 and 1000 units of item 2 is $102,116.

$$f_{q_1}(q_1, q_2) = 0 + 1.2 + 0 + 0.2q_2$$

So $f_{q_1}(500, 1000) = 1.2 + 0.2(1000) = \$201.20$ per unit. When the company is producing at $q_1 = 500$, $q_2 = 1000$, the cost of producing one more unit of item 1 is $201.20.

$$f_{q_2}(q_1, q_2) = 0 + 0 + 1.5 + 0.2q_1$$

So $f_{q_2}(500, 1000) = 1.5 + 0.2(500) = \$101.50$ per unit. When the company is producing at $q_1 = 500$, $q_2 = 1000$, the cost of producing one more unit of item 2 is $101.50.

31. **(a)** $Q_K = 18.75K^{-0.25}L^{0.25}$, $Q_L = 6.25K^{0.75}L^{-0.75}$.

(b) When $K = 60$ and $L = 100$,

$$\begin{aligned} Q &= 25 \cdot 60^{0.75} \cdot 100^{0.25} = 1704.33 \\ Q_K &= 18.75 \cdot 60^{-0.25} 100^{0.25} = 21.3 \\ Q_L &= 6.25 \cdot 60^{0.75} 100^{-0.75} = 4.26 \end{aligned}$$

(c) Q is actual quantity being produced. Q_K is how much more could be produced if you increased K by one unit. Q_L is how much more could be produced if you increased L by 1.

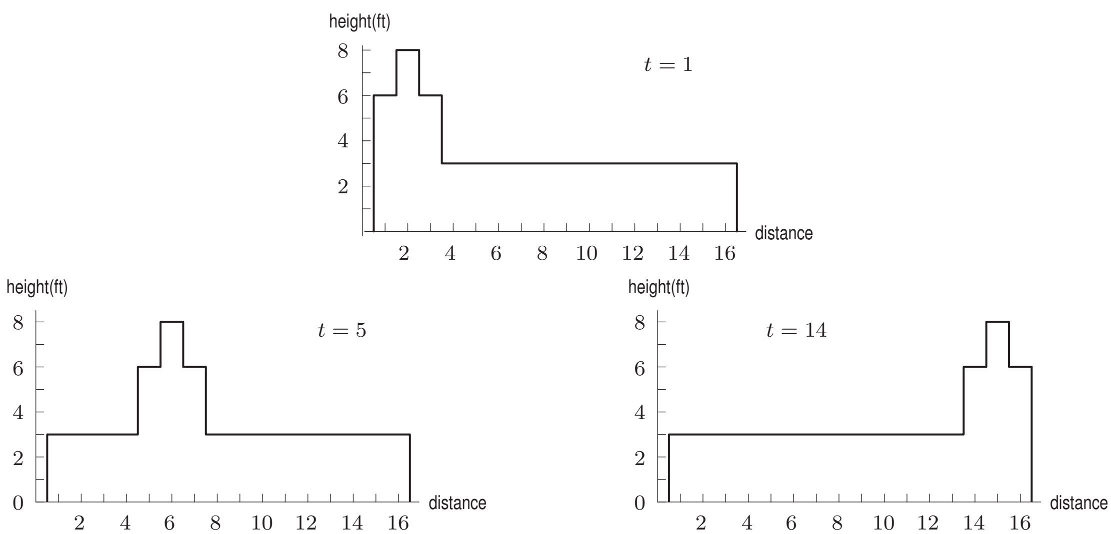
32. **(a)** At Q, R, we have $f_x < 0$ because f decreases as we move in the x-direction.

(b) At Q, P, we have $f_y > 0$ because f increases as we move in the y-direction.

(c) At all four points, P, Q, R, S, we have $f_{xx} > 0$, because f_x is increasing as we move in the x-direction. (At P, S, we see that f_x is positive and getting larger; at Q, R, we see that f_x is negative and getting less negative.)

(d) At all four points, P, Q, R, S, we have $f_{yy} > 0$, so there are none with $f_{yy} < 0$. The reasoning is similar to part (c).

33. **(a)**



(b)

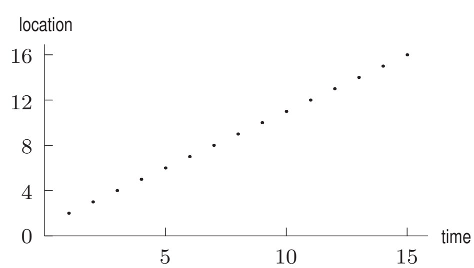


Figure 9.45

(c) The "wave" at a sports arena.

34. The function in $h(x,t)$ tells us the height of the head of the spectator in seat x at time t seconds. Thus, $h_x(2,5)$ is in feet per seat and $h_t(2,5)$ is in feet per second. So

$$h_x(x,t) = -0.5\sin(0.5x - t)$$
$$h_x(2,5) = -0.5\sin(0.5(2) - 5) \approx -0.38 \text{ ft/seat.}$$

and

$$h_t(x,t) = \sin(0.5x - t)$$
$$h_t(2,5) = \sin(0.5(2) - 5) = 0.76 \text{ ft/sec.}$$

The value of $h_x(2,5)$ is the rate of change of height of heads as you move along the row of seats. The value of $h_t(2,5)$ is the vertical velocity of the head of the person at seat 2 at time $t = 5$.

35. Since $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are defined everywhere, a critical point will occur where $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$. So:

$$\frac{\partial f}{\partial x} = 3x^2 - 3 = 0, \qquad \text{so } x^2 = 1 \text{ and } x = \pm 1$$
$$\frac{\partial f}{\partial y} = 2y = 0, \qquad \text{so } y = 0$$

So $(1,0)$ and $(-1,0)$ are the critical points. To determine whether these are local extrema, we can examine values of f for (x,y) near the critical points. Functional values for points near $(1,0)$ are shown in Table 9.22:

Table 9.22

y \ x	0.99	1.00	1.01
−0.01	−1.9996	−1.9999	−1.9996
0.00	−1.9997	−2.0000	−1.9997
0.01	−1.9996	−1.9999	−1.9996

As we can see from the table, all the points close to $(1,0)$ have greater functional value, so $f(1,0)$ is a local minimum. A similar display of points near $(-1,0)$ is shown in Table 9.23:

Table 9.23

y \ x	−1.01	−1.00	−0.99
−0.01	1.9998	2.0001	1.9998
0.00	1.9997	2.0000	1.9997
0.01	1.9998	2.0001	1.9998

As we can see, some points near $(-1,0)$ have greater functional value than $f(-1,0)$ and others have less. So $f(-1,0)$ is neither a local maximum nor a local minimum.

36. Since $\dfrac{\partial f}{\partial y}$ and $\dfrac{\partial f}{\partial y}$ are defined everywhere, a critical point will occur where $\dfrac{\partial f}{\partial x} = 0$ and $\dfrac{\partial f}{\partial y} = 0$. So:

$$\frac{\partial f}{\partial x} = 2x - 4 = 0 \Rightarrow x = 2$$
$$\frac{\partial f}{\partial y} = 6y + 6 = 0 \Rightarrow y = -1$$

$(2,-1)$ is the critical point of $f(x,y)$.

37. **(a)** We first express the revenue R in terms of the prices p_1 and p_2:

$$\begin{aligned} R(p_1,p_2) &= p_1q_1 + p_2q_2 \\ &= p_1(517 - 3.5p_1 + 0.8p_2) + p_2(770 - 4.4p_2 + 1.4p_1) \\ &= 517p_1 - 3.5p_1^2 + 770p_2 - 4.4p_2^2 + 2.2p_1p_2. \end{aligned}$$

(b) We compute the partial derivatives and set them to zero:

$$\begin{aligned} \frac{\partial R}{\partial p_1} &= 517 - 7p_1 + 2.2p_2 = 0, \\ \frac{\partial R}{\partial p_2} &= 770 - 8.8p_2 + 2.2p_1 = 0. \end{aligned}$$

Solving these equations, we find that

$$p_1 = 110 \quad \text{and} \quad p_2 = 115.$$

To see whether or not we have a found a local maximum, we compute the second-order partial derivatives:

$$\frac{\partial^2 R}{\partial p_1^2} = -7, \quad \frac{\partial^2 R}{\partial p_2^2} = -8.8, \quad \frac{\partial^2 R}{\partial p_1 \partial p_2} = 2.2.$$

Therefore,

$$D = \frac{\partial^2 R}{\partial p_1^2}\frac{\partial^2 R}{\partial p_2^2} - \left(\frac{\partial^2 R}{\partial p_1 \partial p_2}\right)^2 = (-7)(-8.8) - (2.2)^2 = 56.76,$$

and so we have found a local maximum point. The graph of $R(p_1, p_2)$ has the shape of an upside down bowl. Therefore, $(110, 115)$ is a global maximum point.

38. The smallest value of $f(x, y)$ on the line $y = 100$ shown in Figure 9.46 occurs at the right endpoint, at the point $(300, 100)$. The minimum value is $f(300, 100) = 35$.

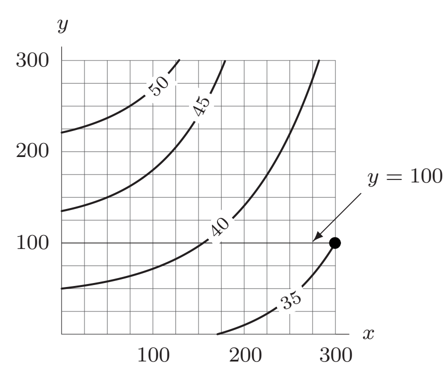


Figure 9.46

39. The largest value of $f(x, y)$ on the line $y = 100$ shown in Figure 9.47 occurs at the left endpoint, at the point $(0, 100)$. The maximum value is $f(0, 100) \approx 43$.

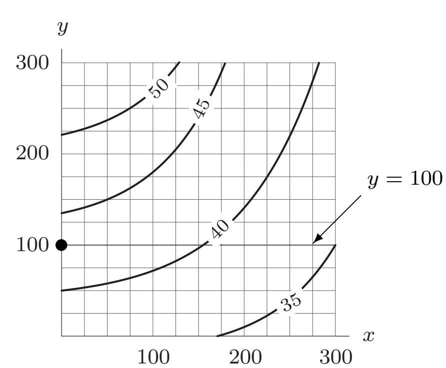


Figure 9.47

40. The constraint is the line $y = x$. The largest value of $f(x, y)$ on the line $y = x$ shown in Figure 9.48 occurs at the point where that line is tangent to a contour. Since the contour is not shown in the figure, we make a rough approximation by imagining intermediate contours. The point $(150, 150)$ seems about right. The maximum value is $f(150, 150) \approx 42$.

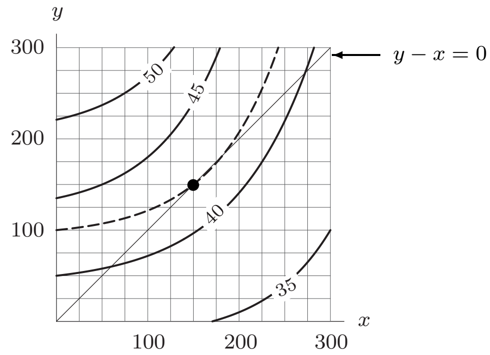


Figure 9.48

41. **(a)** Objective function: $Q = x_1^{0.3}x_2^{0.7}$.
(b) Constraint: $10x_1 + 25x_2 = 50{,}000$.

42. The largest value of $f(x, y)$ on the constraint line $15x + 20y = 300$ shown in Figure 9.49 occurs at the point where that line is tangent to a contour. Since the contour is not shown in the figure, we make a rough approximation by imagining intermediate contours. The point $(12, 6)$ seems about right. The maximum value is $f(12, 6) \approx 9000$, achieved with inputs $x = 12$ and $y = 6$.

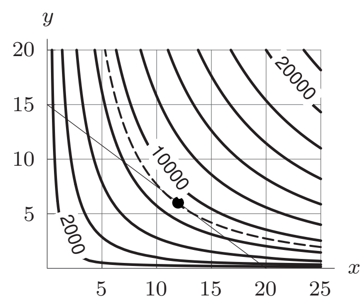


Figure 9.49

43. **(a)** We have 1500 workers and \$15,000,000 per month of capital, so $x = 1500, y = 15{,}000{,}000/4000 = 3750$. Substituting into the equation for Q gives us $Q = (1500)^{0.4}(3750)^{0.6} = 2599$ cars per month.

(b) Now we are only producing 2000 cars per month. We wish to minimize cost subject to the constraint that the monthly production is 2000 cars. So, our objective function is cost $C = 5000x + 4000y$ and our constraint is that $Q = x^{0.4}y^{0.6} = 2000$. To minimize C according to this, we solve the following system of equations:

$$\begin{aligned} C_x &= \lambda Q_x, && \text{so } 0.4x^{-0.6}y^{0.6}\lambda = 5000 \\ C_y &= \lambda Q_x, && \text{so } 0.6x^{0.4}y^{-0.4}\lambda = 4000 \\ Q &= 2000, && \text{so } x^{0.4}y^{0.6} = 2000 \end{aligned}$$

Dividing the first two equations gives

$$\frac{0.4x^{-0.6}y^{0.6}\lambda}{0.6x^{0.4}y^{-0.4}\lambda} = \frac{0.4}{0.6}\frac{y}{x} = \frac{5000}{4000} \Rightarrow y = 1.875x.$$

Substituting this into the constraint gives us $x \approx 1371.606$ and $y \approx 2571.760$. So our new level of production uses 905 workers and \$6,786,916 of equipment. So $1500 - 1371 = 129$ workers will be laid off, and monthly investment in capital will fall by $\$15{,}000{,}000 - \$10{,}287{,}042 = \$4{,}712{,}958$.

(c) Solving for the Lagrange multiplier λ from the above equations gives us

$$\lambda = \frac{5000}{0.4x^{-0.6}y^{0.6}} = \frac{5000}{0.4(1371.606)^{-0.6}(2571.760)^{0.6}} \approx \$8572.54 \text{ per car}$$

This means that to produce on additional car per month would cost about \$8572.54 with the lowest-cost use of capital and labor.

44. (a) We want to minimize C subject to $g = x + y = 39$. We solve the system of equations

$$\begin{aligned} C_x &= \lambda g_x, &\quad \text{so } 10x + 2y = \lambda \\ C_y &= \lambda g_y, &\quad \text{so } 2x + 6y = \lambda \\ g &= 39, &\quad \text{so } x + y = 39. \end{aligned}$$

The first two equations give $y = 2x$. Solving with $x + y = 39$ gives $x = 13, y = 26, \lambda = 182$. Therefore $C = \$4349$.

(b) Since $\lambda = 182$, increasing production by 1 will cause costs to increase by approximately \$182. Similarly, decreasing production by 1 will save approximately \$182.

45. (a) The profit is given by

$$\begin{aligned} \text{Profit} &= \pi(q_1, q_2) \\ &= \text{Total Revenue} - \text{Total Cost} \\ &= p_1q_1 + p_2q_2 - (10q_1 + q_1q_2 + 10q_2) \\ &= (50 - q_1 + q_2)q_1 + (30 + 2q_1 - q_2)q_2 - (10q_1 + q_1q_2 + 10q_2) \\ &= 40q_1 - q_1^2 + 2q_1q_2 + 20q_2 - q_2^2, \end{aligned}$$

subject to the constraint $q_1 + q_2 = 15$.

Then we need to solve the equations

$$\frac{\partial \pi}{\partial q_1} = 0, \quad \frac{\partial \pi}{\partial q_2} = 0, \text{ subject to } q_1 + q_2 = 15.$$

These equations are

$$\begin{aligned} 40 - 2q_1 + 2q_2 &= \lambda \\ 2q_1 + 20 - 2q_2 &= \lambda \\ q_1 + q_2 &= 15. \end{aligned}$$

Adding the first two equations gives

$$60 = 2\lambda$$

so $\lambda = 30$. Substituting this into the first equation gives

$$q_1 - q_2 = 5,$$

therefore, q_1 and q_2 satisfy

$$\begin{aligned} q_1 + q_2 &= 15 \\ q_1 - q_2 &= 5. \end{aligned}$$

Adding these equations gives $2q_1 = 20$ so $q_1 = 10$ and $q_2 = 5$. Substituting these values into the expression for the total profit gives

$$\pi(10, 5) = 40 \cdot 10 - 10^2 + 2 \cdot 10 \cdot 5 + 20 \cdot 5 - 5^2 = 475.$$

The endpoints of the constraint are $(15, 0)$ and $(0, 15)$ giving

$$\begin{aligned} \pi(15, 0) &= 40 \cdot 15 - 15^2 = 375 \\ \pi(0, 15) &= 20 \cdot 15 - 15^2 = 75. \end{aligned}$$

Thus the maximum profit is 475.

(b) The approximate change in the maximum profit due to a one unit increase in the production constraint is $\lambda = 30$. Thus a one unit increase in the production quota increases production by 30 units, to $475 + 30 = 505$ units.

CHECK YOUR UNDERSTANDING

1. False, the units of 1.5 are kilometers.

2. True, since $t = 6$ hours.

3. False, the units of 4.1 are ppm, parts per million.

4. False. We expect the quantity of pollutants to go down as we get farther from the incinerator, so Q is a decreasing function of x.

5. True.

6. False. To be an increasing function of x, we must have that f increases whenever x increases and y is fixed.

7. True. For example, let $f(x, y) = x - y$. Then if y is fixed, f increases as x increases, and if x is fixed, f decreases as y increases.

8. True. If x is fixed, then f decreases as y increases.

9. False. The cross-section with $x = 1$ is $f(1, y) = e^y - y^2$.

10. False. The cross section with $y = 0$ is $f(x, 0) = e^0 - 0 = 1$.

11. True. When we set $f(x, y) = c$, we get $y + 3x - 1 = c$, which is the equation of a line.

12. True. When we set $f(x, y) = c$, we get $x^2 - y = c$, which can be rewritten $y = x^2 - c$, a parabola.

13. True. If they intersected at some point (a, b), we would have simultaneously $f(a, b) = 1$ and $f(a, b) = 2$. This is impossible since a function can have only one output for a given input.

14. True. If we set $f(x, y) = c$, we get $3x + 2y = c$, which can be rewritten as $y = -3/2x + c/2$. Thus all the contours are lines with the same slope.

15. True. A Cobb-Douglas production function for variable N and V has the form $P = cN^aV^b$.

16. False, since the exponents of N and V must be between 0 and 1.

17. True, since a contour is defined as the set of (x, y) such that $f(x, y)$ is constant.

18. True. For example, if $f(x, y) = x^2 + y^2$, the contour $f(x, y) = 0$ consists of the single point $(0, 0)$.

19. True, as illustrated in the text.

20. True. If (a, b) is any point in the domain of g and $f(a, b) = c$, then (a, b) lies on the contour $f(x, y) = c$.

21. True, as specified in the text.

22. False. It has units of dollars per mile.

23. True. As mileage on a used car increases, the price generally decreases, so the partial derivative will be negative.

24. True, since the units of the partial derivative are units of the output (pounds) per units of the changing input (calories).

25. False. In general, as calorie consumption goes up, the weight goes up, so the partial derivative would be positive.

26. True. For a fixed amount of exercise t, the weight should increase as the daily food intake increases. Thus the sign of $\partial W/\partial c$ is positive. Also, for a fixed amount of daily food intake c, the weight should decrease as the amount of daily exercise t increases. Thus the sign of $\partial W/\partial t$ is negative.

27. True, since when y is held constant at 2, we have $f_x(1, 2) \approx \Delta z/\Delta x = 0.5/0.1 = 5$.

28. False, since when x is held constant at 1, we have $f_y(1, 2) \approx \Delta z/\Delta y = 0.8/0.1 = 8$.

29. True, since when y is held constant at 3, we have $f_x(5, 3) \approx \Delta z/\Delta x = 0.9/0.1 = 9$.

30. False. The correct value is -6, since when x is held constant at 5, we have $f_y(5, 3) \approx \Delta z/\Delta y = -0.6/0.1 = -6$.

31. True, since we find the x-partial by treating y as a constant.

32. False, since $f_y = x^2$, so $f_y(1, 2) = 1^2 = 1$.

33. True, since $g_u = e^v$ so $g_u(0, 0) = e^0 = 1$.

34. True. For example, let $f(x, y) = 17$, so $f_x = f_y = 0$.

35. False. We have $Q_y = 2x^3y$, so $Q_y(1, 2) = 4 > 0$. Thus Q is increasing in the y direction near $(1, 2)$, not decreasing.

36. True, since we take the V-partial by treating N as a constant.

37. False. For example, let $f(x, y) = x^2 + y$. Then $f_{xx} = 2$ but $f_{yy} = 0$. It is true that the *mixed* second order partial derivatives are equal: $f_{xy} = f_{yx}$ if f_{xx}, f_{xy}, f_{yx}, and f_{yy} are all continuous.

38. False. We have $f_x = 6xe^{2y}$ and $f_y = 6x^2e^{2y}$, so $f_x(1, 0) = f_y(1, 0) = 6$.

39. True, since we take the A-partial by treating B as a constant.

40. False, since $\partial z/\partial r = 2\pi rh$, so taking the h-partial of this result gives $2\pi r$ not 2π.

41. False. We need also $f_y(1, 2) = 0$.

42. False. We have $f_x(1, 1) = 2(1) \neq 0$, so $(1, 1)$ is not a critical point for f.

43. True. We have $g_u = 2(u - 3)$ and $g_v = 2(v - 2)$. Thus $g_u(3, 2) = g_v(3, 2) = 0$.

44. True. We have $f_x = e^y$, which is never 0.

45. False. For example, let $f(x, y) = x^2 - y^2$. Then critical point $(0, 0)$ is neither a local maximum nor a local minimum.

46. True. Because $D > 0$, we have either a local maximum or a local minimum at (a, b), and because $f_{xx} > 0$ we have a local minimum.

47. True. The second derivative test applies only to critical points.

48. False. It has neither a local maximum not a local minimum at $(0, 0)$ by the second derivative test since $D = -4 < 0$.

49. True. We have $D = (0)(0) - 3^2 < 0$, so the critical point $(0, 0)$ is a saddle point.

50. True. For example, let $f(x, y) = x^2 + y^2$. Then $(0, 0)$ is both a local minimum and a global minimum.

51. False. For example, let $f(x, y) = x$ and let the constraint be $x^2 + y^2 = 1$. Then since the constraint forces $-1 \leq x \leq 1$, the maximum occurs at $P_0 = (0, 1)$. Since $f_x = 1, f_y = 0$, we have that f has no critical points so P_0 is not a critical point.

52. True. Solutions to a constrained optimization problem must satisfy the constraint equation.

53. False. The budget equation is the constraint equation.

54. True, since the production equation is the equation that must be met.

55. True. Since $(0, 0)$ satisfies the constraint and $f(x, y) \geq 0$ for all x, y, the point $(0, 0)$ must be the minimum value of f subject to the constraint.

56. False. The point $(0, 0)$ cannot be a solution to this constrained optimization problem since it doesn't satisfy the constraint $2x + 3y = 12$.

57. True, since in this situation, λ represents the approximate change in cost given a one unit increase in production.

58. False, since in this situation, λ represents the approximate change in production given a one unit increase in the budget. We expect a budget of \$80,001 to generate production of about 50,120 tons.

59. False. The second derivative test does not apply to constrained optimization problems.

60. False, since we also need to know the unit cost of each of the two raw materials in order to set up a budget constraint.

PROJECTS FOR CHAPTER NINE

1. **(a)** About 15 feet along the wall, because that's where there are regions of cold air (55°F and 65°F).
(b) Roughly between 10 am and 12 noon, and between 4 pm and 6 pm.
(c) Roughly between midnight and 2 am, between 10 am and 1 pm, and between 4 pm and 9 pm, since that is when the temperature near the heater is greater than 80°F.

(d)

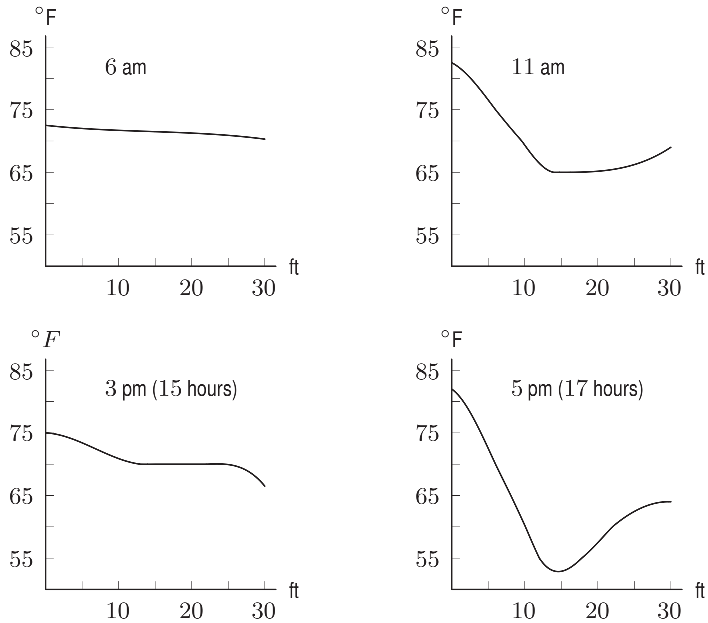


Figure 9.50

(e)

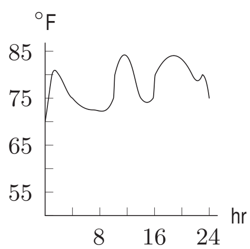


Figure 9.51: Temp. vs. Time at heater

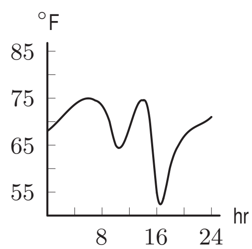


Figure 9.52: Temp. vs. Time at window

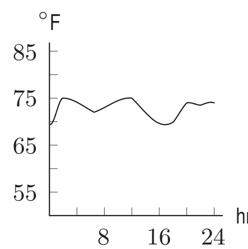


Figure 9.53: Temp. vs. Time midway between heater and window

(f) The temperature at the window is colder at 5 pm than at 11 am because the outside temperature is colder at 5 pm than at 11 am.

(g) The thermostat is set to roughly 70°F. We know this because the temperature in the room stays close to 70°F until we get close (a couple of feet) to the window.

(h) We are told that the thermostat is about 2 feet from the window. Thus, the thermostat is either about 13 feet or about 17 feet from the wall. If the thermostat is set to 70°F, every time the temperature at the thermostat goes over or under 70°F, the heater turns off or on. Look at the point at which the vertical lines at 13 feet or about 17 feet cross the 70°F contours. We need to decide which of these crossings correspond best with the times that the heater turns on and off. (These times can be seen along the wall.) Notice that the 17 foot line does not cross the 70°F contour after 16 hours (4 pm). Thus, if the thermostat were 17 feet from the wall, the heater would not turn off after 4 pm. However, the heater does turn off at about 21 hours (9 pm). Since this is the time that the 13 foot line crosses the 70°F contour, we estimate that the thermostat is about 13 feet away from the wall.

2. We want to maximize the theater's profit, P, as a function of the two variables (prices) p_c and p_a. As always, $P = R - C$, where R is the revenue, $R = q_cp_c + q_ap_a$, and C is the cost, which is of the form $C = k(q_c + q_a)$ for some constant k. Thus,

$$\begin{aligned} P(p_c, p_a) &= q_cp_c + q_ap_a - k(q_c + q_a) \\ &= rp_c^{-3} - krp_c^{-4} + sp_a^{-1} - ksp_a^{-2} \end{aligned}$$

To find the critical points, solve

$$\frac{\partial P}{\partial p_c} = -3rp_c^{-4} + 4krp_c^{-5} = 0$$
$$\frac{\partial P}{\partial p_a} = -sp_a^{-2} + 2ksp_a^{-3} = 0.$$

We get $p_c = 4k/3$ and $p_a = 2k$.

This critical point is a global maximum by the following useful, general argument. Suppose that $F(x, y) = f(x) + g(y)$, where f has a global maximum at $x = b$ and g has a global maximum at $y = d$. Then for all x, y:

$$F(x, y) = f(x) + g(y) \leq f(b) + g(d) = F(b, d),$$

so F has global maximum at $x = b,\ y = d$.

The profit function in this problem has the form

$$P(p_c, p_a) = f(p_c) + g(p_a),$$

and the usual single-variable calculus argument using f' and g' shows that $p_c = 4k/3$ and $p_a = 2k$ are global maxima for f and g, respectively. Thus the maximum profit occurs when $p_c = 4k/3$ and $p_a = 2k$. Thus,

$$\frac{p_c}{p_a} = \frac{4k/3}{2k} = \frac{2}{3}.$$

3. (a) The budget constraint is $4x_1 + 27x_2 = 324$. Maximizing $P = 270x_1^{1/3}x_2^{2/3}$ subject to this constraint, we solve

$$90x_1^{-2/3}x_2^{2/3} = 4\lambda$$
$$180x_1^{1/3}x_2^{-1/3} = 27\lambda$$
$$4x_1 + 27x_2 = 324.$$

Dividing the first two equations gives

$$\frac{90x_1^{-2/3}x_2^{2/3}}{180x_1^{1/3}x_2^{-1/3}} = \frac{4}{27}$$
$$\frac{x_2}{2x_1} = \frac{4}{27}.$$

Substituting $27x_2 = 8x_1$ into $4x_1 + 27x_2 = 324$ gives

$$4x_1 + 8x_1 = 324,$$

so $x_1 = 27$ and $x_2 = 8$. This critical point gives a maximum because the endpoint values on the budget constraint, where either $x_1 = 0$ or $x_2 = 0$, give a production of 0. With $x_1 = 27$ and $x_2 = 8$, the maximum production is

$$P_0 = 270 \cdot 27^{1/3} \cdot 8^{2/3} = 270 \cdot 3 \cdot 4 = 3240.$$

(b) To minimize cost, $C = 4x_1 + 27x_2$ subject to the production constraint $270x_1^{1/3}x_2^{2/3} = 3240$, or $x_1^{1/3}x_2^{2/3} = 12$, we solve

$$4 = \lambda(90x_1^{-2/3}x_2^{2/3})$$
$$27 = \lambda(180x_1^{1/3}x_2^{-1/3})$$
$$x_1^{1/3}x_2^{2/3} = 12.$$

Dividing the first two equations gives

$$\begin{aligned}\frac{4}{27} &= \frac{90x_1^{-2/3}x_2^{2/3}}{180x_1^{1/3}x_2^{-1/3}}\\ \frac{4}{27} &= \frac{x_2}{2x_1}.\end{aligned}$$

Substituting $8x_1 = 27x_2$ into $x_1^{1/3}x_2^{2/3} = 12$ gives

$$\begin{aligned}\left(\frac{27x_2}{8}\right)^{1/3} x_2^{2/3} &= 12\\ \frac{3}{2}x_2^{1/3}x_2^{2/3} = \frac{3}{2}x_2 &= 12,\end{aligned}$$

so $x_2 = 8$ and $x_1 = 27$. Since there are points on this production constraint with arbitrarily large values of either x_1 or x_2, as we move away from the critical point on the constraint, the cost increases without bound. Thus, the critical point gives a minimum. The minimum cost, in dollars, is

$$C = 4 \cdot 27 + 27 \cdot 8 = 324.$$

(c) The maximum production and the minimum cost occur for the same input values of x_1 and x_2. The critical points in parts (a) and (b) are both $x_1 = 27, x_2 = 8$. Then production is $P = 3240$ and cost is $C = 324$ dollars. These input values give a maximum production of 3240 for a cost of \$324, and a minimum cost of \$324 for a production of 3240. The fact that the solution is the same when the objective function and the constraint are interchanged in this way is described as duality by economists.

Solutions to Problems on Deriving the Formula for Regression Lines

1. Let the line be in the form $y = b + mx$. When x equals -1, 0 and 1, then y equals $b - m$, b, and $b + m$, respectively. The sum of the squares of the vertical distances, which is what we want to minimize, is

$$f(m, b) = (2 - (b - m))^2 + (-1 - b)^2 + (1 - (b + m))^2.$$

To find the critical points, we compute the partial derivatives with respect to m and b,

$$\begin{aligned}f_m &= 2(2 - b + m) + 0 + 2(1 - b - m)(-1)\\ &= 4 - 2b + 2m - 2 + 2b + 2m\\ &= 2 + 4m,\\ f_b &= 2(2 - b + m)(-1) + 2(-1 - b)(-1) + 2(1 - b - m)(-1)\\ &= -4 + 2b - 2m + 2 + 2b - 2 + 2b + 2m\\ &= -4 + 6b.\end{aligned}$$

Setting both partial derivatives equal to zero, we get a system of equations:

$$\begin{aligned}2 + 4m &= 0,\\ -4 + 6b &= 0.\end{aligned}$$

The solution is $m = -1/2$ and $b = 2/3$. You can check that it is a minimum. Hence, the regression line is $y = \frac{2}{3} - \frac{1}{2}x$.

2. Let the line be in the form $y = b + mx$. When x equals 0, 1, and 2, then y equals b, $b+m$, and $b+2m$, respectively. The sum of the squares of the vertical distances, which is what we want to minimize, is

$$f(m,b) = (2-b)^2 + (4-(b+m))^2 + (5-(b+2m))^2.$$

To find the critical points, we compute the partial derivatives with respect to m and b,

$$\begin{aligned} f_m &= 0 + 2(4-b-m)(-1) + 2(5-b-2m)(-2) \\ &= -8 + 2b + 2m - 20 + 4b + 8m \\ &= -28 + 6b + 10m \\ f_b &= 2(2-b)(-1) + 2(4-b-m)(-1) + 2(5-b-2m)(-1) \\ &= -4 + 2b - 8 + 2b + 2m - 10 + 2b + 4m \\ &= -22 + 6b + 6m \end{aligned}$$

Setting both partial derivatives equal to zero, we get a system of equations:

$$\begin{aligned} -28 + 6b + 10m &= 0, \\ -22 + 6b + 6m &= 0. \end{aligned}$$

The solution is $m = 1.5$ and $b = 13/6 \approx 2.17$. You can check that it is a minimum. Hence, the regression line is $y = 2.17 + 1.5x$.

3. We have $\sum x_i = 0$, $\sum y_i = 2$, $\sum x_i^2 = 2$, and $\sum y_i x_i = -1$. Thus

$$\begin{aligned} b &= (2\cdot 2 - 0\cdot(-1)) \,/\, \left(3\cdot 2 - 0^2\right) = 2/3 \\ m &= (3\cdot(-1) - 0\cdot 2) \,/\, \left(3\cdot 2 - 0^2\right) = -1/2. \end{aligned}$$

The line is $y = \frac{2}{3} - \frac{1}{2}x$, which agrees with the answer to Problem 1.

4. We have $\sum x_i = 3$, $\sum y_i = 11$, $\sum x_i^2 = 5$, and $\sum y_i x_i = 14$. Thus

$$\begin{aligned} b &= (5\cdot 11 - 3\cdot 14) \,/\, \left(3\cdot 5 - 3^2\right) = 13/6 = 2.17. \\ m &= (3\cdot 14 - 3\cdot 11) \,/\, \left(3\cdot 5 - 3^2\right) = 9/6 = 1.5. \end{aligned}$$

The line is $y = 2.17 + 1.5x$, which agrees with the answer to Problem 2.

5. We have $\sum x_i = 6$, $\sum y_i = 5$, $\sum x_i^2 = 14$, and $\sum y_i x_i = 12$. Thus

$$\begin{aligned} b &= (14\cdot 5 - 6\cdot 12) \,/\, \left(3\cdot 14 - 6^2\right) = -1/3. \\ m &= (3\cdot 12 - 6\cdot 5) \,/\, \left(3\cdot 14 - 6^2\right) = 1. \end{aligned}$$

The line is $y = x - \frac{1}{3}$, which agrees with the answer to Example 1.

6. **(a)** Let t be the number of years since 1960 and let $P(t)$ be the population in millions in the year $1960 + t$. We assume that $P = Ce^{at}$, and therefore

$$\ln P = at + \ln C.$$

So, we plot $\ln P$ against t and find the line of best fit. Our data points are $(0, \ln 180)$, $(10, \ln 206)$, and $(20, \ln 226)$. Applying the method of least squares to find the best-fitting line, we find that

$$\begin{aligned} a &= \frac{\ln 226 - \ln 180}{20} \approx 0.0114, \\ \ln C &= \frac{\ln 206}{3} - \frac{\ln 226}{6} + \frac{5\ln 180}{6} \approx 5.20 \end{aligned}$$

Then, $C = e^{5.20} = 181.3$ and so

$$P(t) = 181.3e^{0.0114t}.$$

In 1990, we have $t = 30$ and the predicted population in millions is

$$P(30) = 181.3e^{0.01141(30)} = 255.3.$$

(b) The difference between the actual and the predicted population is about 6 million or $2\frac{1}{2}\%$. Given that only three data points were used to calculate a and c, this discrepancy is not surprising. Thus, the 1990 census data does not mean that the assumption of exponential growth is unjustified.

(c) In 2010, we have $t = 50$ and $P(50) = 320.7$.

7. (a) (i) Suppose $N = kA^p$. Then the rule of thumb tells us that if A is multiplied by 10, the value of N doubles. Thus

$$2N = k(10A)^p = k10^p A^p.$$

Thus, dividing by $N = kA^p$, we have

$$2 = 10^p$$

so taking logs to base 10 we have

$$p = \log 2 = 0.3010.$$

(where $\log 2$ means $\log_{10} 2$). Thus,

$$N = kA^{0.3010}.$$

(ii) Taking natural logs gives

$$\ln N = \ln(kA^p)$$
$$\ln N = \ln k + p \ln A$$
$$\ln N \approx \ln k + 0.301 \ln A$$

Thus, $\ln N$ is a linear function of $\ln A$.

(b) Table 9.24 contains the natural logarithms of the data:

Table 9.24 $\ln N$ *and* $\ln A$

Island	$\ln A$	$\ln N$
Redonda	1.1	1.6
Saba	3.0	2.2
Montserrat	2.3	2.7
Puerto Rico	9.1	4.3
Jamaica	9.3	4.2
Hispaniola	11.2	4.8
Cuba	11.6	4.8

Using a least squares fit we find the line:

$$\ln N = 1.20 + 0.32 \ln A$$

This yields the power function:

$$N = e^{1.20} A^{0.32} = 3.32 A^{0.32}$$

Since 0.32 is pretty close to $\log 2 \approx 0.301$, the answer does agree with the biological rule.