# CHAPTER TEN

## Solutions for Section 10.1

**1.** (a) = (III), (b) = (IV), (c) = (I), (d) = (II).

**2.** (a) = (I), (b) = (IV), (c) = (III). Graph (II) represents an egg originally at $0^\circ$ C which is moved to the kitchen table ($20^\circ$ C) two minutes after the egg in part (a) is moved.

**3.** The rate of change of $P$ is proportional to $P$ so we have

$$\frac{dP}{dt} = kP,$$

for some constant $k$. Since the population $P$ is increasing, the derivative $dP/dt$ must be positive. Therefore, $k$ is positive.

**4.** The rate at which the balance is changing is 5% times the current balance, so we have

$$\text{Rate of change of } B = 0.05 \cdot \text{ Current balance}$$

so we have

$$\frac{dB}{dt} = 0.05B.$$

**5.** The rate of change of $Q$ is proportional to $Q$ so we have

$$\frac{dQ}{dt} = kQ,$$

for some constant $k$. Since the radioactive substance is decaying, the quantity present, $Q$, is decreasing. The derivative $dQ/dt$ must be negative, so the constant of proportionality $k$ is negative.

**6.** The balance in the account, $B$, is increasing at a rate of 4% times $B$ and is decreasing at a rate of 2000 dollars per year. We have

$$\text{Rate of change of } B = \text{Rate in} - \text{Rate out.}$$
$$\frac{dB}{dt} = 0.04B - 2000.$$

Notice that the initial amount of \$25,000 in the account is not used in the differential equation. The differential equation tells us only how things are changing.

**7.** The amount of the pollutant, $P$, is decreasing at a rate of $0.08$ times $P$ and is also decreasing at a constant rate of 30 gallons per day. Notice that both changes cause $P$ to decrease, so both will have a negative effect on $dP/dt$. We have

$$\frac{dP}{dt} = -0.08P - 30.$$

**8.** The amount of morphine, $M$, is increasing at a rate of $2.5$ mg/hour and is decreasing at a rate of $0.347$ times $M$. We have

$$\text{Rate of change of } M = \text{Rate in} - \text{Rate out.}$$
$$\frac{dM}{dt} = 2.5 - 0.347M.$$

**9.** The amount of alcohol, $A$, is decreasing at a constant rate of 1 ounce per hour, so we have

$$\frac{dA}{dt} = -1.$$

The negative sign indicates that the amount of alcohol is decreasing.

**10.** The amount of toxin, $A$, is increasing at a rate of 10 micrograms per day and is decreasing at a rate of $0.03$ times $A$. We have

$$\text{Rate of change of } A = \text{Rate in} - \text{Rate out.}$$
$$\frac{dA}{dt} = 10 - 0.03A.$$

**11.** **(a)** The amount of caffeine, $A$, is decreasing at a rate of 17% times $A$, so we have

$$\frac{dA}{dt} = -0.17A.$$

The negative sign indicates that the amount of caffeine is decreasing at a rate of 17% times $A$. Notice that the initial amount of caffeine, 100 mg, is not used in the differential equation. The differential equation tells us only how things are changing.

**(b)** At the start of the first hour, we have $A = 100$. Substituting this into the differential equation, we have

$$\frac{dA}{dt} = -0.17A = -0.17(100) = -17 \text{ mg/hour.}$$

We estimate that the amount of caffeine decreases by about $(17 \text{ mg/hr}) \cdot (1 \text{ hr}) = 17$ mg during the first hour. This is only an estimate, however, since the derivative $dA/dt$ will not stay constant at $-17$ throughout the entire first hour.

**12.** **(a)** The balance in the account, $B$, is increasing at a rate of \$6000 per year and is also increasing at a rate of $0.07$ times the balance $B$. Notice that both changes cause $B$ to increase, rather than decrease, so both will have a positive effect on $dB/dt$. We have

$$\frac{dB}{dt} = 6000 + 0.07B.$$

**(b)** If $B = 10{,}000$, we have

$$\frac{dB}{dt} = 6000 + 0.07B = 6000 + 0.07(10{,}000) = 6700.$$

If the balance is \$10,000, we expect the balance to increase at a rate of about \$6700 per year.

If $B = 100{,}000$, we have

$$\frac{dB}{dt} = 6000 + 0.07B = 6000 + 0.07(100{,}000) = 13{,}000.$$

If the balance is \$100,000, we expect the balance to increase at a rate of about \$13,000 per year.

**13.** **(a)** To see if $W$ is increasing or decreasing, we determine whether the derivative $dW/dt$ is positive or negative. When $W = 10$, we have

$$\frac{dW}{dt} = 5W - 20 = 5(10) - 20 = 30 > 0.$$

Since $dW/dt$ is positive when $W = 10$, the quantity $W$ is increasing.

When $W = 2$, we have

$$\frac{dW}{dt} = 5W - 20 = 5(2) - 20 = -10 < 0.$$

Since $dW/dt$ is negative when $W = 2$, the quantity $W$ is decreasing.

**(b)** We set $dW/dt = 0$ and solve:

$$\begin{aligned} \frac{dW}{dt} &= 0 \\ 5W - 20 &= 0 \\ W &= 4. \end{aligned}$$

The rate of change of $W$ is zero when $W = 4$.

**14.** The quantity $y$ is increasing when $dy/dt$ is positive. Using the differential equation, we see that $dy/dt$ is positive when $-0.5y$ is positive, which means $y$ is negative. The quantity $y$ is increasing when $y$ is negative. Similarly, the quantity $y$ is decreasing when $y$ is positive.

**15.** The derivative $dN/dt$ represents the number of new Wikipedia articles per day. The model expresses this number as a sum of two terms. The first term is $B$, the number of articles added every day by dedicated wikipedians. The second term, representing the number of articles per day added by the general public, is proportional to the number of articles in the Wikipedia and can be written $kN$, where $k$ is constant. Putting the terms together, we have:

$$\frac{dN}{dt} = B + kN.$$

**16.** The rate of change of the value of infrastructure, $dK/dt$, is the sum of two terms. One term represents increase due to investment, and can be written $k_1Y$, where $Y$ is national income and $k_1$ is a constant of proportionality. The constant $k_1$ is positive because investment increases the value of infrastructure. The other term represents the decrease due to depreciation, and can be written $k_2K$, where $k_2$ is a second proportionality constant. The constant $k_2$ is positive because depreciation decreases the value of infrastructure, which means that the $-k_2K$ is negative. The differential equation for $K$ is

$$\frac{dK}{dt} = k_1Y - k_2K.$$

## Solutions for Section 10.2

**1.** We are told that $y$ is a function of $t$ (since the derivative is $dy/dt$) with derivative $2t$. We need to think of a function with derivative $2t$. Since $y = t^2$ has derivative $2t$, we see that $y = t^2$ is a solution to this differential equation. Since the function $y = t^2 + 1$ also has derivative $2t$, we see that $y = t^2 + 1$ is also a solution. In fact, $y = t^2 + C$ is a solution for any constant $C$. The general solution is

$$y = t^2 + C.$$

**2.** **(a)** Since $y = x^2$, we have $y' = 2x$. Substituting these functions into our differential equation, we have

$$xy' - 2y = x(2x) - 2(x^2) = 2x^2 - 2x^2 = 0.$$

Therefore, $y = x^2$ is a solution to the differential equation $xy' - 2y = 0$.

**(b)** For $y = x^3$, we have $y' = 3x^2$. Substituting gives:

$$xy' - 2y = x(3x^2) - 2(x^3) = 3x^3 - 2x^3 = x^3.$$

Since $x^3$ does not equal 0 for all $x$, we see that $y = x^3$ is not a solution to the differential equation.

**3.** Since $y = t^4$, the derivative is $dy/dt = 4t^3$. We have

$$\text{Left-side} = t\frac{dy}{dt} = t(4t^3) = 4t^4.$$
$$\text{Right-side} = 4y = 4t^4.$$

Since the substitution $y = t^4$ makes the differential equation true, $y = t^4$ is in fact a solution.

**4.** Since $dy/dx = -1$, the slope of the curve must be $-1$ at all points. Since the slope is constant, the solution curve must be a line with slope $-1$. Graph C is a possible solution curve for this differential equation.

**5.** Since $dy/dx = 0.1$, the slope of the curve is $0.1$ at all points. Thus the curve is a line with positive slope, such as Graph F.

**6.** Since $-y^2$ is always less than or equal to zero, the derivative $dy/dx$ is always less than or equal to zero. A possible solution curve must have slope less than or equal to zero at all points, so possible answers are B or C. The slope of $-y^2$ is steeper for large $y$-values and less steep for $y$-values close to zero, so the only possible solution curve for this differential equation is Graph B.

**7.** Since $dy/dx = 2x$, the slope of the solution curve will be negative when $x$ is negative and positive when $x$ is positive. A solution curve will be decreasing for negative $x$ and increasing for positive $x$. The only graph with these features is Graph E.

**8.** Since $dy/dx = 2$, the slope of the solution curve will be 2 at all points. Any possible solution curve for this differential equation will be a line with slope 2. A possible solution curve for this differential equation is Graph F.

**9.** Since $dy/dx = y$, the slope of the solution curve will be positive for positive $y$-values and negative for negative $y$-values. In addition, the slope will be bigger for large $y$ and closer to zero when the $y$-value is closer to zero. A possible solution curve for this differential equation is Graph A.

**10.** Since $dy/dx$ is negative for all $x$, the slope of the solution curve is negative everywhere and becomes closer to horizontal as $x$ increases, as in Graph B.

**11.** Since $dy/dx = 1 - x$ is positive for $x < 1$ and negative for $x > 1$, the slope of the solution curve is positive for $x < 1$ and negative for $x > 1$. The answer is Graph D.

**12.** Since $dy/dx$ is positive if $y$ is positive, the slope of the solution curve is positive everywhere and increases as $y$ increases, as in Graph A.

**13.** We know that at time $t = 0$, the value of $y$ is 8. Since we are told that $dy/dt = 0.5t$, we know that at time $t = 0$

$$\frac{dy}{dt} = 0.5(0) = 0.$$

As $t$ goes from 0 to 1, $y$ will increase by 0, so at $t = 1$,

$$y = 8 + 0(1) = 8.$$

Likewise, we get that at $t = 1$,

$$\frac{dy}{dt} = 0.5(1) = 0.5$$

and so at $t = 2$

$$y = 8 + 0.5(1) = 8.5.$$

At $t = 2$,

$$\frac{dy}{dt} = 0.5(2) = 1$$

then at $t = 3$

$$y = 8.5 + 1(1) = 9.5.$$

At $t = 3$, $\frac{dy}{dt} = 0.5(3) = 1.5$ so that at $t = 4$, $y = 9.5 + 1.5(1) = 11$.
Thus we get the following table

| $t$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $y$ | 8 | 8 | 8.5 | 9.5 | 11 |

**14.** We know that at time $t = 0$, the value of $y$ is 8. Since we are told that $dy/dt = 0.5y$, we know that at time $t = 0$

$$\frac{dy}{dt} = 0.5(8) = 4.$$

As $t$ goes from 0 to 1, $y$ will increase by 4, so at $t = 1$,

$$y = 8 + 4 = 12.$$

Likewise, we get that at $t = 1$,

$$\frac{dy}{dt} = .5(12) = 6$$

so that at $t = 2$,

$$y = 12 + 6 = 18.$$

At $t = 2$, $\frac{dy}{dt} = .5(18) = 9$ so that at $t = 3$, $y = 18 + 9 = 27$.
At $t = 3$, $\frac{dy}{dt} = .5(27) = 13.5$ so that at $t = 4$, $y = 27 + 13.5 = 40.5$.
Thus we get the following table

| $t$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $y$ | 8 | 12 | 18 | 27 | 40.5 |

**15.** We know that at time $t = 0$, the value of $y$ is 8. Since we are told that $dy/dt = 4 - y$, we know that at time $t = 0$

$$\frac{dy}{dt} = 4 - 8 = -4.$$

As $t$ goes from 0 to 1, $y$ will decrease by 4, so at $t = 1$,

$$y = 8 - 4 = 4$$

Likewise, we get that at $t = 1$,

$$\frac{dy}{dt} = 4 - 4 = 0$$

so that at $t = 2$,

$$y = 4 + 0(1) = 4.$$

At $t = 2$, $\frac{dy}{dt} = 4 - 4 = 0$ so that at $t = 3$, $y = 4 + 0 = 4$.
At $t = 3$, $\frac{dy}{dt} = 4 - 4 = 0$ so that at $t = 4$, $y = 4 + 0 = 4$.
Thus we get the following table

| $t$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $y$ | 8 | 4 | 4 | 4 | 4 |

**16.** When $y = 100$, the rate of change of $y$ is

$$\frac{dy}{dt} = \sqrt{y} = \sqrt{100} = 10$$

The value of $y$ goes up by 10 units as $t$ goes up 1 unit. When $t = 1$, we have

$$y = \text{Old value of y + Change in y} = 100 + 10 = 110.$$

Continuing in this way, we obtain the table:

| $t$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $y$ | 100 | 110 | 120.5 | 131.5 | 143.0 |

**17.** At $t = 0$, we know $P = 70$ and we can compute the value of $dP/dt$:

$$\text{At } t = 0, \qquad \text{we have} \quad \frac{dP}{dt} = 0.2P - 10 = 0.2(70) - 10 = 4.$$

The population is increasing at a rate of 4 million fish per year. At the end of the first year, the fish population will have grown by about 4 million fish, and so we have:

$$\text{At } t = 1, \qquad \text{we estimate} \quad P = 70 + 4 = 74.$$

We can now use this new value of $P$ to calculate $dP/dt$ at $t = 1$:

$$\text{At } t = 1, \qquad \text{we have} \quad \frac{dP}{dt} = 0.2P - 10 = 0.2(74) - 10 = 4.8,$$

and so:

$$\text{At } t = 2, \qquad \text{we estimate} \quad P = 74 + 4.8 = 78.8.$$

Continuing in this way, we obtain the values in Table 10.1.

**Table 10.1**

| $t$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $P$ | 70 | 74 | 78.8 | 84.56 |

**18.** If $P = P_0e^t$, then
$$\frac{dP}{dt} = \frac{d}{dt}(P_0e^t) = P_0e^t = P.$$

**19.** If $Q = Ce^{kt}$, then
$$\frac{dQ}{dt} = Cke^{kt} = k(Ce^{kt}) = kQ.$$
We are given that $\frac{dQ}{dt} = -0.03Q$, so we know that $kQ = -0.03Q$. Thus we either have $Q = 0$ (in which case $C = 0$ and $k$ is anything) or $k = -0.03$. Notice that if $k = -0.03$, then $C$ can be any number.

**20.** Yes. To see why, we substitute $y = x^n$ into the equation $13x\frac{dy}{dx} = y$. We first calculate $\frac{dy}{dx} = \frac{d}{dx}(x^n) = nx^{n-1}$. The differential equation becomes
$$13x(nx^{n-1}) = x^n$$
But $13x(nx^{n-1}) = 13n(x \cdot x^{n-1}) = 13nx^n$, so we have
$$13n(x^n) = x^n$$
This equality must hold for all $x$, so we get $13n = 1$, so $n = 1/13$. Thus, $y = x^{1/13}$ is a solution.

**21.** Since $y = x^2 + k$ we know that
$$y' = 2x.$$
Substituting $y = x^2 + k$ and $y' = 2x$ into the differential equation we get
$$\begin{aligned} 10 &= 2y - xy' \\ &= 2(x^2 + k) - x(2x) \\ &= 2x^2 + 2k - 2x^2 \\ &= 2k \end{aligned}$$
Thus, $k = 5$ is the only solution.

**22.** We first compute $dy/dx$ for each of the functions on the right.

If $y = x^3$ then
$$\begin{aligned} \frac{dy}{dx} &= 3x^2 \\ &= 3\frac{y}{x}. \end{aligned}$$
If $y = 3x$ then
$$\begin{aligned} \frac{dy}{dx} &= 3 \\ &= \frac{y}{x}. \end{aligned}$$
If $y = e^{3x}$ then
$$\begin{aligned} \frac{dy}{dx} &= 3e^{3x} \\ &= 3y. \end{aligned}$$
If $y = 3e^x$ then
$$\begin{aligned} \frac{dy}{dx} &= 3e^x \\ &= y. \end{aligned}$$
Finally, if $y = x$ then
$$\begin{aligned} \frac{dy}{dx} &= 1 \\ &= \frac{y}{x}. \end{aligned}$$
Comparing our calculated derivatives with the right-hand sides of the differential equations we see that (a) is solved by (II) and (V), (b) is solved by (I), (c) is not solved by any of our functions, (d) is solved by (IV) and (e) is solved by (III).

## Solutions for Section 10.3

**1. (a)** (i), (ii), (iii)

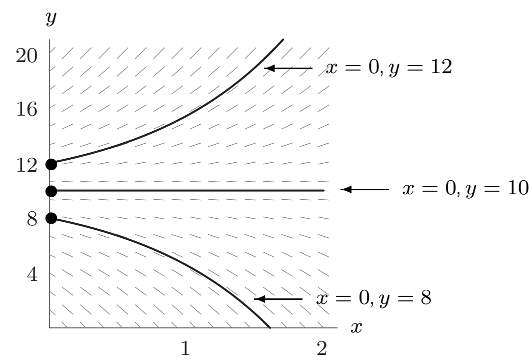



**Figure 10.1**

**(b)** When $y = 10$, we have $dy/dx = 0$ and so the solution curve is horizontal. This is why the solution curve through $y = 10$ is a horizontal line.

**2.** See Figure 10.2. Other choices of solution curves are, of course, possible.

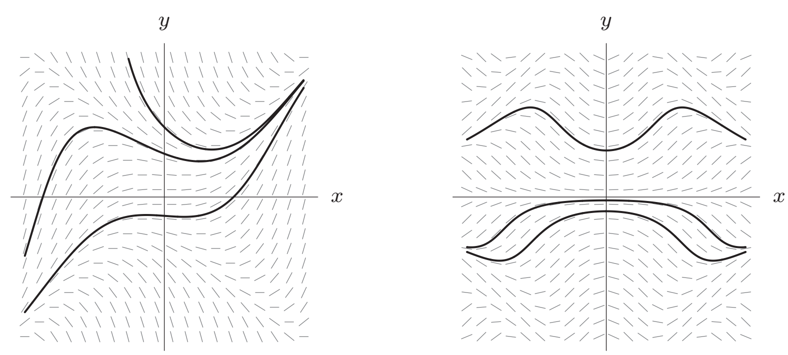



**Figure 10.2**

**3. (a)** See Figure 10.3.

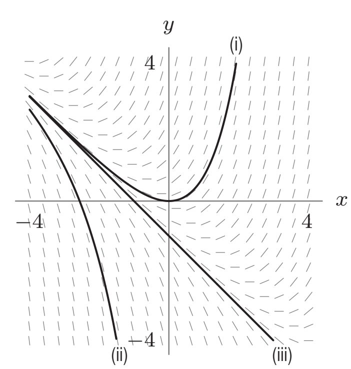



**Figure 10.3**

**(b)** The solution through $(-1, 0)$ appears to be linear with equation $y = -x - 1$.

**(c)** If $y = -x - 1$, then $y' = -1$ and $x + y = x + (-x - 1) = -1$, so this checks as a solution.

**4.** III. The slope field appears to be near zero at $P = 1$ and $P = 0$, so this rules out $dP/dt = P - 1$, which has a slope of $-1$ at $P = 0$. Between $P = 0$ and $P = 1$, the slopes in the figure are positive, so this rules out $dP/dt = P(P-1)$, which has negative values for $0 < P < 1$. To decide between the remaining two possibilities, note that when $P = 1/2$, the slopes in the figure appear to be about 1. The differential equation $dP/dt = 3P(1-P)$ gives a slope of $3 \cdot 1/2 \cdot (1 - 1/2) = 3/4$, while the differential equation $dP/dt = 1/3P(1-P)$ gives a slope of $1/12$ which is clearly too small. Thus the best answer is $dP/dt = 3P(1-P)$.

**5.** **(a)** The slope at any point $(x, y)$ is equal to $dy/dx$, which is $xy$.Then:

$$\begin{aligned}
&\text{At point } (2,1), \text{ slope } = 2 \cdot 1 = 2, \\
&\text{At point } (0,2), \text{ slope } = 0 \cdot 2 = 0, \\
&\text{At point } (-1,1), \text{ slope } = -1 \cdot 1 = -1, \\
&\text{At point } (2,-2), \text{ slope } = 2(-2) = -4.
\end{aligned}$$

**(b)** See Figure 10.4.

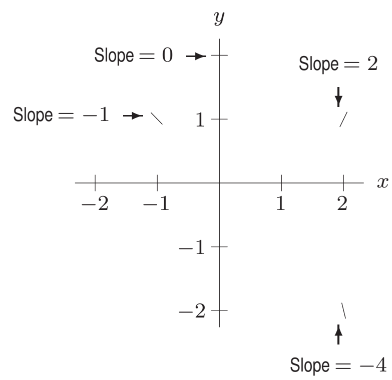



**Figure 10.4**

**6.** **(a)** Since $y' = -y$, the slope is negative above the $x$-axis (when $y$ is positive) and positive below the $x$-axis (when $y$ is negative). The only slope field for which this is true is II.

**(b)** Since $y' = y$, the slope is positive for positive $y$ and negative for negative $y$. This is true of both I and III. As $y$ get larger, the slope should get larger, so the correct slope field is I.

**(c)** Since $y' = x$, the slope is positive for positive $x$ and negative for negative $x$. This corresponds to slope field V.

**(d)** Since $y' = \dfrac{1}{y}$, the slope is positive for positive $y$ and negative for negative $y$. As $y$ approaches 0, the slope becomes larger in magnitude, which correspond to solution curves close to vertical. The correct slope field is III.

**(e)** Since $y' = y^2$, the slope is always positive, so this must correspond to slope field IV.

**7.** (a) II (b) VI (c) IV (d) I (e) III (f) V

**8.** If the starting point has $y > 0$, then $y \to \infty$ as $x \to \infty$. If the starting point has $y = 0$, then the solution is constant; $y = 0$. If the starting point has $y < 0$, then $y \to -\infty$ as $x \to \infty$.

**9.** As $x$ increases, $y \to \infty$.

**10.** As $x \to \infty$, $y \to \infty$, no matter what the starting point is.

**11.** As $x \to \infty$, $y$ seems to oscillate within a certain range. The range will depend on the starting point, but the *size* of the range appears independent of the starting point.

**12.** If $y = 4$ for the starting point, then $y = 4$ always, so $y \to 4$ as $x \to \infty$. If $y \neq 4$ for the starting point, then $y \to 4$ as $x \to \infty$.

**13.** From the slope field, the function looks like a parabola of the form $y = x^2 + C$, where $C$ depends on the starting point. In any case, $y \to \infty$ as $x \to \infty$.

**14.** When $a = 1$ and $b = 2$, the Gompertz equation is $y' = -y\ln(y/2) = y\ln(2/y) = y(\ln 2 - \ln y)$. This differential equation is similar to the differential equation $y' = y(2-y)$ in certain ways. For example, in both equations $y'$ is positive for $0 < y < 2$ and negative for $y > 2$. Also, for $y$-values close to 2, the quantities $(\ln 2 - \ln y)$ and $(2-y)$ are both close to 0, so $y(\ln 2 - \ln y)$ and $y(2-y)$ are approximately equal to zero. Thus around $y = 2$ the slope fields look almost the same. This happens again around $y = 0$, since around $y = 0$ both $y(2-y)$ and $y(\ln 2 - \ln y)$ go to 0. Finally, for $y > 2$, $\ln y$ grows much slower than $y$, so the slope field for $y' = y(\ln 2 - \ln y)$ is less steep, negatively, than for $y' = y(2-y)$.

## Solutions for Section 10.4

**1.** The equation is in the form $dw/dr = kw$, so the general solution is the exponential function

$$w = Ce^{3r}.$$

We find $C$ using the initial condition that $w = 30$ when $r = 0$.

$$\begin{aligned} w &= Ce^{3r} \\ 30 &= Ce^{0} \\ C &= 30. \end{aligned}$$

The solution is

$$w = 30e^{3r}.$$

**2.** The equation is in the form $dy/dx = ky$, so the general solution is the exponential function

$$y = Ce^{-0.14x}.$$

We find $C$ using the initial condition that $y = 5.6$ when $x = 0$.

$$\begin{aligned} y &= Ce^{-0.14x} \\ 5.6 &= Ce^{0} \\ C &= 5.6. \end{aligned}$$

The solution is

$$y = 5.6e^{-0.14x}.$$

**3.** The equation given is in the form

$$\frac{dP}{dt} = kP.$$

Thus we know that the general solution to this equation will be

$$P = Ce^{kt}.$$

And in our case, with $k = 0.02$ and $C = 20$ we get

$$P = 20e^{0.02t}.$$

**4.** The equation is in the form $dp/dq = kp$, so the general solution is the exponential function

$$p = Ce^{-0.1q}.$$

We find $C$ using the condition that $p = 100$ when $q = 5$.

$$\begin{aligned} p &= Ce^{-0.1q} \\ 100 &= Ce^{-0.1(5)} \\ C &= \frac{100}{e^{-0.5}} = 164.87. \end{aligned}$$

The solution is

$$p = 164.87e^{-0.1q}.$$

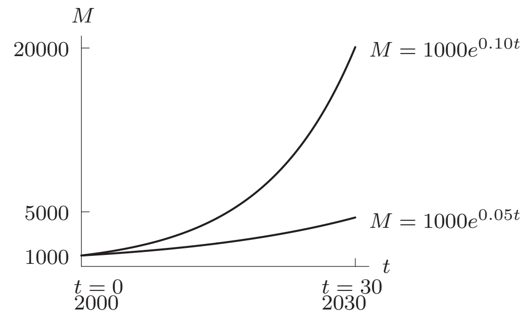
**5.** The equation given is in the form
$$\frac{dQ}{dt} = kQ.$$
Thus we know that the general solution to this equation will be
$$Q = Ce^{kt}.$$
And in our case, with $k = \frac{1}{5}$, we get
$$Q = Ce^{\frac{1}{5}t}.$$
We know that $Q = 50$ when $t = 0$. Thus we get
$$\begin{aligned} Q(t) &= Ce^{\frac{1}{5}t} \\ Q(0) &= 50 = Ce^0 \\ 50 &= C \end{aligned}$$
Thus we get
$$Q = 50e^{\frac{1}{5}t}.$$

**6.** Rewriting we get
$$\frac{dy}{dx} = -\frac{1}{3}y.$$
We know that the general solution to an equation in the form
$$\frac{dy}{dx} = ky$$
is
$$y = Ce^{kx}.$$
Thus in our case we get
$$y = Ce^{-\frac{1}{3}x}.$$
We are told that $y(0) = 10$ so we get
$$\begin{aligned} y(x) &= Ce^{-\frac{1}{3}x} \\ y(0) &= 10 = Ce^0 \\ C &= 10 \end{aligned}$$
Thus we get
$$y = 10e^{-\frac{1}{3}x}.$$

**7.** **(a)** The rate of growth of the money in the account is proportional to the amount of money in the account. Thus
$$\frac{dM}{dt} = rM.$$

**(b)** We know that the equation
$$\frac{dM}{dt} = rM$$
has the general solution
$$M = Ae^{rt}.$$
We know that in 2000 (i.e. $t = 0$) we have $M = 1000$. Thus we get
$$\begin{aligned} M &= Ae^{rt} \\ M(0) &= 1000 = Ae^{0r} \\ 1000 &= Ae^0 \\ A &= 1000. \end{aligned}$$
Thus we get
$$M = 1000e^{rt}.$$

**(c)**




**8.** **(a)** Since interest is earned continuously,

$$\text{Rate of change of balance} = 1.5\%(\text{Balance})$$

so

$$\frac{dB}{dt} = 0.015B.$$

**(b)** $B = Ce^{0.015t}$ is the general solution. Since $B = 5000$ when $t = 0$, we have $C = 5000$. The solution is $B = 5000e^{0.015t}$

**(c)** When $t = 10$, $B = 5000e^{0.015(10)} = \$5809.17$.

**9.** **(a)** If $B = f(t)$ (where $t$ is in years)

$$\begin{aligned}\frac{dB}{dt} &= \text{Rate at which interest is earned} + \text{Rate at which money is deposited}\\ &= 0.10B + 1000.\end{aligned}$$

**(b)**

$$\frac{dB}{dt} = 0.1(B + 10{,}000)$$

We know that a differential equation of the form

$$\frac{dB}{dt} = k(B - A)$$

has general solution:

$$B = Ce^{kt} + A.$$

Thus, in our case

$$B = Ce^{0.1t} - 10{,}000.$$

For $t = 0$, $B = 0$, hence $C = 10{,}000$. Therefore, $B = 10{,}000e^{0.1t} - 10{,}000$.

**10.** Since the rate of change is proportional to the amount present, we have $\frac{dQ}{dt} = kQ$. We know the constant of proportionality is $k = -0.0025$, so a differential equation for $Q$ as a function of $t$ is

$$\frac{dQ}{dt} = -0.0025Q.$$

The solution to this differential equation is

$$Q = Ce^{-0.0025t},$$

for some constant $C$. When $t = 20$, we have $Q = Ce^{-0.0025(20)} = C(0.951)$, so approximately 95% of the current ozone will still be here in 20 years. Approximately 5% will decay during this time.

**11.** Michigan:

$$\frac{dQ}{dt} = -\frac{r}{V}Q = -\frac{158}{4.9 \times 10^3}Q \approx -0.032Q$$

so

$$Q = Q_0e^{-0.032t}.$$

We want to find $t$ such that

$$0.1Q_0 = Q_0e^{-0.032t}$$

so

$$t = \frac{-\ln(0.1)}{0.032} \approx 72 \text{ years.}$$

Ontario:

$$\frac{dQ}{dt} = -\frac{r}{V}Q = \frac{-209}{1.6 \times 10^3}Q = -0.131Q$$

so

$$Q = Q_0 e^{-0.131t}.$$

We want to find $t$ such that

$$0.1Q_0 = Q_0 e^{-0.131t}$$

so

$$t = \frac{-\ln(0.1)}{0.131} \approx 18 \text{ years.}$$

Lake Michigan will take longer because it is larger (4900 km$^3$ compared to 1600 km$^3$) and water is flowing through it at a slower rate (158 km$^3$/year compared to 209 km$^3$/year).

**12.** Lake Superior will take the longest, because the lake is largest ($V$ is largest) and water is moving through it most slowly ($r$ is smallest). Lake Erie looks as though it will take the least time because $V$ is smallest and $r$ is close to the largest. For Erie, $k = r/V = 175/460 \approx 0.38$. The lake with the largest value of $r$ is Ontario, where $k = r/V = 209/1600 \approx 0.13$. Since $e^{-kt}$ decreases faster for larger $k$, Lake Erie will take the shortest time for any fixed fraction of the pollution to be removed.

For Lake Superior,

$$\frac{dQ}{dt} = -\frac{r}{V}Q = -\frac{65.2}{12{,}200}Q \approx -0.0053Q$$

so

$$Q = Q_0 e^{-0.0053t}.$$

When 80% of the pollution has been removed, 20% remains so $Q = 0.2Q_0$. Substituting gives us

$$0.2Q_0 = Q_0 e^{-0.0053t}$$

so

$$t = -\frac{\ln(0.2)}{0.0053} \approx 301 \text{ years.}$$

(Note: The 301 is obtained by using the exact value of $\frac{r}{V} = \frac{65.2}{12{,}200}$, rather than 0.0053. Using 0.0053 gives 304 years.)
For Lake Erie, as in the text

$$\frac{dQ}{dt} = -\frac{r}{V}Q = -\frac{175}{460}Q \approx -0.38Q$$

so

$$Q = Q_0 e^{-0.38t}.$$

When 80% of the pollution has been removed

$$0.2Q_0 = Q_0 e^{-0.38t}$$
$$t = -\frac{\ln(0.2)}{0.38} \approx 4 \text{ years.}$$

So the ratio is

$$\frac{\text{Time for Lake Superior}}{\text{Time for Lake Erie}} \approx \frac{301}{4} \approx 75.$$

In other words it will take about 75 times as long to clean Lake Superior as Lake Erie.

**13.** **(a)** Since the amount leaving the blood is proportional to the quantity in the blood,

$$\frac{dQ}{dt} = -kQ \quad \text{for some positive constant } k.$$

Thus $Q = Q_0 e^{-kt}$, where $Q_0$ is the initial quantity in the bloodstream. Only 20% is left in the blood after 3 hours. Thus $0.20 = e^{-3k}$, so $k = \frac{\ln 0.20}{-3} \approx 0.5365$. Therefore $Q = Q_0 e^{-0.5365t}$.

**(b)** Since 20% is left after 3 hours, after 6 hours only 20% of that 20% will be left. Thus after 6 hours only 4% will be left, so if the patient is given 100 mg, only 4 mg will be left 6 hours later.

**14.** **(a)** Suppose $Y(t)$ is the quantity of oil in the well at time $t$. We know that the oil in the well decreases at a rate proportional to $Y(t)$, so

$$\frac{dY}{dt} = -kY.$$

Integrating, and using the fact that initially $Y = Y_0 = 10^6$, we have

$$Y = Y_0 e^{-kt} = 10^6 e^{-kt}.$$

In six years, $Y = 500,000 = 5 \cdot 10^5$, so

$$5 \cdot 10^5 = 10^6 e^{-k \cdot 6}$$

so

$$\begin{aligned} 0.5 &= e^{-6k} \\ k &= -\frac{\ln 0.5}{6} = 0.1155. \end{aligned}$$

When $Y = 600,000 = 6 \cdot 10^5$,

$$\text{Rate at which oil decreasing } = \left|\frac{dY}{dt}\right| = kY = 0.1155(6 \cdot 10^5) = 69{,}300 \text{ barrels/year.}$$

**(b)** We solve the equation

$$\begin{aligned} 5 \cdot 10^4 &= 10^6 e^{-0.1155t} \\ 0.05 &= e^{-0.1155t} \\ t &= \frac{\ln 0.05}{-0.1155} = 25.9 \text{ years.} \end{aligned}$$

**15.** **(a)** Since the rate of change is proportional to the amount present, $dy/dt = ky$ for some constant $k$.

**(b)** Solving the differential equation, we have $y = Ae^{kt}$, where $A$ is the initial amount. Since 100 grams become 54.9 grams in one hour, $54.9 = 100e^k$, so $k = \ln(54.9/100) \approx -0.5997$.
Thus, after 10 hours, there remains $100e^{(-0.5997)10} \approx 0.2486$ grams.

**16.** **(a)** Since we are told that the rate at which the quantity of the drug decreases is proportional to the amount of the drug left in the body, we know the differential equation modeling this situation is

$$\frac{dQ}{dt} = -kQ.$$

Since we are told that the quantity of the drug is decreasing, we include the negative sign.

**(b)** We know that the general solution to the differential equation

$$\frac{dQ}{dt} = -kQ$$

is

$$Q = Ce^{-kt}.$$

**(c)** We are told that the half life of the drug is 3.8 hours. This means that at $t = 3.8$ the amount of the drug in the body is half the amount that was in the body at $t = 0$, or in other words

$$0.5Q(0) = Q(3.8).$$

Solving this equation gives

$$\begin{aligned} 0.5Ce^{-k(0)} &= Ce^{-k(3.8)} \\ 0.5C &= Ce^{-k(3.8)} \\ 0.5 &= e^{-k(3.8)} \\ \ln(0.5) &= -k(3.8) \\ k &= \frac{-\ln(0.5)}{3.8} \\ &\approx 0.182. \end{aligned}$$

**(d)** From part (c) we know that the formula for $Q$ is

$$Q = Ce^{-0.182t}.$$

We are told that initially there are 10 mg of the drug in the body. Thus at $t = 0$ we get

$$10 = Ce^{-0.182(0)}$$

$$C = 10.$$

Thus the formula is

$$Q(t) = 10e^{-0.182t}.$$

Substituting in $t = 12$ gives

$$\begin{aligned} Q(12) &= 10e^{-0.182(12)} \\ &= 10e^{-2.184} \\ Q(12) &\approx 1.126 \text{ mg} \end{aligned}$$

**17. (a)** Assuming that the world's population grows exponentially, satisfying $dP/dt = cP$, and that the land in use for crops is proportional to the population, we expect $A$ to satisfy $dA/dt = kA$.

**(b)** We have $A(t) = A_0 e^{kt} = (1 \times 10^9)e^{kt}$, where $t$ is the number of years after 1950. Since $2 \times 10^9 = (1 \times 10^9)e^{k(30)}$, we have $e^{30k} = 2$, so $k = \frac{\ln 2}{30} \approx 0.023$. Thus, $A \approx (1 \times 10^9)e^{0.023t}$. We want to find $t$ such that $3.2 \times 10^9 = A(t) = (1 \times 10^9)e^{0.023t}$. Taking logarithms yields

$$t = \frac{\ln(3.2)}{0.023} \approx 50.6 \text{ years.}$$

Thus this model predicts land will have run out by the year 2001.

## Solutions for Section 10.5

**1.** We know that the general solution to a differential equation of the form

$$\frac{dH}{dt} = k(H - A)$$

is

$$H = A + Ce^{kt}.$$

Thus in our case we get

$$H = 75 + Ce^{3t}.$$

We know that at $t = 0$ we have $H = 0$, so solving for $C$ we get

$$\begin{aligned} H &= 75 + Ce^{3t} \\ 0 &= 75 + Ce^{3(0)} \\ -75 &= Ce^{0} \\ C &= -75. \end{aligned}$$

Thus we get

$$H = 75 - 75e^{3t}.$$

**2.** We know that the general solution to a differential equation of the form

$$\frac{dy}{dt} = k(y - A)$$

is

$$y = A + Ce^{kt}.$$

Thus in our case we get

$$y = 200 + Ce^{0.5t}.$$

We know that at $t = 0$ we have $y = 50$, so solving for $C$ we get

$$\begin{aligned} y &= 200 + Ce^{0.5t} \\ 50 &= 200 + Ce^{0.5(0)} \\ -150 &= Ce^{0} \\ C &= -150. \end{aligned}$$

Thus we get

$$y = 200 - 150e^{0.5t}.$$

**3.** We know that the general solution to a differential equation of the form

$$\frac{dP}{dt} = k(P - A)$$

is

$$P = Ce^{kt} + A.$$

Thus in our case we have $k = 1$, so we get

$$P = Ce^{t} - 4.$$

We know that at $t = 0$ we have $P = 100$ so solving for $C$ we get

$$\begin{aligned} P &= Ce^{t} - 4 \\ 100 &= Ce^{0} - 4 \\ 104 &= Ce^{0} \\ C &= 104. \end{aligned}$$

Thus we get

$$P = 104e^{t} - 4.$$

**4.** We know that the general solution to a differential equation of the form

$$\frac{dB}{dt} = k(B - A)$$

is

$$B = A + Ce^{kt}.$$

To get our equation in this form we factor out a $4$ to get

$$\frac{dB}{dt} = 4\left(B - \frac{100}{4}\right) = 4(B - 25).$$

Thus in our case we get

$$B = Ce^{4t} + 25.$$

We know that at $t = 0$ we have $B = 20$, so solving for $C$ we get

$$\begin{aligned} B &= 25 + Ce^{4t} \\ 20 &= 25 + Ce^{4(0)} \\ -5 &= Ce^{0} \\ C &= -5. \end{aligned}$$

Thus we get

$$B = 25 - 5e^{4t}.$$

**5.** We know that the general solution to a differential equation of the form

$$\frac{dQ}{dt} = k(Q - A)$$

is

$$H = A + Ce^{kt}.$$

To get our equation in this form we factor out a 0.3 to get

$$\frac{dQ}{dt} = 0.3\left(Q - \frac{120}{0.3}\right) = 0.3(Q - 400).$$

Thus in our case we get

$$Q = 400 + Ce^{0.3t}.$$

We know that at $t = 0$ we have $Q = 50$, so solving for $C$ we get

$$\begin{aligned} Q &= 400 + Ce^{0.3t} \\ 50 &= 400 + Ce^{0.3(0)} \\ -350 &= Ce^{0} \\ C &= -350. \end{aligned}$$

Thus we get

$$Q = 400 - 350e^{0.3t}.$$

**6.** We know that the general solution to a differential equation of the form

$$\frac{dm}{dt} = k(m - A)$$

is

$$m = Ce^{kt} + A.$$

Factoring out a 0.1 on the left side we get

$$\frac{dm}{dt} = 0.1\left(m - \frac{-200}{0.1}\right) = 0.1(m - (-2000)).$$

Thus in our case we get

$$m = Ce^{0.1t} - 2000.$$

We know that at $t = 0$ we have $m = 1000$ so solving for $C$ we get

$$\begin{aligned} m &= Ce^{0.1t} - 2000 \\ 1000 &= Ce^{0} - 2000 \\ 3000 &= Ce^{0} \\ C &= 3000. \end{aligned}$$

Thus we get

$$m = 3000e^{0.1t} - 2000.$$

**7.** We know that the general solution to a differential equation of the form

$$\frac{dB}{dt} = k(B - A)$$

is

$$B = Ce^{kt} + A.$$

Rewriting we get

$$\frac{dB}{dt} = -2B + 50.$$

Factoring out a $-2$ on the right side we get

$$\frac{dB}{dt} = -2\left(B - \frac{-50}{-2}\right) = -2(B-25).$$

Thus in our case we get

$$B = Ce^{-2t} + 25.$$

We know that at $t = 1$ we have $B = 100$ so solving for $C$ we get

$$\begin{aligned} B &= Ce^{-2t} + 25 \\ 100 &= Ce^{-2} + 25 \\ 75 &= Ce^{-2} \\ C &= 75e^{2}. \end{aligned}$$

Thus we get

$$B = 75e^{2}e^{-2t} + 25 = 75e^{2-2t} + 25.$$

**8.** Rewrite the differential equation as

$$\frac{dB}{dt} = -0.1B + 10$$

Factoring out $-0.1$ gives

$$\frac{dB}{dt} = -0.1(B - 100),$$

which has solution

$$B = 100 + Ce^{-0.1t}.$$

Substituting $B = 3$ and $t = 2$ gives

$$3 = 100 + Ce^{-0.1(2)}.$$

Solving for $C$ we get

$$C = -\frac{97}{e^{-0.02}} \approx -99$$

So the solution is $B = 100 - 99e^{-0.1t}$.

**9.** In order to check that $y = A + Ce^{kt}$ is a solution to the differential equation

$$\frac{dy}{dt} = k(y - A),$$

we must show that the derivative of $y$ with respect to $t$ is equal to $k(y - A)$:

$$\begin{aligned} y &= A + Ce^{kt} \\ \frac{dy}{dt} &= 0 + (Ce^{kt})(k) \\ &= kCe^{kt} \\ &= k(Ce^{kt} + A - A) \\ &= k\left((Ce^{kt} + A) - A\right) \\ &= k(y - A) \end{aligned}$$

**10. (a)** We know that the rate by which the account changes every year is

Rate of change of balance = Rate of increase − Rate of decrease.

Since \$1000 will be withdrawn every year, we know that the account decreases by \$1000 every year. We also know that the account accumulates interest at 7% compounded continuously. Thus the amount by which the account increases each year is

Rate balance increases per year = 7%(Account balance) = 0.07(Account balance).

Denoting the account balance by $B$ we get

$$\text{Rate balance increases per year} = 0.07B.$$

Thus we get

$$\text{Rate of change of balance} = 0.07B - 1000.$$

or

$$\frac{dB}{dt} = 0.07B - 1000,$$

with $t$ measured in years.

**(b)** The equilibrium solution makes the derivative 0, so

$$\begin{aligned} \frac{dB}{dt} &= 0 \\ 0.07B - 1000 &= 0 \\ B &= \frac{1000}{0.07} \approx \$14{,}285.71. \end{aligned}$$

**(c)** We know that the general solution to a differential equation of the form

$$\frac{dB}{dt} = k(B - A)$$

is

$$B = Ce^{kt} + A.$$

To get our equation in this form we factor out a 0.07 to get

$$\frac{dB}{dt} = 0.07\left(B - \frac{1000}{0.07}\right) \approx 0.07(B - 14{,}285.71).$$

Thus in our case we get

$$B = Ce^{0.07t} + 14{,}285.71.$$

We know that at $t = 0$ we have $B = \$10{,}000$ so solving for $C$ we get

$$\begin{aligned} B &= Ce^{0.07t} + 14{,}285.71 \\ 10{,}000 &= Ce^{4(0)} + 14{,}285.71 \\ -4285.71 &= Ce^{0} \\ C &= -4285.71. \end{aligned}$$

Thus we get

$$B = 14{,}285.71 - (4285.71)e^{0.07t}.$$

**(d)** Substituting the value $t = 5$ into our function for $B$ we get

$$\begin{aligned} B(t) &= 14{,}285.71 - (4285.71)e^{0.07t} \\ B(5) &= 14{,}285.71 - (4285.71)e^{0.07(5)} \\ &= 14{,}285.71 - (4285.71)e^{0.35} \\ B(5) &\approx \$8204 \end{aligned}$$

**(e)** From Figure 10.5 we see that in the long run there is no money left in the account.

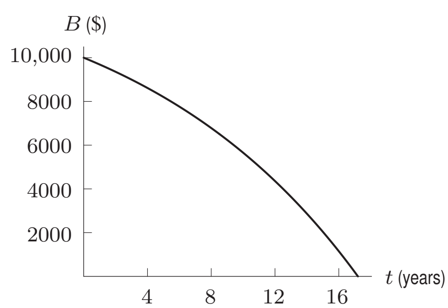



Figure 10.5

**11.** **(a)** The value of the company satisfies

$$\text{Rate of change of value} = \text{Rate interest earned} - \text{Rate expenses paid}$$

so

$$\frac{dV}{dt} = 0.02V - 80{,}000.$$

**(b)** We find $V$ when

$$\begin{aligned} \frac{dV}{dt} &= 0 \\ 0.02V - 80{,}000 &= 0 \\ 0.02V &= 80{,}000 \\ V &= 4{,}000{,}000 \end{aligned}$$

There is an equilibrium solution at $V = \$4{,}000{,}000$. If the company has $\$4{,}000{,}000$ in assets, its earnings will exactly equal its expenses.

**(c)** The general solution is

$$V = 4{,}000{,}000 + Ce^{0.02t}.$$

**(d)** If $V = 3{,}000{,}000$ when $t = 0$, we have $C = -1{,}000{,}000$. The solution is

$$V = 4{,}000{,}000 - 1{,}000{,}000e^{0.02t}.$$

When $t = 12$, we have

$$\begin{aligned} V &= 4{,}000{,}000 - 1{,}000{,}000e^{0.02(12)} \\ &= 4{,}000{,}000 - 1{,}271{,}249 \\ &= \$2{,}728{,}751. \end{aligned}$$

The company is losing money.

**12.** The bank account is earning money at a rate of 8% times the current balance, and it is losing money at a constant rate of \$5000 a year. We have

$$\begin{aligned} \text{Rate of change of } B &= \text{Rate in} - \text{Rate out} \\ \frac{dB}{dt} &= 0.08B - 5000 = 0.08(B - 62{,}500). \end{aligned}$$

The solution to this differential equation is $B = 62{,}500 + Ce^{0.08t}$, for some constant $C$. To find $C$, we use the fact that $B = 50{,}000$ when $t = 0$:

$$\begin{aligned} B &= 62{,}500 + Ce^{0.08t} \\ 50{,}000 &= 62{,}500 + Ce^{0} \\ C &= -12{,}500. \end{aligned}$$

The solution is

$$B = 62{,}500 - 12{,}500e^{0.08t}.$$

This solution is shown in Figure 10.6. We see that the account loses money, and runs out of money in about 20 years. Algebraically, $B = 0$ where

$$\begin{aligned} 12{,}500e^{0.08t} &= 62{,}500 \\ e^{0.08t} &= \frac{62{,}500}{12{,}500} = 5 \\ t &= \frac{\ln 5}{0.08} \approx 20.1. \end{aligned}$$

So the account runs out of money in 20.1 years.

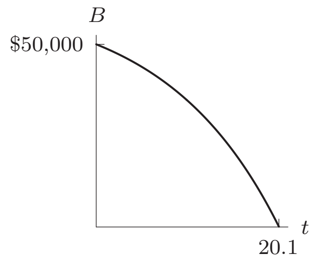



Figure 10.6

**13.** **(a)** For this situation,

$$\begin{pmatrix}\text{Rate money added}\\ \text{to account}\end{pmatrix} = \begin{pmatrix}\text{Rate money added}\\ \text{via interest}\end{pmatrix} + \begin{pmatrix}\text{Rate money}\\ \text{deposited}\end{pmatrix}.$$

Translating this into an equation yields

$$\frac{dB}{dt} = 0.05B + 1200.$$

**(b)** We know that the general solution to the differential equation

$$\frac{dB}{dt} = k(B + A)$$

is

$$B = Ce^{kt} - A.$$

We factor out 0.05 to put our equation in the form

$$\frac{dB}{dt} = 0.05\left(B + \frac{1200}{0.05}\right) = 0.05(B + 24{,}000).$$

This equation has the solution

$$B = Ce^{0.05t} - 24{,}000.$$

Solving for $C$ with $B(0) = 0$ we get $C = 24{,}000$ and so

$$B = f(t) = 24{,}000(e^{0.05t} - 1).$$

The solution is

$$B = 24{,}000e^{0.05t} - 24{,}000.$$

**(c)** After 5 years, the balance is

$$B = f(5) = 24{,}000(e^{0.05(5)} - 1) = 6816.61 \text{ dollars}.$$

**14.** **(a)** The quantity increases with time. As the quantity increases, the rate at which the drug is excreted also increases, and so the rate at which the drug builds up in the blood decreases; thus the graph of quantity against time is concave down. The quantity rises until the rate of excretion exactly balances the rate at which the drug is entering; at this quantity there is a horizontal asymptote.

**(b)** Theophylline enters at a constant rate of $43.2$mg/hour and leaves at a rate of $0.082Q$, so we have

$$\frac{dQ}{dt} = 43.2 - 0.082Q$$

**(c)** We know that the general solution to a differential equation of the form

$$\frac{dy}{dt} = k(y - A)$$

is

$$y = Ce^{kt} + A.$$

Thus in our case, since

$$\frac{dQ}{dt} = 43.2 - 0.082Q \approx -0.082(Q - 526.8),$$

we have

$$Q = 526.8 + Ce^{-0.082t}.$$

Since $Q = 0$ when $t = 0$, we can solve for $C$:

$$Q = 526.8 + Ce^{-0.082t}$$
$$0 = 526.8 + Ce^{0}$$
$$C = -526.8$$

The solution is

$$Q = 526.8 - 526.8e^{-0.082t}.$$

In the long run, the quantity in the body approaches 526.8 mg. See Figure 10.7.

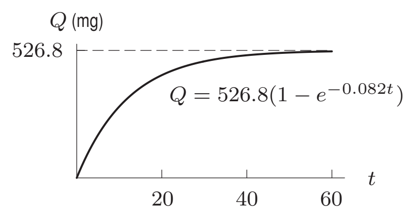



**Figure 10.7**

**15. (a)** We know that the general solution to a differential equation of the form

$$\frac{dy}{dt} = k(y - A)$$

is

$$y = Ce^{kt} + A.$$

Factoring out a $-1$ on the left side we get

$$\frac{dy}{dt} = -(y - 100)$$

Thus in our case we get

$$y = Ce^{-t} + 100.$$

This is meaningful if $C \leq 0$, since one cannot know more than 100%.

**(b)** See Figure 10.8.

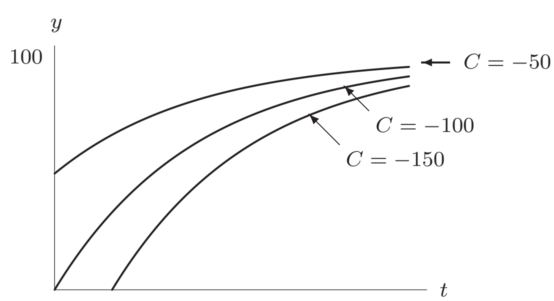



**Figure 10.8**

**(c)** Substituting $y = 0$ when $t = 0$ gives

$$0 = 100 - Ce^{-0}$$

so $C = 100$. The solution is

$$y = 100 - 100e^{-t}$$

**16.** **(a)** The smoker smokes 5 cigarettes per hour, and each cigarette contributes $0.4$ mg of nicotine, so, every hour, the amount of nicotine is increasing by $5(0.4) = 2.0$ mg. At the same time, the nicotine is being eliminated at a rate of $-0.346$ times the amount of nicotine. Thus, we have

$$\frac{dN}{dt} = \text{Rate in} - \text{Rate out} = 2.0 - 0.346N.$$

**(b)** We have

$$\frac{dN}{dt} = 2.0 - 0.346N = -0.346(N - 5.78),$$

so the solution is

$$N = 5.78 - 5.78e^{-0.346t}.$$

**(c)** At $t = 16$, we have $N = 5.78 - 5.78e^{-0.346(16)} = 5.76$ mg.

**17.** **(a)** $\frac{dy}{dt} = -k(y - a)$, where $k > 0$ and $a$ are constants.

**(b)** We know that the general solution to a differential equation of the form

$$\frac{dy}{dt} = -k(y - a)$$

is

$$y = Ce^{-kt} + a.$$

We can assume that right after the course is over (at $t = 0$) 100% of the material is remembered. Thus we get

$$\begin{aligned} y &= Ce^{-kt} + a \\ 1 &= Ce^{0} + a \\ C &= 1 - a. \end{aligned}$$

Thus

$$y = (1 - a)e^{-kt} + a.$$

**(c)** As $t \to \infty$, $e^{-kt} \to 0$, so $y \to a$.

Thus, $a$ represents the fraction of material which is remembered in the long run. The constant $k$ tells us about the rate at which material is forgotten.

**18.** **(a)** We know that the equilibrium solution are the functions satisfying the differential equation whose derivative everywhere is 0. Thus we must solve the equation

$$\frac{dy}{dt} = 0.$$

Solving we get

$$\begin{aligned} \frac{dy}{dt} &= 0 \\ 0.2(y - 3)(y + 2) &= 0 \\ (y - 3)(y + 2) &= 0 \end{aligned}$$

Thus the solutions are $y = 3$ and $y = -2$.

**(b)** Looking at Figure 10.9 we see that the line $y = 3$ is an unstable solution while the line $y = -2$ is a stable solution.

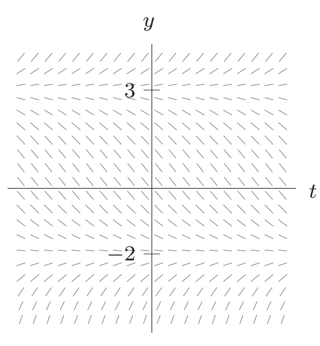



**Figure 10.9**

**19.** **(a)** We know that the equilibrium solution is the solution satisfying the differential equation whose derivative is everywhere 0. Thus we must solve

$$\frac{dy}{dt} = 0.$$

Solving this gives

$$\begin{aligned} \frac{dy}{dt} &= 0 \\ 0.5y - 250 &= 0 \\ y &= 500 \end{aligned}$$

**(b)** We know that the general solution to a differential equation of the form

$$\frac{dy}{dt} = k(y - A)$$

is

$$y = A + Ce^{kt}.$$

To get our equation in this form we factor out a 0.5 to get

$$\frac{dy}{dt} = 0.5\left(y - \frac{250}{0.5}\right) = 0.5(y - 500).$$

Thus in our case we get

$$y = 500 + Ce^{0.5t}.$$

**(c)** The graphs of several solutions is shown in Figure 10.10.

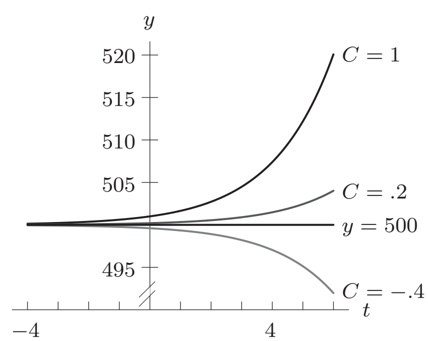



**Figure 10.10**

**(d)** Looking at Figure 10.10 we see that as $t \to \infty$, the value of $y$ gets further and further away from the line $y = 500$. The equilibrium solution $y = 500$ is unstable.

**20.** Graphically, a function is an equilibrium solution if its graph is a horizontal line. From the slope field we see that the equilibrium solutions are $y = 1$ and $y = 3$. An equilibrium solution is stable if a small change in the initial value conditions gives a solution which tends toward the equilibrium as $t$ tends to positive infinity. Thus, by looking at the given slope fields, we see that $y = 3$ is a stable solution while $y = 1$ is an unstable solution.

**21.** **(a)** We know that the general solution to a differential equation of the form

$$\frac{dH}{dt} = -k(H - 200)$$

is

$$H = Ce^{-kt} + 200.$$

We know that at $t = 0$ we have $H = 20$ so solving for $C$ we get

$$\begin{aligned} H &= Ce^{-kt} + 200 \\ 20 &= Ce^{0} + 200 \\ C &= -180. \end{aligned}$$

Thus we get

$$H = -180e^{-kt} + 200.$$

**(b)** Using part (a), we have $120 = 200 - 180e^{-k(30)}$. Solving for $k$, we have $e^{-30k} = \frac{-80}{-180}$, giving

$$k = \frac{\ln \frac{4}{9}}{-30} \approx 0.027.$$

Note that this $k$ is correct if $t$ is given in *minutes*. (If $t$ is given in hours, $k = \frac{\ln \frac{4}{9}}{-\frac{1}{2}} \approx 1.62$.)

**22. (a)** The differential equation is

$$\frac{dT}{dt} = -k(T - A),$$

where $A = 10^\circ$F is the outside temperature.

**(b)** We know that the general solution to a differential equation of the form

$$\frac{dT}{dt} = -k(T - 10)$$

is

$$T = Ce^{-kt} + 10.$$

We know that initially $T = 68^\circ$F. Thus, letting $t = 0$ correspond to 1 pm, we get

$$\begin{aligned} T &= Ce^{-kt} + 10 \\ 68 &= Ce^{0} + 10 \\ C &= 58. \end{aligned}$$

Thus

$$T = 10 + 58e^{-kt}.$$

Since 10:00 pm corresponds to $t = 9$,

$$\begin{aligned} 57 &= 10 + 58e^{-9k} \\ \frac{47}{58} &= e^{-9k} \\ \ln \frac{47}{58} &= -9k \\ k &= -\frac{1}{9} \ln \frac{47}{58} \approx 0.0234. \end{aligned}$$

At 7:00 the next morning $(t = 18)$ we have

$$\begin{aligned} T &\approx 10 + 58e^{18(-0.0234)} \\ &= 10 + 58(0.66) \\ &\approx 48^\circ\text{F}, \end{aligned}$$

so the pipes won't freeze.

**(c)** We assumed that the temperature outside the house stayed constant at $10^\circ$F. This is probably incorrect because the temperature was most likely warmer during the day (between 1 pm and 10 pm) and colder after (between 10 pm and 7 am). Thus, when the temperature in the house dropped from $68^\circ$F to $57^\circ$F between 1 pm and 10 pm, the outside temperature was probably higher than $10^\circ$F, which changes our calculation of the value of the constant $k$. The house temperature will most certainly be lower than $48^\circ$F at 7 am, but not by much—not enough to freeze.

**23. (a)** $\frac{dT}{dt} = -k(T - A)$, where $A = 68^\circ$F is the temperature of the room, and $t$ is time since 9 am.

**(b)** We know that the general solution to a differential equation of the form

$$\frac{dT}{dt} = -k(T - 68)$$

is

$$T = Ce^{-kt} + 68.$$

We know that the temperature of the body is $90.3^\circ$F at 9 am. Thus, letting $t = 0$ correspond to 9 am, we get

$$\begin{aligned} T &= Ce^{-kt} + 68 \\ T(0) &= 90.3 = Ce^{-k(0)} + 68 \\ 90.3 &= Ce^{0} + 68 \\ C &= 90.3 - 68 = 22.3 \end{aligned}$$

Thus

$$T = 68 + 22.3e^{-kt}.$$

At $t = 1$, we have

$$\begin{aligned} 89.0 &= 68 + 22.3e^{-k} \\ 21 &= 22.3e^{-k} \\ k &= -\ln\frac{21}{22.3} \approx 0.06. \end{aligned}$$

Thus $T = 68 + 22.3e^{-0.06t}$.

We want to know when $T$ was equal to $98.6^\circ$F, the temperature of a live body, so

$$\begin{aligned} 98.6 &= 68 + 22.3e^{-0.06t} \\ \ln\frac{30.6}{22.3} &= -0.06t \\ t &= \left(-\frac{1}{0.06}\right)\ln\frac{30.6}{22.3} \\ t &\approx -5.27. \end{aligned}$$

The victim was killed approximately $5\frac{1}{4}$ hours prior to 9 am, at 3:45 am.

**24. (a)** We have

$$\frac{dQ}{dt} = r - \alpha Q.$$

We know that the general solution to a differential equation of the form

$$\frac{dQ}{dt} = k(Q - A)$$

is

$$Q = Ce^{kt} + A.$$

Factoring out a $-\alpha$ on the left side we get

$$\frac{dQ}{dt} = -\alpha\left(Q - \frac{r}{\alpha}\right).$$

Thus in our case we get

$$Q = Ce^{-\alpha t} + \frac{r}{\alpha}.$$

We know that at $t = 0$ we have $Q = 0$ so solving for $C$ we get

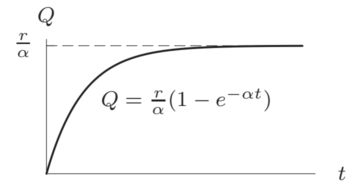
$$\begin{aligned} Q &= Ce^{-\alpha t} + \frac{r}{\alpha} \\ 0 &= Ce^{0} + \frac{r}{\alpha} \\ C &= -\frac{r}{\alpha}. \end{aligned}$$

Thus we get

$$Q = -\frac{r}{\alpha}e^{-\alpha t} + \frac{r}{\alpha}.$$

So,

$$Q_\infty = \lim_{t\to\infty} Q = \frac{r}{\alpha}.$$




**(b)** Doubling $r$ doubles $Q_\infty$. Since $Q_\infty = r/\alpha$, the time to reach $\frac{1}{2}Q_\infty$ is obtained by solving

$$\begin{aligned} \frac{r}{2\alpha} &= \frac{r}{\alpha}(1 - e^{-\alpha t}) \\ \frac{1}{2} &= 1 - e^{-\alpha t} \\ e^{-\alpha t} &= \frac{1}{2} \\ t &= -\frac{\ln(1/2)}{\alpha} = \frac{\ln 2}{\alpha}. \end{aligned}$$

So altering $r$ does not alter the time it takes to reach $\frac{1}{2}Q_\infty$. See Figure 10.11.

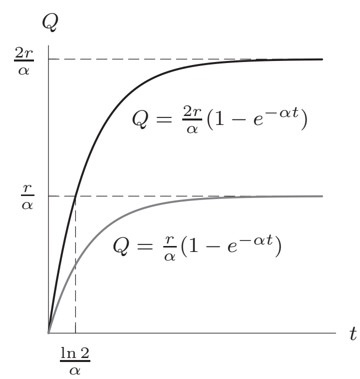



**Figure 10.11**

**(c)** $Q_\infty$ is halved by doubling $\alpha$, and so is the time, $t = \frac{\ln 2}{\alpha}$, to reach $\frac{1}{2}Q_\infty$.

**25.** Differentiate with respect to $t$ on both sides of the equation:

$$\frac{y - A}{y_0 - A} = e^{kt}.$$

Since $A$ and $y_0$ are constant, we have

$$\frac{y'}{y_0 - A} = ke^{kt} = k\frac{y - A}{y_0 - A}.$$

Multiplication by $y_0 - A$ gives

$$y' = k(y - A)$$

which shows that $y$ satisfies the given differential equation.

Now we need to show that the initial condition, $y(0) = y_0$, is satisfied. Substituting $t = 0$ gives

$$\begin{aligned} \frac{y(0) - A}{y_0 - A} &= e^{k \cdot 0} = 1 \\ y(0) - A &= y_0 - A \\ y(0) &= y_0 \end{aligned}$$

which shows that $y$ satisfies the given initial condition.

## Solutions for Section 10.6

**1.** **(a)** If alone, the $x$ population grows exponentially, since if $y = 0$ we have $dx/dt = 0.01x$. If alone, the $y$ population decreases to 0 exponentially, since if $x = 0$ we have $dy/dt = -0.2y$.
**(b)** This is a predator-prey relationship: interaction between populations $x$ and $y$ decreases the $x$ population and increases the $y$ population. The interaction of lions and gazelles might be modeled by these equations.

**2.** **(a)** If alone, the $x$ and $y$ populations each grow exponentially, because the equations become $dx/dt = 0.01x$ and $dy/dt = 0.2y$.
**(b)** For each population, the presence of the other decreases their growth rate. The two populations are therefore competitors—they may be eating each other's food, for instance. The interaction of gazelles and zebras might be modeled by these equations.

**3.** **(a)** The $x$ population is unaffected by the $y$ population—it grows exponentially no matter what the $y$ population is, even if $y = 0$. If alone, the $y$ population decreases to zero exponentially, because its equation becomes $dy/dt = -0.1y$.
**(b)** Here, interaction between the two populations helps the $y$ population but does not effect the $x$ population. This is not a predator-prey relationship; instead, this is a one-way relationship, where the $y$ population is helped by the existence of $x$'s. These equations could, for instance, model the interaction of rhinoceroses $(x)$ and dung beetles $(y)$.

**4.** **(a)** The species need each other to survive. Both would die out without the other, and they help each other.
**(b)** If $x = 2$ and $y = 1$,

$$\frac{dx}{dt} = -3x + 2xy = -3(2) + 2(2)(1) < 0,$$

and so population $x$ decreases. If $x = 2$ and $y = 1$,

$$\frac{dy}{dx} = -y + 5xy = -1 + 5(2)(1) > 0,$$

and so population $y$ increases.
**(c)** $\dfrac{dy}{dx} = \dfrac{-y + 5xy}{-3x + 2xy}$.
**(d)** See Figure 10.12.
**(e)** See Figure 10.12.

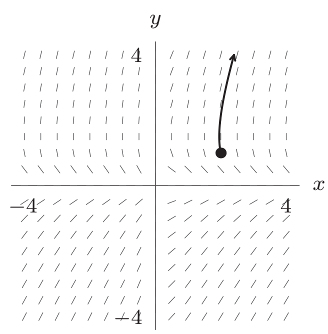



**Figure 10.12**

**5.** $\dfrac{dx}{dt} = x - xy, \quad \dfrac{dy}{dt} = y - xy$

**6.** $\dfrac{dx}{dt} = -x + xy, \quad \dfrac{dy}{dt} = y$

**7.** $\dfrac{dx}{dt} = -x - xy, \quad \dfrac{dy}{dt} = -y - xy$

**8.** (I) Both companies start with about 4 million dollars, and both initially lose money. In the long run, however, Company $A$ makes money and Company $B$ looks like it goes out of business.
(II) Initially, Company $A$ has 2 million dollars and Company $B$ has 4 million dollars. Both companies lose money in the beginning. Company $A$ continues to lose money and probably goes out of business, but Company $B$ eventually makes money and does well.

(III) Company $A$ starts with well under 1 million dollars and Company $B$ starts with 1 million dollars. Company $B$ makes money the whole time and does well. Company $A$ shows a small profit for a while but then loses money and probably goes out of business.

(IV) Both companies start with well under 1 million dollars. Company $A$ makes money and does well, Company $B$ holds steady for a time but then loses money and probably goes out of business.

**9.** This is an example of a predator-prey relationship. Normally, we would expect the worm population, in the absence of predators, to increase without bound. As the number of worms $w$ increases, so would the rate of increase $dw/dt$; in other words, the relation $dw/dt = w$ might be a reasonable model for the worm population in the absence of predators.

However, since there are predators (robins), $dw/dt$ won't be that big. We must lessen $dw/dt$. It makes sense that the more interaction there is between robins and worms, the more slowly the worms are able to increase their numbers. Hence we lessen $dw/dt$ by the amount $wr$ to get $dw/dt = w - wr$. The term $-wr$ reflects the fact that more interactions between the species means slower reproduction for the worms.

Similarly, we would expect the robin population to decrease in the absence of worms. We'd expect the population decrease at a rate related to the current population, making $dr/dt = -r$ a reasonable model for the robin population in absence of worms. The negative term reflects the fact that the greater the population of robins, the more quickly they are dying off. The $wr$ term in $dr/dt = -r + wr$ reflects the fact that the more interactions between robins and worms, the greater the tendency for the robins to increase in population.

**10.** If there are no worms, then $w = 0$, and $\frac{dr}{dt} = -r$ giving $r = r_0e^{-t}$, where $r_0$ is the initial robin population. If there are no robins, then $r = 0$, and $\frac{dw}{dt} = w$ giving $w = w_0e^{t}$, where $w_0$ is the initial worm population.

**11.** There is symmetry across the line $r = w$. Indeed, since $\frac{dr}{dw} = \frac{r(w-1)}{w(1-r)}$, if we switch $w$ and $r$ we get $\frac{dw}{dr} = \frac{w(r-1)}{r(1-w)}$, so $\frac{dr}{dw} = \frac{r(1-w)}{w(r-1)}$. Since switching $w$ and $r$ changes nothing, the slope field must be symmetric across the line $r = w$. The slope field shows that the solution curves are either spirals or closed curves. Since there is symmetry about the line $r = w$, the solutions must in fact be closed curves.

**12.** If $w = 2$ and $r = 2$, then $\frac{dw}{dt} = -2$ and $\frac{dr}{dt} = 2$, so initially the number of worms decreases and the number of robins increases. In the long run, however, the populations will oscillate; they will even go back to $w = 2$ and $r = 2$. See Figure 10.13.

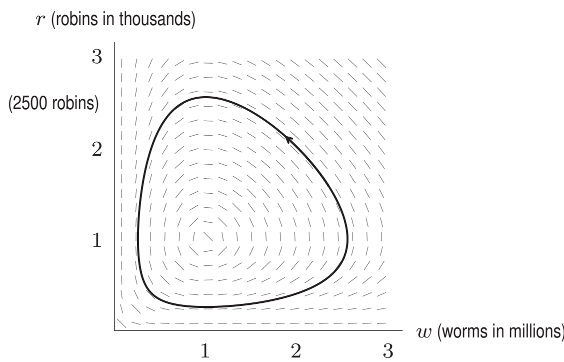



**Figure 10.13**

**13.** Sketching the trajectory through the point $(2, 2)$ on the slope field given shows that the maximum robin population is about 2500, and the minimum robin population is about 500. When the robin population is at its maximum, the worm population is about 1,000,000.

**14.**

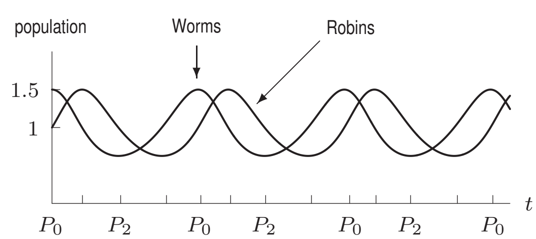



**Figure 10.14**

**15.** It will work somewhat; the maximum number the robins reach will increase. However, the minimum number the robins reach will decrease as well. (See graph of slope field.) In the long term, the robin-worm populations will again fall into a cycle. Notice, however, if the extra robins are added during the part of the cycle where there are the fewest robins, the new cycle will have smaller variation. See Figure 10.15.

Note that if too many robins are added, the minimum number may get so small the model may fail, since a small number of robins are more susceptible to disaster.

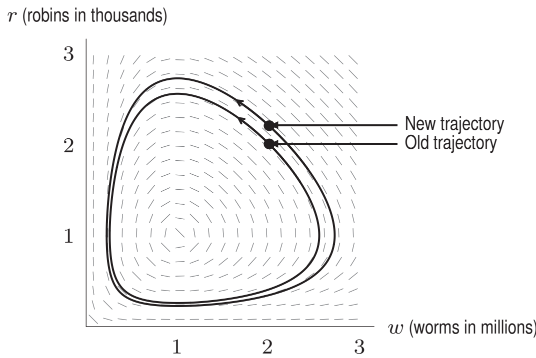



**Figure 10.15**

**16.** The numbers of robins begins to increase while the number of worms remains approximately constant. See Figure 10.16.

The numbers of robins and worms oscillate periodically between 0.2 and 3, with the robin population lagging behind the worm population.

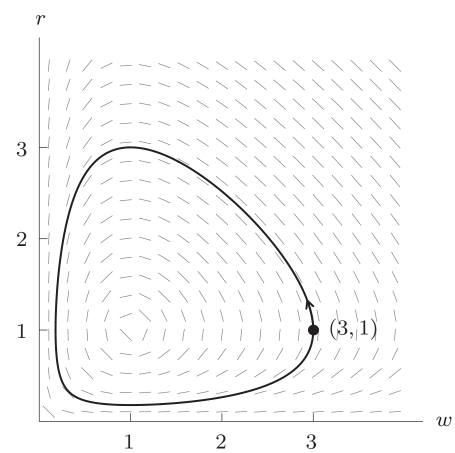



**Figure 10.16**

**17.** **(a)** Substituting $w = 2.2$ and $r = 1$ into the differential equations gives

$$\frac{dw}{dt} = 2.2 - (2.2)(1) = 0$$
$$\frac{dr}{dt} = -1 + 1(2.2) = 1.2.$$

**(b)** Since the rate of change of $w$ with time is 0,

$$\text{At } t = 0.1, \text{ we estimate } w = 2.2$$

Since the rate of change of $r$ is 1.2 thousand robins per unit time,

$$\text{At } t = 0.1, \text{ we estimate } r = 1 + 1.2(0.1) = 1.12 \approx 1.1.$$

**(c)** We must recompute the derivatives. At $t = 0.1$, we have

$$\frac{dw}{dt} = 2.2 - 2.2(1.12) = -0.264$$
$$\frac{dr}{dt} = -1.12 + 1.12(2.2) = 1.344.$$

Then at $t = 0.2$, we estimate

$$w = 2.2 - 0.264(0.1) = 2.1736 \approx 2.2$$
$$r = 1.12 + 1.344(0.1) = 1.2544 \approx 1.3$$

Recomputing the derivatives at $t = 0.2$ gives

$$\frac{dw}{dt} = 2.1736 - 2.1736(1.2544) = -0.553$$
$$\frac{dr}{dt} = -1.2544 + 1.2544(2.1736) = 1.472$$

Then at $t = 0.3$, we estimate

$$w = 2.1736 - 0.553(0.1) = 2.1183 \approx 2.1$$
$$r = 1.2544 + 1.472(0.1) = 1.4016 \approx 1.4.$$

**18. (a)** See Figure 10.17.

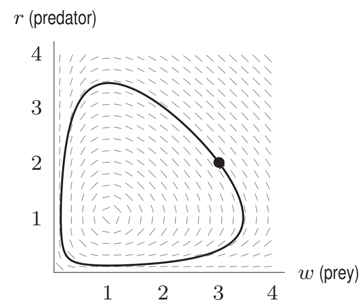



**Figure 10.17**

**(b)** The point moves in a counterclockwise direction. If $w = 3$ and $r = 2$, we have

$$\frac{dw}{dt} = w - wr = 3 - (3)(2) = 3 - 6 < 0$$
$$\frac{dr}{dt} = -r + wr = -2 + (3)(2) = -2 + 6 > 0.$$

So $w$ is decreasing and $r$ is increasing. The point moves up and to the left (counterclockwise).

**(c)** We see in the trajectory that $r$ achieves a maximum value of about 3.5, so the population of robins goes as high as 3.5 thousand robins. At this time, the worm population is at about 1 million.

**(d)** The worm population goes as high as 3.5 million worms. At this time, the robin population is about 1 thousand.

**19. (a)** See Figure 10.18.

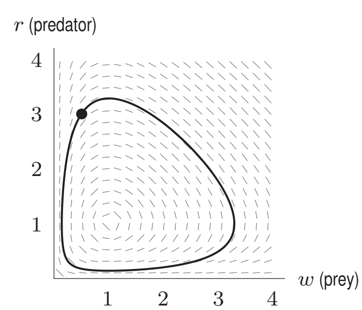



**Figure 10.18**

**(b)** If $w = 0.5$ and $r = 3$, we have

$$\frac{dw}{dt} = w - wr = 0.5 - (0.5)(3) < 0$$
$$\frac{dr}{dt} = -r + wr = -3 + (0.5)(3) < 0$$

Both $w$ and $r$ are decreasing, so the point is moving down and to the left (counterclockwise).

**(c)** The robin population goes up to about 3.3 thousand robins. At this time, the worm population is about $1$ million.

**(d)** The worm population goes up to about 3.3 million worm. At this point, the robin population is about $1$ thousand.

**20. (a)** Both $x$ and $y$ decrease, since

$$\frac{dx}{dt} = 0.2x - 0.5xy = 0.2(2) - 0.5(2)(2) < 0,$$
$$\frac{dy}{dt} = 0.6y - 0.8xy = 0.6(2) - 0.8(2)(2) < 0.$$

**(b)** Population $x$ increases and population $y$ decreases, since

$$\frac{dx}{dt} = -2x + 5xy = -2(2) + 5(2)(2) > 0,$$
$$\frac{dy}{dt} = -y + 0.2xy = -2 + 0.2(2)(2) < 0.$$

**(c)** Both $x$ and $y$ increase, since

$$\frac{dx}{dt} = 0.5x = 0.5(2) > 0,$$
$$\frac{dy}{dt} = -1.6y + 2xy = -1.6(2) + 2(2)(2) > 0$$

**(d)** Population $x$ decreases and population $y$ increases, since

$$\frac{dx}{dt} = 0.3x - 1.2xy = 0.3(2) - 1.2(2)(2) < 0$$
$$\frac{dy}{dt} = -0.7x + 2.5xy = 0.7(2) + 2.5(2)(2) > 0$$

**21. (a)** $\frac{dy}{dt} = \frac{0.6y - 0.8xy}{0.2x - 0.5xy}$; See Figure 10.19.

**(b)** $\frac{dy}{dx} = \frac{-y + 0.2xy}{-2x + 5xy}$; See Figure 10.20.

**(c)** $\frac{dy}{dx} = \frac{-1.6y + 2xy}{0.5x}$; See Figure 10.21.

**(d)** $\frac{dy}{dx} = \frac{-0.7y + 2.5xy}{0.3x - 1.2xy}$; See Figure 10.22.

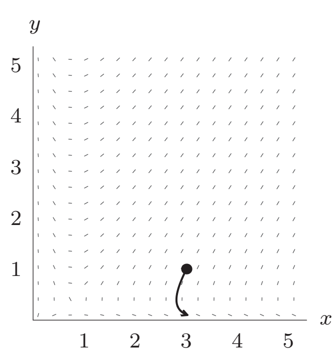



**Figure 10.19**: $\frac{dy}{dx} = \frac{0.6y - 0.8xy}{0.2x - 0.5xy}$

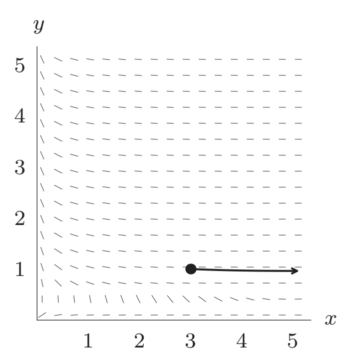



**Figure 10.20**: $\frac{dy}{dx} = \frac{-y + 0.2xy}{-2x + 5xy}$

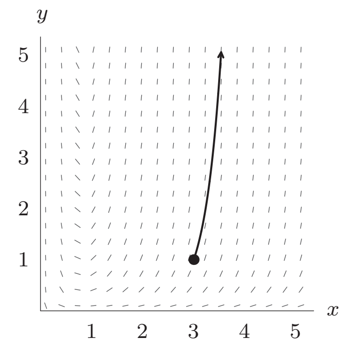



Figure 10.21: $\frac{dy}{dx} = \frac{-1.6y+2xy}{0.5x}$

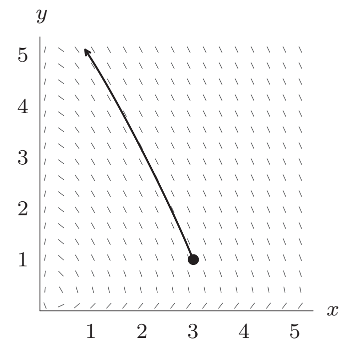



Figure 10.22: $\frac{dy}{dx} = \frac{-0.7y+2.5xy}{0.3x-1.2xy}$

## Solutions for Section 10.7

**1.** Susceptible people are infected at a rate proportional to the product of $S$ and $I$. As susceptible people become infected, $S$ decreases at a rate of $aSI$ and (since these same people are now infected) $I$ increases at the same rate. At the same time, infected people are recovering at a rate proportional to the number infected, so $I$ is decreasing at a rate of $bI$.

**2.** Since

$$\frac{dS}{dt} = -aSI,$$
$$\frac{dI}{dt} = aSI - bI,$$
$$\frac{dR}{dt} = bI$$

we have

$$\frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = -aSI + aSI - bI + bI = 0.$$

Thus $\frac{d}{dt}(S + I + R) = 0$, so $S + I + R =$ constant.

**3.** The epidemic is over when the number of infected people is zero, that is, at the horizontal intercept of the trajectory. This intercept is a very small $S$ value. When the epidemic is over, there are almost no susceptibles left— that is, almost everyone has become infected.

**4.** **(a)** The initial values are $I_0 = 1, S_0 = 149$.

**(b)** Substituting for $I_0$ and $S_0$, initially we have

$$\frac{dI}{dt} = 0.0026SI - 0.5I = 0.0026(149)(1) - 0.5(1) < 0.$$

So, $I$ is decreasing. The number of infected people goes down from 1 to 0. The disease does not spread.

**5.** **(a)** $I_0 = 1, S_0 = 349$

**(b)** Since $\frac{dI}{dt} = 0.0026SI - 0.5I = 0.0026(349)(1) - 0.5(1) > 0$, so $I$ is increasing. The number of infected people will increase, and the disease will spread. This is an epidemic.

**6.** **(a)** See Figure 10.23.

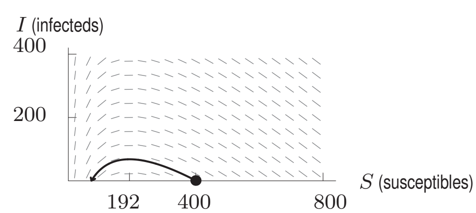



Figure 10.23

**(b)** $I$ is at a maximum when $S = 192$.

**7.** The maximum value of $I$ is approximately 300 boys. This represents the maximum number of infected boys who have not (yet) been removed from circulation. It occurs at about $t \approx 6$ days.

**8.** In the system

$$\frac{dS}{dt} = -aSI$$
$$\frac{dI}{dt} = aSI - bI$$

the constant $a$ represents how infectious the disease is; the larger $a$, the more infectious. The constant $b$ represents $1/(\text{number of days before removal})$. Thus, the larger $b$ is, the quicker the infecteds are removed. For the flu example, $a = 0.0026$ and $b = 0.5$.

(I) Since $0.0026 < 0.04$, this is more infectious. Since $0.2 < 0.5$, infecteds are being removed more slowly. So system (I) corresponds to (a).

(II) Since $0.002 < 0.0026$, this is less infectious. Since $0.3 < 0.5$, infecteds are being removed more slowly. This corresponds to (c).

(III) Since the second equation has no $-bI$ term, we have $b = 0$. The infecteds are never removed. This corresponds to (e).

A system of equations corresponding to (b) is

$$\frac{dS}{dt} = -0.04SI$$
$$\frac{dI}{dt} = 0.04SI - 0.7I.$$

A system of equations corresponding to (d) is

$$\frac{dS}{dt} = -0.002SI$$
$$\frac{dI}{dt} = 0.002SI - 0.7I.$$

**9.** The threshold value of $S$ is the value at which $I$ is a maximum. When $I$ is a maximum,

$$\frac{dI}{dt} = 0.04SI - 0.2I = 0,$$

so

$$S = 0.2/0.04 = 5.$$

**10.** Since the threshold value of $S$ is given by

$$\frac{dI}{dt} = 0.002SI - 0.3I = 0,$$

we have

$$S = \frac{0.3}{0.002} = 150.$$

So, if $S_0 = 100$, the disease does not spread initially. If $S_0 = 200$, the disease does spread initially.

**11.** **(a)** Setting $dI/dt = 0$,

$$\frac{dI}{dt} = aSI - bI = (aS - b)I = 0$$
$$aS - b = 0$$
$$S = \frac{b}{a}.$$

**(b)** For $S > b/a$, we know $aS - b > 0$, so

$$\frac{dI}{dt} = (aS - b)I > 0.$$

Thus, for $S > b/a$, we know $I$ is increasing. Similarly, for $S < b/a$, we can show $I$ is decreasing.

**(c)** Since $I$ increases initially if $S_0 > b/a$, we have an epidemic if $S_0 > b/a$. If $S_0 < b/a$, there is no epidemic as $I$ decreases initially. Thus, the threshold value is $b/a$.

**12.** **(a)** $I_0 = 1$, so $S_0 = 45{,}000 - 1 = 44{,}999$.

**(b)** If $S_0$ is greater than the threshold value, we expect an epidemic.

$$\text{Threshold} = \frac{b}{a} = \frac{9.865}{0.000267} = 36{,}948 \text{ people.}$$

Since the camp contained 45,000 soldiers, more than the threshold, an epidemic is predicted.

**(c)** Using the chain rule, and the values of $a$ and $b$, we have

$$\frac{dI}{dS} = \frac{dI/dt}{dS/dt} = \frac{aSI - bI}{-aSI} = -1 + \frac{b}{aS} = -1 + \frac{36{,}948}{S}.$$

Figure 10.24 shows the slope field for this differential equation and the solution curve for these initial conditions. The total number of soldiers who did not get the disease is the $S$-intercept, or about 30,000. Thus, approximately 15,000 soldiers were infected.

**(d)** To solve the differential equation, we integrate

$$\frac{dI}{dS} = -1 + \frac{36{,}948}{S}$$

giving, since $S > 0$,

$$I = -S + 36{,}948 \ln S + C.$$

To find $C$, substitute $I_0 = 1, S_0 = 44{,}999$, so

$$1 = -44{,}999 + 36{,}948 \ln(44{,}999) + C$$
$$C = 45{,}000 - 36{,}948 \ln(44{,}999),$$

thus

$$I = -S + 36{,}948 \ln S + 45{,}000 - 36{,}948 \ln(44{,}999)$$
$$I = -S + 36{,}948 \ln\left(\frac{S}{44{,}999}\right) + 45{,}000.$$

We find the $S$-intercept, giving the number of people unaffected, by setting $I = 0$. Then

$$0 = -S + 36{,}948 \ln\left(\frac{S}{44{,}999}\right) + 45{,}000.$$

This equation cannot be solved algebraically. However, numerical methods, or tracing along a graph, gives $S \approx 30{,}000$. Thus, about 15,000 soldiers were infected.

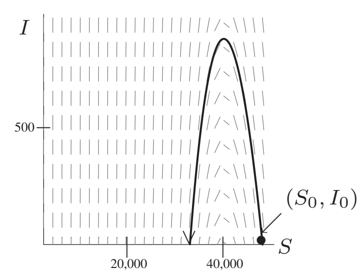



**Figure 10.24**

## Solutions for Chapter 10 Review

**1.** **(a)** (III) An island can only sustain the population up to a certain size. The population will grow until it reaches this limiting value.

**(b)** (V) The ingot will get hot and then cool off, so the temperature will increase and then decrease.

**(c)** (I) The speed of the car is constant, and then decreases linearly when the breaks are applied uniformly.

**(d)** (II) Carbon-14 decays exponentially.

**(e)** (IV) Tree pollen is seasonal, and therefore cyclical.

**2.** **(a)** (i) If $y = Cx^2$, then $\frac{dy}{dx} = C(2x) = 2Cx$. We have

$$x\frac{dy}{dx} = x(2Cx) = 2Cx^2$$

and

$$3y = 3(Cx^2) = 3Cx^2$$

Since $x\frac{dy}{dx} \neq 3y$, this is not a solution.

(ii) If $y = Cx^3$, then $\frac{dy}{dx} = C(3x^2) = 3Cx^2$. We have

$$x\frac{dy}{dx} = x(3Cx^2) = 3Cx^3,$$

and

$$3y = 3Cx^3.$$

Thus $x\frac{dy}{dx} = 3y$, and $y = Cx^3$ is a solution.

(iii) If $y = x^3 + C$, then $\frac{dy}{dx} = 3x^2$. We have

$$x\frac{dy}{dx} = x(3x^2) = 3x^3$$

and

$$3y = 3(x^3 + C) = 3x^3 + 3C.$$

Since $x\frac{dy}{dx} \neq 3y$, this is not a solution.

**(b)** The solution is $y = Cx^3$. If $y = 40$ when $x = 2$, we have

$$\begin{aligned} 40 &= C(2^3) \\ 40 &= C \cdot 8 \\ C &= 5. \end{aligned}$$

**3.** Since $y = x^3$, we know that $y' = 3x^2$. Substituting $y = x^3$ and $y' = 3x^2$ into the differential equation we get

$$\text{Left-side} = xy' - 3y = x(3x^2) - 3(x^3) = 3x^3 - 3x^3 = 0.$$

Since the left and right sides are equal for all $x$, we see that $y = x^3$ is a solution.

**4.** When $y = 125$, the rate of change of $y$ is

$$\frac{dy}{dt} = -0.20y = -0.20(125) = -25.$$

The value of $y$ goes down by 25 as $t$ goes up by 1, so when $t = 1$, we have

$$\begin{aligned} y &= \text{Old value of } y + \text{Change in } y \\ &= 125 + (-25) \\ &= 100. \end{aligned}$$

Continuing in this way , we fill in the table as shown:

| $t$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $y$ | 125 | 100 | 80 | 64 | 51.2 |

**5.** **(a)** Slope field I corresponds to $\dfrac{dy}{dx} = 1 + y$ and slope field II corresponds to $\dfrac{dy}{dx} = 1 + x$.
**(b)** See Figures 10.25 and 10.26.

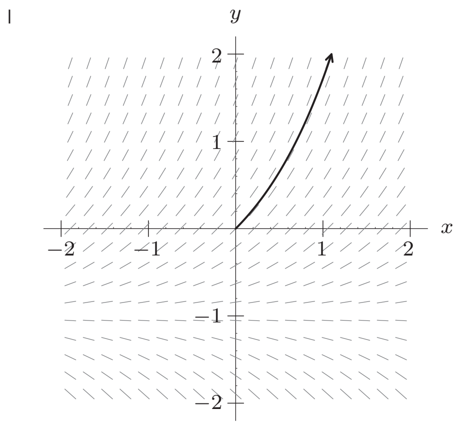



Figure 10.25: $\frac{dy}{dx} = 1 + y$

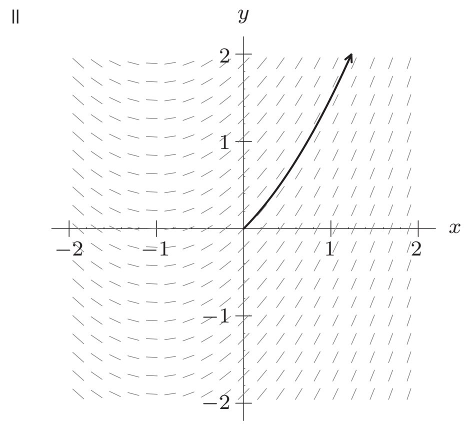



Figure 10.26: $\frac{dy}{dx} = 1 + x$

**(c)** Slope field I has an equilibrium solution at $y = -1$, since $\dfrac{dy}{dx} = 0$ at $y = -1$. We see in the slope field that this equilibrium solution is unstable. Slope field II does not have any equilibrium solutions.

**6.** **(a)** See Figure 10.27.

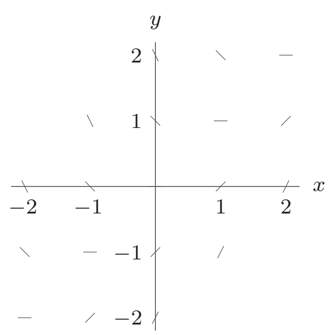



Figure 10.27

**(b)** The point $(1, 0)$ satisfies the equation $y = x - 1$. If $y = x - 1$, then $y' = 1$ and $x - y = x - (x - 1) = 1$, so $y = x - 1$ is the solution to the differential equation through $(1, 0)$.

**7.** III. The slope field appears to be constant for a fixed value of $y$, regardless of the value of $x$. This feature says that $y'$ does not depend on $x$, ruling out the formulas $y' = 1+x$ and $y' = xy$. The differential equation $y' = 1+y$ would have a slope field with zero slope at $y = -1$ and nowhere else, but the given slope field has two areas of zero slope, so $y' = 1 + y$ is ruled out and so is $y' = 2 - y$ for the same reason. This leaves $y' = (1 + y)(2 - y)$ as the correct answer, which fits the slope field as it has zero slopes at $y = 2$ and $y = -1$, positive slopes for $-1 < y < 2$ and negative slopes for $y < -1$ and $y > 2$.

**8.** Figure (I) shows a line segment at $(4, 0)$ with positive slope. The only possible differential equation is (b), since at $(4, 0)$ we have $y' = \cos 0 = 1$. Note that (a) is not possible as $y'(4, 0) = e^{-16} = 0.0000001$, a much smaller positive slope than that shown.

Figure (II) shows a line segment at $(0, 4)$ with zero slope. The possible differential equations are (d), since at $(0, 4)$ we have $y' = 4(4 - 4) = 0$, and (f), since at $(0, 4)$ we have $y' = 0(3 - 0) = 0$.

Figure (III) shows a line segment at $(4, 0)$ with negative slope of large magnitude. The only possible differential equation is (f), since at $(4, 0)$ we have $y' = 4(3-4) = -4$. Note that (c) is not possible as $y'(4, 0) = \cos(4-0) = -0.65$, a negative slope of smaller magnitude than that shown.

Figure (IV) shows a line segment at $(4, 0)$ with a negative slope of small magnitude. The only possible differential equation is (c), since at $(4, 0)$ we have $y' = \cos(4-0) = -0.65$. Note that (f) is not possible as $y'(4, 0) = 4(3-4) = -4$, a negative slope of larger magnitude than that shown.

Figure (V) shows a line segment at $(0, 4)$ with positive slope. Possible differential equations are (a), since at $(0, 4)$ we have $y' = e^{0^2} = 1$, and (c), since at $(0, 4)$ we have $y' = \cos(4 - 4) = 1$.

Figure (VI) shows a line segment at $(0, 4)$ with a negative slope of large magnitude. The only possible differential equation is (e), since at $(0, 4)$ we have $y' = 4(3 - 4) = -4$. Note that (b) is not possible as $y'(0, 4) = \cos 4 = -0.65$, a negative slope of smaller magnitude than that shown.

**9.** **(a)** We know that the balance, $B$, increases at a rate proportional to the current balance. Since interest is being earned at a rate of $7\%$ compounded continuously we have

$$\text{Rate at which interest is earned} = 7\% \text{ (Current balance)}$$

or in other words, if $t$ is time in years,

$$\frac{dB}{dt} = 7\%(B) = 0.07B.$$

**(b)** The equation is in the form

$$\frac{dB}{dt} = kB$$

so we know that the general solution will be

$$B = B_0 e^{kt}$$

where $B_0$ is the value of $B$ when $t = 0$, i.e., the initial balance. In our case we have $k = 0.07$ so we get

$$B = B_0 e^{0.07t}.$$

**(c)** We are told that the initial balance, $B_0$, is \$5000 so we get

$$B = 5000e^{0.07t}.$$

**(d)** Substituting the value $t = 10$ into our formula for $B$ we get

$$\begin{aligned} B &= 5000e^{0.07t} \\ B(10) &= 5000e^{0.07(10)} \\ &= 5000e^{0.7} \\ B(10) &\approx \$10{,}068.76 \end{aligned}$$

**10.** If we let $Q$ represent the amount of radioactive iodine present at time $t$, with $t$ measured in days, then we have

$$\frac{dQ}{dt} = -0.09Q.$$

The $-0.09$ is negative because the quantity of iodine is decreasing. The solution to this differential equation is

$$Q = Ce^{-0.09t},$$

for some constant $C$.

**11.** Integrating both sides gives

$$P = \frac{1}{2}t^2 + C,$$

where $C$ is some constant.

**12.** The general solution is

$$y = Ce^{5t}.$$

**13.** Integrating both sides we get

$$y = \frac{5}{2}t^2 + C,$$

where $C$ is a constant.

**14.** We know that the general solution to an equation of the form
$$\frac{dP}{dt} = kP$$
is
$$P = Ce^{kt}.$$
Thus in our case the solution is
$$P = Ce^{0.03t},$$
where $C$ is some constant.

**15.** For some constant $C$, the general solution is
$$A = Ce^{-0.07t}.$$

**16.** Multiplying both sides by $Q$ gives
$$\frac{dQ}{dt} = 2Q, \quad \text{where } Q \neq 0.$$
We know that the general solution to an equation of the form
$$\frac{dQ}{dt} = kQ$$
is
$$Q = Ce^{kt}.$$
Thus in our case the solution is
$$Q = Ce^{2t},$$
where $C$ is some constant, $C \neq 0$.

**17.** We know that the general solution to the differential equation
$$\frac{dP}{dt} = k(P - A)$$
is
$$P = Ce^{kt} + A.$$
Thus in our case we factor out $-2$ to get
$$\frac{dP}{dt} = -2\left(P + \frac{10}{-2}\right) = -2(P - 5).$$
Thus the general solution to our differential equation is
$$P = Ce^{-2t} + 5,$$
where $C$ is some constant.

**18.** Since $\frac{dy}{dt} = -(y - 100)$, the general solution is $y = 100 + Ce^{-t}$.

**19.** We know that the general solution to the differential equation
$$\frac{dy}{dx} = k(y - A)$$
is
$$y = Ce^{kx} + A.$$
Thus in our case we factor out $0.2$ to get
$$\frac{dy}{dx} = 0.2\left(y - \frac{8}{0.2}\right) = 0.2(y - 40).$$
Thus the general solution to our differential equation is
$$y = Ce^{0.2x} + 40,$$
where $C$ is some constant.

**20.** We know that the general solution to the differential equation

$$\frac{dH}{dt} = k(H - A)$$

is

$$H = Ce^{kt} + A.$$

Thus in our case we factor out 0.5 to get

$$\frac{dH}{dt} = 0.5\left(H + \frac{10}{0.5}\right) = 0.5(H - (-20)).$$

Thus the general solution to our differential equation is

$$H = Ce^{0.5t} - 20,$$

where $C$ is some constant.

**21.** We find the temperature of the orange juice as a function of time. Newton's Law of Heating says that the rate of change of the temperature is proportional to the temperature difference. If $S$ is the temperature of the juice, this gives us the equation

$$\frac{dS}{dt} = -k(S - 65) \qquad \text{for some constant } k.$$

Notice that the temperature of the juice is increasing, so the quantity $dS/dt$ is positive. In addition, $S = 40$ initially, making the quantity $(S - 65)$ negative.

We know that the general solution to a differential equation of the form

$$\frac{dS}{dt} = -k(S - 65)$$

is

$$S = Ce^{-kt} + 65.$$

Since at $t = 0$, $S = 40$, we have $40 = 65 + C$, so $C = -25$. Thus, $S = 65 - 25e^{-kt}$ for some positive constant $k$. See Figure 10.28 for the graph.

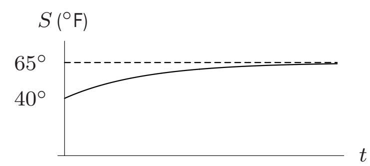



**Figure 10.28**: Graph of $S = 65 - 25e^{-kt}$ for $k > 0$

**22.** **(a)** Since the growth rate of the tumor is proportional to its size, we should have

$$\frac{dS}{dt} = kS.$$

**(b)** We can solve this differential equation by separating variables and then integrating:

$$\int \frac{dS}{S} = \int k\,dt$$
$$\ln|S| = kt + B$$
$$S = Ce^{kt}.$$

**(c)** This information is enough to allow us to solve for $C$:

$$5 = Ce^{0t}$$
$$C = 5.$$

**(d)** Knowing that $C = 5$, this second piece of information allows us to solve for $k$:

$$8 = 5e^{3k}$$
$$k = \frac{1}{3}\ln\left(\frac{8}{5}\right) \approx 0.1567.$$

So the tumor's size is given by

$$S = 5e^{0.1567t}.$$

**23.** **(a)** If $C' = -kC$, and then $C = C_0e^{-kt}$. Since the half-life is 5730 years, $\frac{1}{2}C_0 = C_0e^{-5730k}$. Solving for $k$, we have $-5730k = \ln(1/2)$ so $k = \frac{-\ln(1/2)}{5730} \approx 0.000121$.

**(b)** From the given information, we have $0.91 = e^{-kt}$, where $t$ is the age of the shroud. Solving for $t$, we have $t = \frac{-\ln 0.91}{k} \approx 779.4$ years.

**24.** **(a)** Use the fact that

$$\begin{array}{c}\text{Rate balance} \\ \text{changing}\end{array} = \begin{array}{c}\text{Rate interest} \\ \text{accrued}\end{array} - \begin{array}{c}\text{Rate payments} \\ \text{made}\end{array}.$$

Thus

$$\frac{dB}{dt} = 0.05B - 12{,}000.$$

**(b)** We know that the general solution to a differential equation of the form

$$\frac{dB}{dt} = k(B - A)$$

is

$$B = Ce^{kt} + A.$$

Factoring out a 0.05 on the left side we get

$$\frac{dB}{dt} = 0.05\left(B - \frac{12{,}000}{0.05}\right) = 0.05(B - 240{,}000).$$

Thus in our case we get

$$B = Ce^{0.05t} + 240{,}000.$$

We know that the initial balance is $B_0$, thus we get

$$B_0 = Ce^0 + 240{,}000$$
$$C = B_0 - 240{,}000.$$

Thus we get

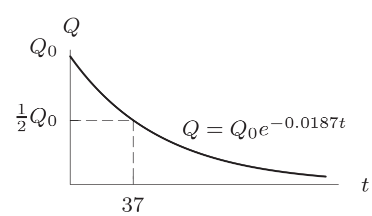
$$B = (B_0 - 240{,}000)e^{0.05t} + 240{,}000.$$

**(c)** To find the initial balance such that the account has a 0 balance after 20 years, we solve

$$0 = (B_0 - 240{,}000)e^{(0.05)20} + 240{,}000 = (B_0 - 240{,}000)e^1 + 240{,}000,$$

$$B_0 = 240{,}000 - \frac{240{,}000}{e} \approx \$151{,}708.93.$$

**25.** **(a)**




**(b)** $\dfrac{dQ}{dt} = -kQ$

**(c)** Since $25\% = 1/4$, it takes two half-lives $= 74$ hours for the drug level to be reduced to $25\%$. Alternatively, $Q = Q_0e^{-kt}$ and $\frac{1}{2} = e^{-k(37)}$, we have

$$k = -\frac{\ln(1/2)}{37} \approx 0.0187.$$

Therefore $Q = Q_0e^{-0.0187t}$. We know that when the drug level is $25\%$ of the original level that $Q = 0.25Q_0$. Setting these equal, we get

$$0.25 = e^{-0.0187t}.$$

giving

$$t = -\frac{\ln(0.25)}{0.0187} \approx 74 \text{ hours} \approx 3 \text{ days}.$$

**26.** Let $D(t)$ be the quantity of dead leaves, in grams per square centimeter. Then $\frac{dD}{dt} = 3 - 0.75D$, where $t$ is in years. We know that the general solution to a differential equation of the form

$$\frac{dD}{dt} = k(D - B)$$

is

$$D = Ae^{kt} + B.$$

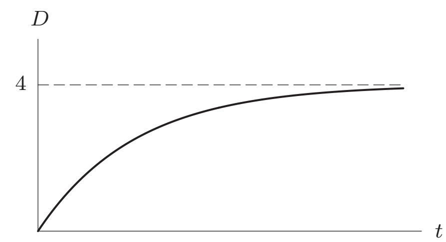
Factoring out a $-0.75$ on the left side we get

$$\frac{dD}{dt} = -0.75\left(D - \frac{-3}{-0.75}\right) = -0.75(D - 4).$$

Thus in our case we get

$$D = Ae^{-0.75t} + 4.$$

If initially the ground is clear, the solution looks like the following graph:




The equilibrium level is 4 grams per square centimeter, regardless of the initial condition.

**27.** **(a)** We know that the rate at which morphine leaves the body is proportional to the amount of morphine in the body at that particular instant. If we let $Q$ be the amount of morphine in the body, we get that

$$\text{Rate of morphine leaving the body} = kQ,$$

where $k$ is the rate of proportionality. The solution is $Q = Q_0e^{kt}$ (neglecting the continuously incoming morphine). Since the half-life is 2 hours, we have

$$\frac{1}{2}Q_0 = Q_0e^{k\cdot 2},$$

so

$$k = \frac{\ln(1/2)}{2} = -0.347.$$

**(b)** Since

$$\text{Rate of change of quantity } = \text{Rate in } - \text{Rate out},$$

we have

$$\frac{dQ}{dt} = -0.347Q + 2.5.$$

**(c)** Equilibrium occurs when $dQ/dt = 0$, that is, when $0.347Q = 2.5$ or $Q = 7.2$ mg.

**28.** **(a)** We know that the equilibrium solutions are those functions which satisfy the differential equation and whose derivative is everywhere 0. Thus we must solve

$$\begin{aligned} 0 &= \frac{dy}{dx} \\ &= 0.5y(y-4)(2+y) \end{aligned}$$

Thus the equilibrium solutions are $y = 0$, $y = 4$, and $y = -2$.

**(b)** The slope field of the differential equation is shown in Figure 10.29. An equilibrium solution is stable if a small change in the initial conditions gives a solution which tends toward the equilibrium as the independent variable tends to positive infinity. Looking at Figure 10.29 we see that the only stable solution is $y = 0$.

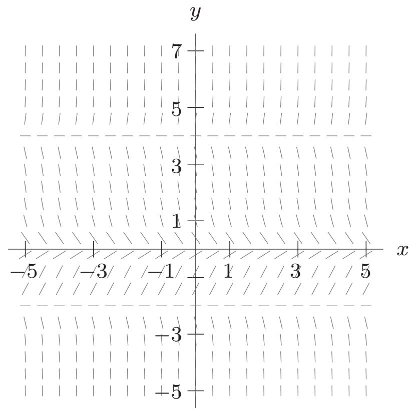



Figure 10.29

**29.** **(a)** Since the rate of change of the weight is equal to

$$\frac{1}{3500}(\text{Intake} - \text{Amount to maintain weight})$$

we have

$$\frac{dW}{dt} = \frac{1}{3500}(I - 20W).$$

**(b)** We know that the general solution to a differential equation of the form

$$\frac{dW}{dt} = k(W - A)$$

is

$$W = Ce^{kt} + A.$$

Factoring out a $-20$ on the left side we get

$$\frac{dW}{dt} = \frac{-20}{3500}\left(W - \frac{-I}{-20}\right) = -\frac{2}{350}\left(W - \frac{I}{20}\right).$$

Thus in our case we get

$$W = Ce^{-\frac{2}{350}t} + \frac{I}{20}.$$

Let us call the person's initial weight $W_0$ at $t = 0$. Then $W_0 = \frac{I}{20} + Ce^0$, so $C = W_0 - \frac{I}{20}$. Thus

$$W = \frac{I}{20} + \left(W_0 - \frac{I}{20}\right)e^{-\frac{1}{175}t}.$$

**(c)** Using part (b), we have $W = 150 + 10e^{-\frac{1}{175}t}$. This means that $W \to 150$ as $t \to \infty$. See Figure 10.30.

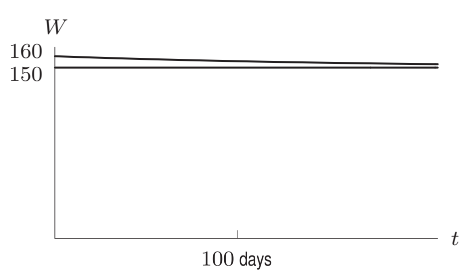



Figure 10.30

**30.** **(a)** The graph of $f(y) = y - y^2$ is shown Figure 10.31.
**(b)** The slope field $dy/dx = y - y^2$ is shown in Figure 10.32.

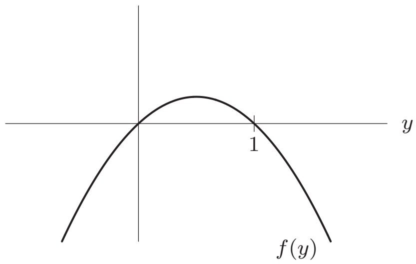



Figure 10.31

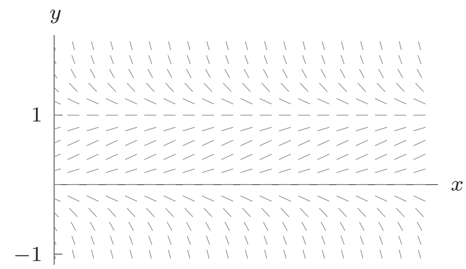



Figure 10.32

**(c)** The slopes in Figure 10.32 are positive for $0 < y < 1$, where $f(y)$ is above the horizontal axis in Figure 10.31. The slopes in Figure 10.32 are negative for $y < 0$ and $y > 1$, where $f(y)$ is negative.

The equilibrium solutions are $y = 0$ and $y = 1$, where the graph of $f(y)$ crosses the horizontal axis. The equilibrium $y = 1$ is stable, which can be seen in Figure 10.31 from the fact that $f(y)$ is positive for $y < 1$ and negative for $y > 1$. Thus, the solution curves increase toward $y = 1$ from below and decrease toward $y = 1$ from above.

The equilibrium at $y = 0$ is unstable. Figure 10.31 shows that $f(y)$ is negative for $y < 0$ and positive for $y > 0$. Thus, solution curves below $y = 0$ decrease away from $y = 0$; solution curves above $y = 0$ increase away from $y = 0$.

**31.** **(a)** The equilibrium solutions are $y = 1, y = 8, y = 16$. For these values, $f(y) = 0$, so $y' = 0$.
**(b)** See Figure 10.33. Since $f(y) > 0$ for $0 < y < 1$, the slopes are upward for these $y$ values; similarly for $8 < y < 16$ and $y > 16$. For $1 < y < 8$, the slopes are downward, since $f(y) < 0$ for these $y$ values.

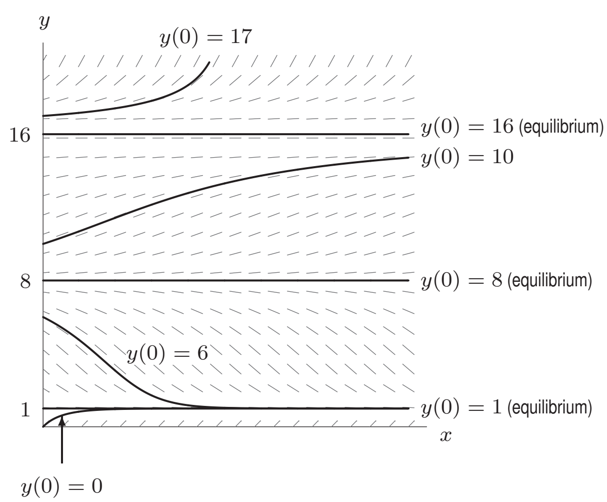



Figure 10.33

**(c)** See Figure 10.33. The equilibrium solutions for the initial conditions $y(0) = 1, y(0) = 8, y(0) = 16$ are horizontal lines. The other solution curves follow the slope field.
**(d)** The equilibrium $y = 1$ is stable; $y = 8$ and $y = 16$ are unstable.

**32.** Here $x$ and $y$ both increase at about the same rate.

**33.** Initially $x = 0$, so we start with only $y$. Then $y$ decreases while $x$ increases. Then $x$ continues to increase while $y$ starts to increase as well. Finally $y$ continues to increase while $x$ decreases.

**34.** $x$ decreases quickly while $y$ increases more slowly.

**35.** The closed trajectory represents populations which oscillate repeatedly.

**36.** **(a)** The rate $dQ_1/dt$ is the sum of three terms that represent the three changes in $Q_1$:

$$\frac{dQ_1}{dt} = A - k_1Q_1 + k_2Q_2.$$

The term $A$ is the rate at which $Q_1$ increases due to creation of new toxin.

The term $-k_1Q_1$ is the rate at which $Q_1$ decreases due to flow of toxin into the blood. The constant $k_1$ is a positive constant of proportionality.

The term $k_2Q_2$ is the rate at which $Q_1$ increases due to flow of toxin out of the blood. The constant $k_2$ is a second positive constant.

**(b)** The rate $dQ_2/dt$ is the sum of three terms that represent the three changes in $Q_2$:

$$\frac{dQ_2}{dt} = -k_3Q_2 + k_1Q_1 - k_2Q_2.$$

The term $-k_3Q_2$ is the rate at which $Q_2$ decreases due to removal of toxin by dialysis. The constant $k_3$ is a positive constant of proportionality.

The term $k_1Q_1$ is the rate at which $Q_2$ increases due to flow of toxin into the blood. The constant $k_1$ is the same positive constant as in part (a).

The term $-k_2Q_2$ is the rate at which $Q_2$ decreases due to flow of toxin out of the blood. The constant $k_2$ is the same positive constant as in part (a).

## CHECK YOUR UNDERSTANDING

**1.** True, since $dQ/dt$ represents the rate of change of $Q$.

**2.** False. Since the population is decreasing by 5% (rather than 5 units of $P$), the correct differential equation is $dP/dt = -0.05P$.

**3.** False. Since the population is *decreasing*, the rate of change must be negative. The correct differential equation is $dP/dt = -0.05P$.

**4.** True. The balance increases at a rate of 3% of $B$, or $0.03B$, each year so the rate of change of $B$ is $0.03B$.

**5.** True, since when two quantities are proportional, one is a constant times the other.

**6.** False. The 200 is not a percent, but in units of $P$ over units of $t$, so the correct differential equation is $dP/dt = -200$.

**7.** True. The balance is increasing due to the interest at a rate of $0.05B$ and is decreasing due to the payments at a rate of 8000.

**8.** False, the correct differential equation is $dQ/dt = 200 - kQ$ (where $k > 0$).

**9.** True. The rate the drug is entering the body is 12 mg per hour and the rate the drug is leaving the body is $0.063Q$ mg per hour.

**10.** False. The deposit of \$10,000 was a one-time deposit, and not a rate of change. This differential equation would be correct if the deposits were being made at a continuous rate of \$10,000 *per year*.

**11.** True, since the derivative of $P$ is zero, so $dP/dt = 0$ and also substituting $P = 10$ in the expression $3P(10 - P)$ gives zero, so substituting $P = 10$ on both sides of the differential equation gives 0, and we have $0 = 0$, a true equation.

**12.** True, since substituting $P = 0$ on both sides of the differential equation gives 0.

**13.** False, since substituting $P = 5$ on the left-hand side of the differential equation gives 0, but on the right-hand side gives 75.

**14.** True. When we substitute $t = 0$ and $y = 40$ in the general solution, we have $40 = Ce^{0.05(0)}$. Since $e^0 = 1$, this gives $C = 40$.

**15.** False. When we substitute $t = 0$ and $y = 40$ in the general solution, we have $40 = 25 + Ce^{0.05(0)}$. Since $e^0 = 1$, this gives $40 = 25 + C$, which implies $C = 15$.

**16.** True. When we substitute $t = 0$ and $y = 40$ in the general solution, we have $40 = 25 + Ce^{0.05(0)}$. Since $e^0 = 1$, this gives $40 = 25 + C$, which implies $C = 15$.

**17.** False. Since $y' = 0.2y$, when $y = 100$ we have $y' = 0.2 \cdot 100 = 20$. The variable $y$ is changing at a rate of 20 units per unit of time. This tells us that $y$ increases approximately 20 units between $t = 0$ and $t = 1$, so we expect $y(1) \approx 100 + 20 = 120$.

**18.** True. Since $y' = 0.2y$, when $y = 100$ we have $y' = 0.2 \cdot 100 = 20$. The variable $y$ is changing at a rate of 20 units per unit of time. This tells us that $y$ increases approximately 20 units between $t = 0$ and $t = 1$, so we expect $y(1) \approx 100 + 20 = 120$.

**19.** True, since when $Q = 10$ we have $dQ/dt = 5 \cdot 10 - 200 = -150 < 0$.

**20.** True, since when $Q = 40$ both sides of the differential equation give $0$.

**21.** False, since when $x = 3$, we have $dy/dx = 2 \cdot 3 = 6$.

**22.** True, since when $x = 3$, we have $dy/dx = 2 \cdot 3 = 6$.

**23.** True, since when $x = 1$ and $y = -2$, we have $dy/dx = 3 \cdot 1 \cdot (-2) = -6$.

**24.** False, since when $x = 2$ and $y = 2$, we have $dy/dx = 3 \cdot 2 \cdot 2 = 12$.

**25.** True, since when $x = 3$ and $y = 2$, we have $dy/dx = 3 \cdot 3 \cdot 2 = 18$.

**26.** False, since whenever $x < 0$ we have $dy/dx < 0$ regardless of the sign of $y$.

**27.** True, since when $y = 1, dy/dx = 2 \cdot 1 = 2$, regardless of the value of $x$.

**28.** True, since when $x = 3, dy/dx = 5 \cdot 0 \cdot (y - 2) = 0$.

**29.** False, since when $P > 3$ we have $12 - 4P < 0$ so $dP/dt < 0$.

**30.** True, since the slope field lines have slope $dy/dx$ which is the derivative of a solution $y = f(x)$.

**31.** False; the general solution is $y = Ce^{kt}$.

**32.** True.

**33.** False. That solution would be the solution to the differential equation $dw/dr = 0.3w$.

**34.** True. The general solution is $H = Ce^{0.5t}$ and the particular solution $H = 57e^{0.5t}$ satisfies the initial condition $H(0) = 57$.

**35.** True, since the general solution is $y = Ce^{3t}$ and setting $t = 0, y = 5$ gives $C = 5$.

**36.** False, since $y = Ce^{-2t}$ is a general solution. The particular solution is $y = 3e^{-2t}$.

**37.** False, the correct differential equation is $dB/dt = 0.03B$.

**38.** False, since if $k$ were negative, $-k$ would be positive, and $Q(t)$ would grow exponentially, rather than decay.

**39.** False. The function $Q = Ce^{kt}$ is the *solution* not the differential equation. The differential equation is $dQ/dt = kQ$.

**40.** True.

**41.** True, as explained in the text.

**42.** False. We need to factor out a 2 to write the differential equation in the form $dP/dt = 2(P - 50)$. Then the general solution is $P = 50 + Ce^{2t}$.

**43.** False. The function $Q = 20 + Ce^{0.5t}$ is a solution to the differential equation $dQ/dt = 0.5(Q - 20)$.

**44.** True. We factor out the coefficient of $W$ to rewrite the differential equation as $dW/dt = -3(\frac{600}{-3} + W) = -3(W - 200)$, which has the solution shown.

**45.** False. The initial condition gives $C = 10$ so the correct solution is $A = 40 + 10e^{0.25t}$.

**46.** True.

**47.** False; the correct differential equation is $dB/dt = 0.04B - 12{,}000$. The equation given has no derivatives so it cannot be a differential equation.

**48.** True, since the drug is entering the body at a rate of 12 and leaving the body at a rate of $0.18A$.

**49.** True, since setting $dH/dt = 0$ gives $H = 225$.

**50.** True.

**51.** False. If $X$ would do fine if $Y$ didn't exist, then the parameter $a$ must be positive.

**52.** False. If species $X$ eats species $Y$, then the parameter $d$ must be negative.

**53.** True. Since $X$ has a negative impact on $Y$, the coefficient of the interaction term must be negative.

**54.** True. Since $X$ would die out alone, the parameter $a$ must be negative. Since $Y$ helps $X$, the parameter $b$ of the interaction term must be positive.

**55.** False. If $Y$ is absent then the population of $X$ grows at a continuous rate of $2\%$, and if $X$ is absent the population of $Y$ grows at a continuous rate of $5\%$

**56.** True, since the coefficients $0.02$ and $0.05$ are positive.

**57.** True, since the coefficients $-0.15$ and $-0.18$ of the interaction terms are negative.

**58.** True. Since the coefficients $-0.12$ and $-0.10$ are negative, both populations would die out in the absence of the other. Since the coefficients $0.07$ and $0.25$ of the interaction terms are positive, both populations are helped by the other.

**59.** True. Substituting the values for $x$ and $y$ into the differential equations gives $dx/dt = -3(1) + (1)(2) = -3 + 2 = -1 < 0$ and $dy/dt = 2(2) - 5(1)(2) = 4 - 10 = -6 < 0$. Since the derivatives of $x$ and $y$ are both negative, both populations are decreasing.

**60.** False. Substituting the values for $x$ and $y$ into the differential equations gives $dx/dt = -3(2) + (2)(5) = -6 + 10 = 4 > 0$ and $dy/dt = 2(5) - 5(2)(5) = 10 - 50 = -40 < 0$. Since the derivative of $y$ is negative, the $y$ population is decreasing. However, the derivative of $x$ is positive, so the $x$ population is increasing.

**61.** False. It is negative because people in the susceptible group are becoming sick and moving to the group of infected people.

**62.** True. People in the susceptible group are becoming sick and moving to the group of infected people.

**63.** False, since the negative of this quantity does not appear in the expression for $dS/dt$.

**64.** True.

**65.** False. The parameter $a$ will be larger for Type I flu.

**66.** True. The parameter $a$ measures how infectious or contagious the disease is.

**67.** False. The parameter $b$ will be *smaller* for Type I flu, since a smaller percentage recover per unit time.

**68.** True. The parameter $b$ will be smaller for Type I flu, since a smaller percentage recover per unit time.

**69.** True. To see if $I$ will increase, we see if $dI/dt$ is positive. We have $dI/dt = 0.001SI - 0.3I = 0.001(500)(100) - 0.3(100) = 50 - 30 = 20 > 0$.

**70.** False. To see if $I$ will increase, we see if $dI/dt$ is positive. We have $dI/dt = 0.001SI - 0.3I = 0.001(100)(500) - 0.3(500) = 50 - 150 = -100 < 0$.

## PROJECTS FOR CHAPTER TEN

**1. (a)** Equilibrium values are $N = 0$ (unstable) and $N = 200$ (stable). The graphs are shown in Figures 10.34 and 10.35.

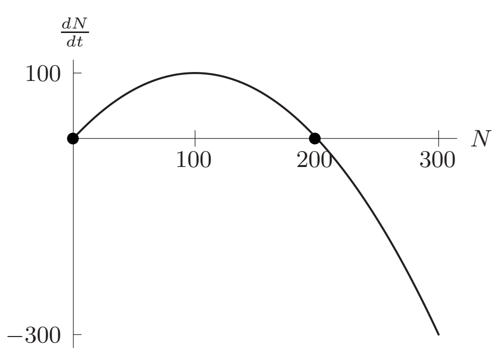



**Figure 10.34:** $dN/dt = 2N - 0.01N^2$

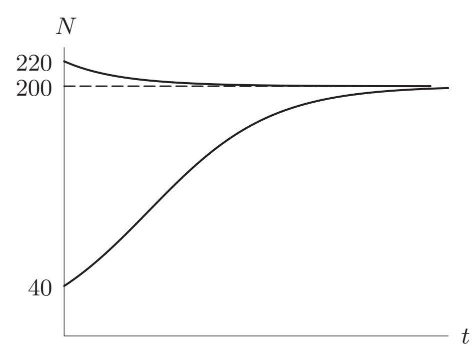



**Figure 10.35:** Solutions to $dN/dt = 2N - 0.01N^2$

**(b)** When there is no fishing the rate of population change is given by $\frac{dN}{dt} = 2N - 0.01N^2$. If fishermen remove fish at a rate of 75 fish/year, then this results in a decrease in the growth rate, $\frac{dP}{dt}$, by 75 fish/year. This is reflected in the differential equation by including the $-75$.

**(c)**

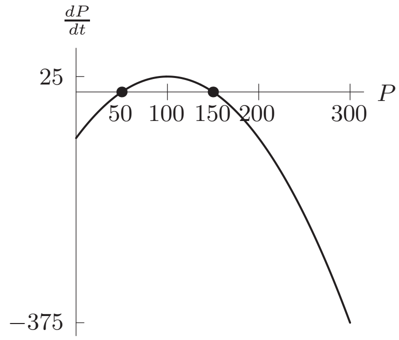



**Figure 10.36**: $dP/dt = 2P - 0.01P^2 - 75$

**(d)**

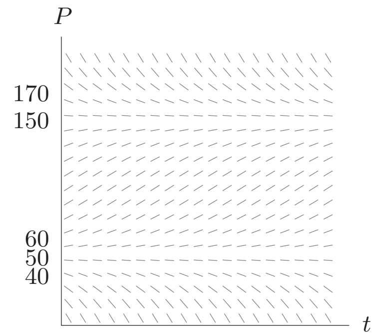



**Figure 10.37**

**(e)** In Figure 10.36, we see that $dP/dt = 0$ when $P = 50$ and when $P = 150$, that $dP/dt$ is positive when $P$ is between 50 and 150, and that $dP/dt$ is negative when $P$ is less than 50 or greater than 150.

(i) Since $dP/dt = 0$ at $P = 50$ and at $P = 150$, these are the two equilibrium values.

(ii) Since $dP/dt$ is positive when $P$ is between 50 and 150, we know that $P$ increases for initial values in this interval. It increases toward the equilibrium value of $P = 150$.

(iii) Since $dP/dt$ is negative for $P$ less than 50 or $P$ greater than 150, we know $P$ decreases for starting values in these intervals. If the initial value of $P$ is less than 50, then $P$ decreases to zero and the fish all die out. If the initial value of $P$ is greater than 150, then the fish population decreases toward the equilibrium value of 150.

The solutions look like those shown in Figure 10.38.

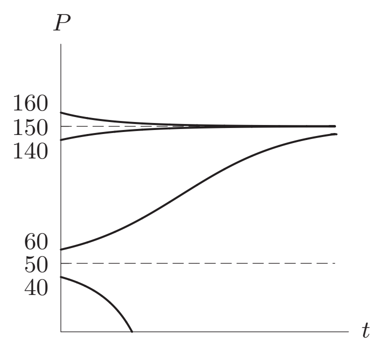



**Figure 10.38**: Solutions to $dP/dt = 2P - 0.01P^2 - 75$

**(f)**

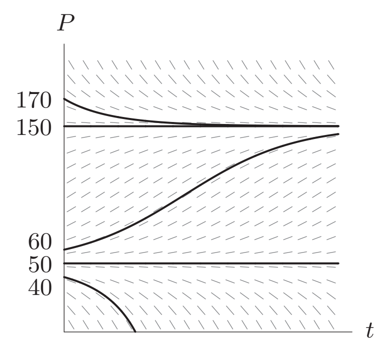


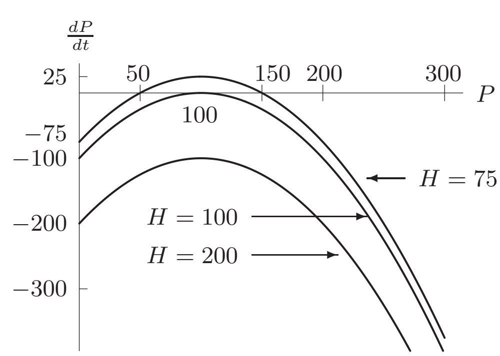

**Figure 10.39**

**(g)** The two equilibrium populations are $P = 50,\ 150$. The stable equilibrium is $P = 150$, while $P = 50$ is unstable.

Notice that $P = 50$ and $P = 150$ are solutions of $dP/dt = 0$:

$$\frac{dP}{dt} = 2P - 0.01P^2 - 75 = -0.01(P^2 - 200P + 7500) = -0.01(P - 50)(P - 150).$$

**(h)** (i)



(ii) For $H = 75$, the equilibrium populations (where $dP/dt = 0$) are $P = 50$ and $P = 150$. If the population is between 50 and 150, $dP/dt$ is positive. This means that when the initial population is between 50 and 150, the population will increase until it reaches 150, when $dP/dt = 0$ and the population no longer increases or decreases. If the initial population is greater than 150, then $dP/dt$ is negative, and the population decreases until it reaches 150. Thus 150 is a stable equilibrium. However, 50 is unstable.

For $H = 100$, the equilibrium population (where $dP/dt = 0$) is $P = 100$. For all other populations, $dP/dt$ is negative and so the population decreases. If the initial population is greater than 100, it will decrease to the equilibrium value, $P = 100$. However, for populations less than 100, the population decreases until the species dies out.

For $H = 200$, there are no equilibrium points where $dP/dt = 0$, and $dP/dt$ is always negative. Thus, no matter what the initial population, the population always dies out eventually.

(iii) If the population is not to die out, looking at the three cases above, there must be an equilibrium value where $dP/dt = 0$, i.e. where the graph of $dP/dt$ crosses the $P$ axis. This happens if $H \leq 100$. Thus provided fishing is not more than 100 fish/year, there are initial values of the population for which the population will not be depleted.

(iv) Fishing should be kept below the level of 100 per year.

**2. (a)** In each generation, mutation causes the fraction of $b$ genes to decrease $k_1$ times the fraction of $b$ genes (as $b$ genes mutate to $B$ genes). Likewise, in every generation, mutation causes the fraction of $b$ genes to increase by $k_2$ times the fraction of $B$ genes (as $B$ genes mutate to $b$ genes). Therefore, $q$ decreases by $k_1 q$ and increases by $k_2(1-q)$, and we have:

$$\frac{dq}{dt} = -k_1 q + k_2(1-q).$$

**(b)** We have

$$\begin{aligned}\frac{dq}{dt} &= -0.0001q + 0.0004(1-q)\\ &= -0.0001q + 0.0004 - 0.0004q\\ &= -0.0005q + 0.0004\\ &= -0.0005(q-0.8).\end{aligned}$$

The solution to this differential equation is

$$q = 0.8 + Ce^{-0.0005t}.$$

If $q_0 = 0.1$, then $C = -0.7$ and the solution is $q = 0.8 - 0.7e^{-0.0005t}$. If $q_0 = 0.9$, then $C = 0.1$ and the solution is $q = 0.8 + 0.1e^{-0.0005t}$. These solution are in Figure 10.40.

The equilibrium value is $q = 0.8$. From Figure 10.40, we see that as generations pass, the fraction of genes responsible for the recessive trait gets closer to $0.8$.

The equilibrium is given by the solution to the equation

$$\frac{dq}{dt} = -0.0005q + 0.0004 = 0.0005(q-0.8) = 0.$$

Therefore the equilibrium is given by $0.0004/0.0005 = 0.8$ and so is completely determined by the values of $k_1$ and $k_2$.

**(c)** We have

$$\begin{aligned}\frac{dq}{dt} &= -0.0003q + 0.0001(1-q)\\ &= -0.0003q + 0.0001 - 0.0001q\\ &= -0.0004q + 0.0001\\ &= -0.0004(q-0.25).\end{aligned}$$

The solution to this differential equation is

$$q = 0.25 + Ce^{-0.0004t}.$$

If $q_0 = 0.1$, then $C = -0.15$ and the solution is $q = 0.25 - 0.15e^{-0.0004t}$. If $q_0 = 0.9$, then $C = 0.65$ and the solution is $q = 0.25 + 0.65e^{-0.0004t}$. These solutions are shown in Figure 10.41.

The equilibrium value is $q = 0.25$. As more generations pass, the fraction of genes responsible for the recessive trait gets and closer to 0.25.

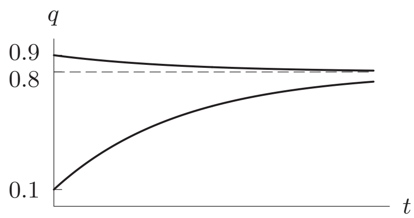



**Figure 10.40**

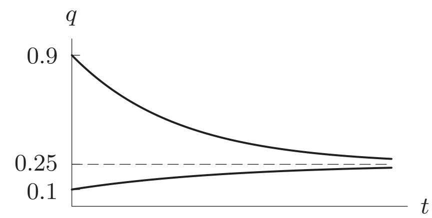



**Figure 10.41**

**3. (a)** Since $I_0$ is the number of infecteds on day $t = 0$, March 17, we have $I_0 = 95$. Since $S_0$ is the initial number of susceptibles, which is the whole population of Hong Kong, $S_0 \approx 6.8$ million.

**(b)** For $a = 1.25 \cdot 10^{-8}$ and $b = 0.06$, the system of equations is

$$\frac{dS}{dt} = -1.25 \cdot 10^{-8} SI$$
$$\frac{dI}{dt} = 1.25 \cdot 10^{-8} SI - 0.06I.$$

So, by the chain rule,

$$\frac{dI}{dS} = \frac{dI/dt}{dS/dt} = \frac{1.25 \cdot 10^{-8} SI - 0.06I}{-1.25 \cdot 10^{-8} SI} = -1 + \frac{4.8 \cdot 10^6}{S}.$$

The slope field and trajectory are in Figure 10.42.

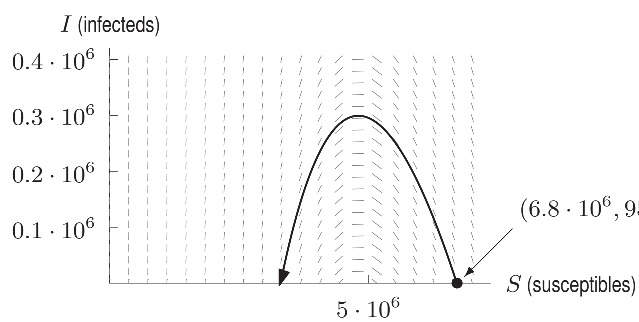



**Figure 10.42**

**(c)** The maximum value of $I$ is about 300,000; this gives us the maximum number of infecteds at any one time—the total number of people infected during the course of the disease is much greater than this. The trajectory meets the $S$-axis at about 3.3 million; this tells us that when the disease dies out, there are still 3.3 million susceptibles who have never had the disease. Therefore $6.8 - 3.3 = 3.5$ million people are predicted to have had the disease.

The threshold value of $S$ occurs where $dI/dt = 0$ and $I \neq 0$, so, for $a = 1.25 \cdot 10^{-8}$ and $b = 0.06$,

$$\frac{dI}{dt} = 1.25 \cdot 10^{-8} SI - 0.06I = 0,$$

giving

$$\text{Threshold value} = S = \frac{0.06}{1.25 \cdot 10^{-8}} = 4.8 \cdot 10^6 \text{ people.}$$

The threshold value tells us that if the initial susceptible population, $S_0$ is more than $4.8$ million, there will be an epidemic. If $S_0$ is less than 4.8 million, there will not be an epidemic. Since the population of Hong Kong is over $4.8$ million, an epidemic is predicted.

**(d)** The value of $b$ represents the rate at which infecteds are removed from circulation. Quarantine increases the rate people are removed and thus increases $b$.

**(e)** For $a = 1.25 \cdot 10^{-8}$ and $b = 0.24$, the system of differential equations is

$$\frac{dS}{dt} = -1.25 \cdot 10^{-8} SI$$
$$\frac{dI}{dt} = 1.25 \cdot 10^{-8} SI - 0.24I.$$

So, by the chain rule,

$$\frac{dI}{dS} = \frac{dI/dt}{dS/dt} = \frac{1.25 \cdot 10^{-8} SI - 0.24I}{-1.25 \cdot 10^{-8} SI} = -1 + \frac{19.2 \cdot 10^6}{S}.$$

The slope field is in Figure 10.43. The solution trajectory does not show as the disease dies out right away.

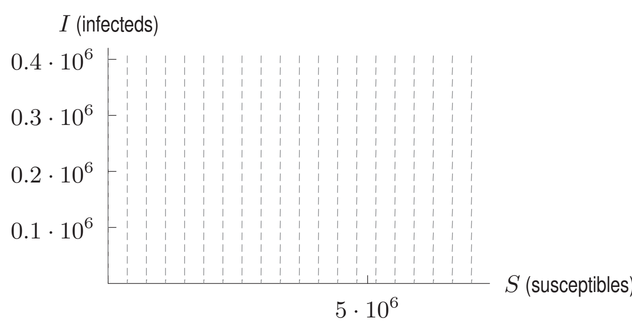



**Figure 10.43**

**(f)** The threshold value of $S$ occurs where $dI/dt = 0$ and $I \neq 0$, so, for $b = 0.24$ and the same value of $a$,

$$\frac{dI}{dt} = 1.25 \cdot 10^{-8} SI - 0.24I = 0,$$

giving

$$\text{Threshold value } = S = \frac{0.24}{1.25 \cdot 10^{-8}} = 19.2 \cdot 10^6 \text{ people.}$$

The threshold value tells us that if $S_0$ is less than 19.2 million, there will be no epidemic. The population of Hong Kong is $6.8$ million, so $S_0$ is below this value. Thus no epidemic is predicted.

Policies, such as quarantine, which raise the value of $b$ can be effective in preventing an epidemic. In this case, the value of $b$ increased sufficiently that the population of Hong Kong fell below the threshold value, and a potential epidemic was averted. However, we do not have evidence that the quarantine policy was responsible for the increase in $b$.

**(g)** Policy I: Closing off the city changes the initial values of $S_0$ and $I_0$ but not the values of $a$ and $b$. If not one infected person enters the city, then $I_0 = 0$ and the solution trajectory is an equilibrium point on the $S$-axis. However, in practice it is almost impossible to cut off a city completely, so usually $I_0 > 0$. Also, by the time a policy to close off a city is put into effect, there may already be infected people inside the city, so again $I_0 > 0$. Thus, whether or not there is an epidemic depends on whether $S_0$ is greater than the threshold value, not on the value of $I_0$ (provided $I_0 > 0$).

For example, in the case of Hong Kong with the March values of $a$ and $b$, changing the value of $I_0$ to 1 leaves the solution trajectory much as before; see Figure 10.44. The main difference is that the epidemic occurs slightly later. So a policy of isolating a city only works if it keeps the disease out of the city of the city entirely. Thus, Policy I does not help the city except in the exceptional case that *every* infected person is kept out.

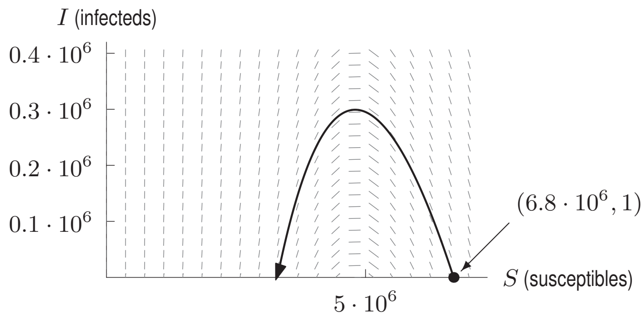



**Figure 10.44**

Policy II: From the analysis of the Hong Kong data, we see that a quarantine policy can help prevent an epidemic if the value of $b$ is increased enough to bring $S_0$ below the threshold value. Thus, Policy II can be very effective.

## Solutions to Problems on Separation of Variables

**1.** Separating variables gives

$$\int \frac{1}{P} dP = -\int 2dt,$$

so

$$\ln |P| = -2t + C.$$

Therefore

$$P = \pm e^{-2t+C} = Ae^{-2t}.$$

The initial value $P(0) = 1$ gives $1 = A$, so

$$P = e^{-2t}.$$

**2.** Separating variables and integrating both sides gives

$$\int \frac{1}{L} dL = \frac{1}{2} \int dp$$

or

$$\ln |L| = \frac{1}{2}p + C.$$

This can be written

$$L = \pm e^{(1/2)p+C} = Ae^{p/2}.$$

The initial condition $L(0) = 100$ gives $100 = A$, so

$$L = 100e^{p/2}.$$

**3.** Separating variables gives

$$\int P dP = \int dt$$

so that

$$\frac{P^2}{2} = t + C$$

or

$$P = \pm\sqrt{2t + D}$$

(where $D = 2C$).

The initial condition $P(0) = 1$ implies we must take the positive root and that $1 = D$, so

$$P = \sqrt{2t + 1}.$$

**4.** Separating variables gives

$$\int \frac{1}{m} dm = \int ds.$$

Hence

$$\ln |m| = s + C$$

which gives

$$m = \pm e^{s+C} = Ae^s.$$

The initial condition $m(1) = 2$ gives $2 = Ae^1$ or $A = 2/e$, so

$$m = \frac{2}{e} e^s = 2e^{s-1}.$$

**5.** Separating variables gives

$$\int \frac{1}{u^2} du = \int \frac{1}{2} dt$$

or

$$-\frac{1}{u} = \frac{1}{2} t + C.$$

The initial condition gives $C = -1$ and so

$$u = \frac{1}{1 - (1/2)t}.$$

**6.** Separating variables and integrating gives

$$\int \frac{1}{z} dz = \int y dy$$

which gives

$$\ln |z| = \frac{1}{2} y^2 + C$$

or

$$z = \pm e^{(1/2)y^2 + C} = Ae^{y^2/2}.$$

The initial condition $y = 0$, $z = 1$ gives $A = 1$. Therefore

$$z = e^{y^2/2}.$$

**7.** Rearrange and write

$$\int \frac{1}{1-R} dR = \int dy$$

or

$$-\ln|1-R| = y + C$$

which can be written as

$$1 - R = \pm e^{-C-y} = Ae^{-y}$$

or

$$R = 1 - Ae^{-y}.$$

The initial condition $R(1) = 0.1$ gives $0.1 = 1 - Ae^{-1}$ and so

$$A = 0.9e.$$

Therefore

$$R = 1 - 0.9e^{1-y}.$$

**8.** Write

$$\int \frac{1}{y} dy = \int \frac{1}{3+t} dt$$

and so

$$\ln|y| = \ln|3+t| + C$$

or

$$\ln|y| = \ln D|3+t|$$

where $\ln D = C$. Therefore

$$y = D(3+t).$$

The initial condition $y(0) = 1$ gives $D = \frac{1}{3}$, so

$$y = \frac{1}{3}(3+t).$$

**9.** Separating variables gives

$$\frac{dz}{dt} = te^{z}$$
$$e^{-z} dz = t dt$$
$$\int e^{-z}\, dz = \int t\, dt,$$

so

$$-e^{-z} = \frac{t^2}{2} + C.$$

Since the solution passes through the origin, $z = 0$ when $t = 0$, we must have

$$-e^{-0} = \frac{0}{2} + C, \text{ so } C = -1.$$

Thus

$$-e^{-z} = \frac{t^2}{2} - 1,$$

or

$$z = -\ln\left(1 - \frac{t^2}{2}\right).$$

**10.** Separating variables gives

$$\frac{dy}{dx} = \frac{5y}{x}$$
$$\int \frac{dy}{y} = \int \frac{5}{x}dx$$
$$\ln|y| = 5\ln|x| + C.$$

Thus

$$|y| = e^{5\ln|x|}e^C = e^C e^{\ln|x|^5} = e^C|x|^5,$$

giving

$$y = Ax^5, \quad \text{where} \quad A = \pm e^C.$$

Since $y = 3$ when $x = 1$, so $A = 3$. Thus

$$y = 3x^5.$$

**11.** Separating variables gives

$$\frac{dy}{dt} = y^2(1+t)$$
$$\int \frac{dy}{y^2} = \int (1+t)\,dt,$$

so

$$-\frac{1}{y} = t + \frac{t^2}{2} + C,$$

giving

$$y = -\frac{1}{t + t^2/2 + C}.$$

Since $y = 2$ when $t = 1$, we have

$$2 = -\frac{1}{1 + 1/2 + C}, \quad \text{so} \quad 2C + 3 = -1, \quad \text{and} \quad C = -2.$$

Thus

$$y = -\frac{1}{t^2/2 + t - 2} = -\frac{2}{t^2 + 2t - 4}.$$

**12.** Separating variables gives

$$\frac{dz}{dt} = z + zt^2 = z(1+t^2)$$
$$\int \frac{dz}{z} = \int (1+t^2)dt,$$

so

$$\ln|z| = t + \frac{t^3}{3} + C,$$

giving

$$z = Ae^{t+t^3/3}.$$

We have $z = 5$ when $t = 0$, so $A = 5$ and

$$z = 5e^{t+t^3/3}.$$

**13.** **(a)** Yes **(b)** No **(c)** Yes
**(d)** No **(e)** Yes **(f)** Yes
**(g)** No **(h)** Yes **(i)** No
**(j)** Yes **(k)** Yes **(l)** No

**14.** Separating variables gives

$$\int \frac{dP}{P-a} = \int dt.$$

Integrating yields

$$\ln|P-a| = t + C,$$

so

$$\begin{aligned} |P-a| &= e^{t+C} = e^t e^C \\ P &= a + Ae^t, \quad \text{where } A = \pm e^C \quad \text{or } A = 0. \end{aligned}$$

**15.** Separating variables gives

$$\int \frac{dQ}{b-Q} = \int dt.$$

Integrating yields

$$-\ln|b-Q| = t + C,$$

so

$$\begin{aligned} |b-Q| &= e^{-(t+C)} = e^{-t} e^{-C} \\ Q &= b - Ae^{-t}, \quad \text{where } A = \pm e^{-C} \quad \text{or } A = 0. \end{aligned}$$

**16.** Separating variables gives

$$\int \frac{dP}{P-a} = \int k\,dt.$$

Integrating yields

$$\ln|P-a| = kt + C,$$

so

$$P = a + Ae^{kt} \quad \text{where } A = \pm e^C \quad \text{or } A = 0.$$

**17.** Factoring and separating variables gives

$$\begin{aligned} \frac{dR}{dt} &= a\left(R + \frac{b}{a}\right) \\ \int \frac{dR}{R + b/a} &= \int a\,dt \\ \ln\left|R + \frac{b}{a}\right| &= at + C \\ R &= -\frac{b}{a} + Ae^{at}, \quad \text{where } A \text{ can be any constant.} \end{aligned}$$

**18.** Separating variables and integrating gives

$$\int \frac{1}{aP+b}dP = \int dt.$$

This gives

$$\begin{aligned} \frac{1}{a}\ln|aP+b| &= t + C \\ \ln|aP+b| &= at + aC \\ aP + b &= \pm e^{at+aC} = Ae^{at}, \quad \text{where } A = \pm e^{aC} \quad \text{or} \quad A = 0, \end{aligned}$$

or

$$P = \frac{1}{a}(Ae^{at} - b).$$

**19.** Separating variables and integrating gives

$$\int \frac{1}{y^2} dy = \int k(1+t^2) dt$$

or

$$-\frac{1}{y} = k\left(t + \frac{1}{3}t^3\right) + C.$$

Hence,

$$y = \frac{-1}{k(t + \frac{1}{3}t^3) + C}.$$

**20.** **(a)** Separating variables and integrating gives

$$\int \frac{1}{100-y} dy = \int dt$$

so that

$$-\ln|100-y| = t + C$$

or

$$y(t) = 100 - Ae^{-t}.$$

**(b)** See Figure 10.45.

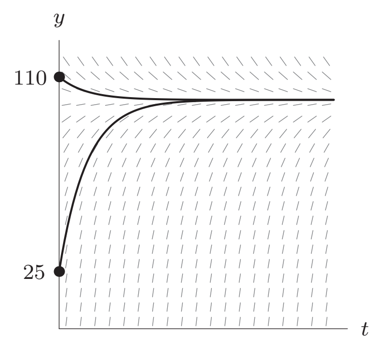



**Figure 10.45**

**(c)** The initial condition $y(0) = 25$ gives $A = 75$, so the solution is

$$y(t) = 100 - 75e^{-t}.$$

The initial condition $y(0) = 110$ gives $A = -10$ so the solution is

$$y(t) = 100 + 10e^{-t}.$$

**(d)** The increasing function, $y(t) = 100 - 75e^{-t}$.

**21.** **(a)** The slope field for $dy/dx = xy$ is in Figure 10.46.

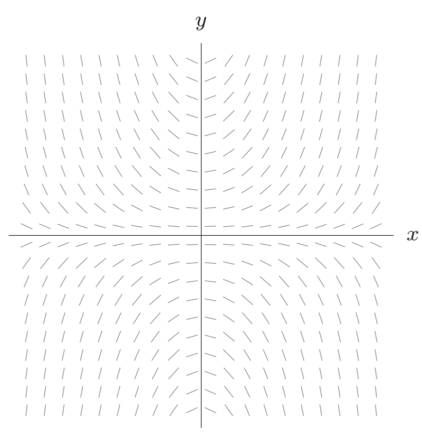



**Figure 10.46**

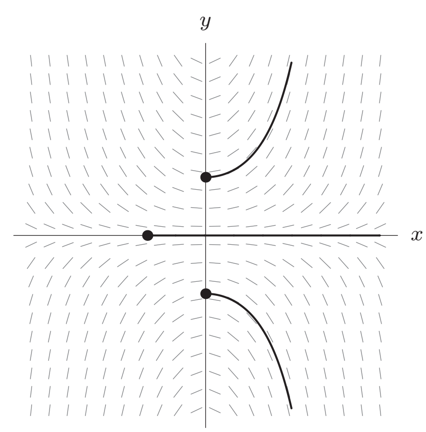



**Figure 10.47**

**(b)** Some solution curves are shown in Figure 10.47.

**(c)** Separating variables gives

$$\int \frac{1}{y} dy = \int x dx$$

or

$$\ln |y| = \frac{1}{2}x^2 + C.$$

Solving for $y$ gives

$$y(x) = Ae^{x^2/2}$$

where $A = \pm e^C$. In addition, $y(x) = 0$ is a solution. So $y(x) = Ae^{x^2/2}$ is a solution for any $A$.