CHAPTER ELEVEN

Solutions for Section 11.1

1. Adding the terms, we see that

$$3 + 3 \cdot 2 + 3 \cdot 2^2 = 3 + 6 + 12 = 21.$$

We can also find the sum using the formula for a finite geometric series with $a = 3$, $r = 2$, and $n = 3$:

$$3 + 3 \cdot 2 + 3 \cdot 2^2 = \frac{3(1 - 2^3)}{1 - 2} = 3(8 - 1) = 21.$$

2. Adding the terms, we see that

$$50 + 50(0.9) + 50(0.9)^2 + 50(0.9)^3 = 50 + 45 + 40.5 + 36.45 = 171.95.$$

We can also find the sum using the formula for a finite geometric series with $a = 50$, $r = 0.9$, and $n = 4$:

$$50 + 50(0.9) + 50(0.9)^2 + 50(0.9)^3 = \frac{50(1 - (0.9)^4)}{1 - 0.9} = 171.95.$$

3. The formula for the sum of a finite geometric series with $a = 5$, $r = 3$, and $n = 13$ gives

$$\text{Sum } = 5 + 5 \cdot 3 + 5 \cdot 3^2 + \cdots + 5 \cdot 3^{12} = \frac{5(1 - 3^{13})}{1 - 3} = 3{,}985{,}805.$$

4. The formula for the sum of a finite geometric series with $a = 20$, $r = 1.45$, and $n = 15$ gives

$$\text{Sum } = 20 + 20(1.45) + \cdots + 20(1.45)^{14} = \frac{20(1 - (1.45)^{15})}{1 - 1.45} = 11{,}659.64.$$

5. The formula for the sum of a finite geometric series with $a = 100$, $r = 0.85$, and $n = 11$ gives

$$\text{Sum } = 100 + 100(0.85) + \cdots + 100(0.85)^{10} = \frac{100(1 - (0.85)^{11})}{1 - 0.85} = 555.10.$$

6. This is an infinite geometric series with $a = 1000$ and $r = 1.05$. Since $r > 1$, the series diverges and the sum does not exist.

7. This is an infinite geometric series with $a = 75$ and $r = 0.22$. Since $-1 < r < 1$, the series converges and the sum is given by:

$$\text{Sum } = 75 + 75(0.22) + 75(0.22)^2 + \cdots = \frac{75}{1 - 0.22} = 96.154.$$

8. Notice that the first term in this series is $500(0.4) = 200$ and each term is 0.4 times the preceding term. This is an infinite geometric series with $a = 500(0.4) = 200$ and $r = 0.4$. Since $-1 < r < 1$, the series converges and the sum is given by:

$$\text{Sum } = \frac{200}{1 - 0.4} = 333.33.$$

9. Since $6300/31500 = 0.2$ and $1260/6300 = 0.2$ and $252/1260 = 0.2$, this is an infinite geometric series with $a = 31500$ and $r = 0.2$:

$$\text{Sum} = 31500 + 6300 + 1260 + 252 + \cdots = 31500 + 31500(0.2) + 31500(0.2)^2 + 31500(0.2)^3 + \cdots.$$

Since $-1 < r < 1$, the series converges and the sum is given by:

$$\text{Sum} = \frac{31500}{1 - 0.2} = 39{,}375.$$

10. This is a finite geometric series with $a = 3$, $r = 1/2$, and $n - 1 = 10$, so $n = 11$. Thus

$$\text{Sum} = \frac{3(1 - (1/2)^{11})}{1 - 1/2} = 3 \cdot 2\frac{(2^{11} - 1)}{2^{11}} = \frac{3(2^{11} - 1)}{2^{10}}.$$

11. The formula for the sum of a finite geometric series with $a = 65$, $r = 1/(1.02)$, and $n = 19$ gives

$$\text{Sum} = 65 + \frac{65}{1.02} + \frac{65}{(1.02)^2} + \cdots + \frac{65}{(1.02)^{18}} = \frac{65(1 - 1/(1.02)^{19})}{1 - 1/1.02} = 1039.482.$$

12. Each term in this series is 1.5 times the preceding term, so this is an infinite geometric series with $a = 1000$ and $r = 1.5$. Since $r > 1$, the series diverges and the sum does not exist.

13. Each term in this series is half the preceding term, so this is an infinite geometric series with $a = 200$ and $r = 0.5$. Since $-1 < r < 1$, the series converges and we have

$$\text{Sum} = \frac{200}{1 - 0.5} = 400.$$

14. This is an infinite geometric series with $a = -2$ and $r = -1/2$. Since $-1 < r < 1$, the series converges and

$$\text{Sum} = \frac{-2}{1 - (-1/2)} = -\frac{4}{3}.$$

15. Since $a = 10$ and $r = 0.75$, we find the partial sums using the formula

$$S_n = \frac{10(1 - (0.75)^n)}{1 - 0.75}.$$

For $n = 5$, we have

$$S_5 = \frac{10(1 - (0.75)^5)}{1 - 0.75} = 30.51.$$

For $n = 10$, we have

$$S_{10} = \frac{10(1 - (0.75)^{10})}{1 - 0.75} = 37.75.$$

For $n = 15$, we have

$$S_{15} = \frac{10(1 - (0.75)^{15})}{1 - 0.75} = 39.47.$$

For $n = 20$, we have

$$S_{20} = \frac{10(1 - (0.75)^{20})}{1 - 0.75} = 39.87.$$

As n gets larger, the partial sums appear to be approaching 40, as we expect.

16. We find the partial sums using the formula

$$S_n = \frac{250(1-(1.2)^n)}{1-1.2}.$$

For $n = 5$, we have

$$S_5 = \frac{250(1-(1.2)^5)}{1-1.2} = 1{,}860.40.$$

For $n = 10$, we have

$$S_{10} = \frac{250(1-(1.2)^{10})}{1-1.2} = 6{,}489.67.$$

For $n = 15$, we have

$$S_{15} = \frac{250(1-(1.2)^{15})}{1-1.2} = 18{,}008.78.$$

For $n = 20$, we have

$$S_{20} = \frac{250(1-(1.2)^{20})}{1-1.2} = 46{,}672.00.$$

As n gets larger, the partial sums appear to be growing without bound, as we expect, since $r > 1$.

17. **(a)** Notice that the 6^{th} deposit is made 5 months after the first deposit, so the first deposit has grown to $500(1.005)^5$ at that time. The balance in the account right after the 6^{th} deposit is the sum

$$\text{Balance } = 500 + 500(1.005) + 500(1.005)^2 + \cdots + 500(1.005)^5.$$

We find the sum using the formula for a finite geometric series with $a = 500$, $r = 1.005$, and $n = 6$:

$$\text{Balance right after } 6^{\text{th}} \text{ deposit } = \frac{500(1-(1.005)^6)}{1-1.005} = \$3037.75.$$

Since each deposit is \$500, the balance in the account right before the 6^{th} deposit is $3037.75 - 500 = \$2537.75$.

(b) Similarly, the 12^{th} deposit is made 11 months after the first deposit, so the first deposit has grown to $500(1.005)^{11}$ at that time. The balance in the account right after the 12^{th} deposit is the sum

$$\text{Balance } = 500 + 500(1.005) + 500(1.005)^2 + \cdots + 500(1.005)^{11}.$$

We find the sum using the formula for a finite geometric series with $a = 500$, $r = 1.005$, and $n = 12$:

$$\text{Balance right after } 12^{\text{th}} \text{ deposit } = \frac{500(1-(1.005)^{12})}{1-1.005} = \$6167.78.$$

Since each deposit is \$500, the balance in the account right before the 12^{th} deposit is $6167.78 - 500 = \$5667.78$.

18. Notice that the 20^{th} deposit is made 19 years after the first deposit, so the first deposit has grown to $5000(1.0812)^{19}$ at that time. The balance in the account from all 20 deposits is

$$\text{Balance } = 5000 + 5000(1.0812) + 5000(1.0812)^2 + \cdots + 5000(1.0812)^{19}.$$

We find the sum using the formula for a finite geometric series with $a = 5000$, $r = 1.0812$, and $n = 20$:

$$\text{Balance } = \frac{5000(1-(1.0812)^{20})}{1-1.0812} = \$231{,}874.05.$$

19. The quantity of drug in the body after the n^{th} injection is given by

$$\text{Quantity after } n^{\text{th}} \text{ injection } = 25 + 25(0.4) + 25(0.4)^2 + \cdots + 25(0.4)^{n-1}.$$

This is a finite geometric series with $a = 25$ and $r = 0.4$. Using the formula for the sum, we have

$$\text{Sum } = \frac{25(1-(0.4)^n)}{1-0.4}.$$

(a) We have

$$\text{Quantity after } 3^{\text{rd}} \text{ injection} = 25 + 25(0.4) + 25(0.4)^2 = \frac{25(1-(0.4)^3)}{1-0.4} = 39 \text{ mg}.$$

Notice that we could also have found the sum by adding the three terms.

(b) We have

$$\begin{aligned}\text{Quantity after } 6^{\text{th}} \text{ injection} &= 25 + 25(0.4) + 25(0.4)^2 + \cdots + 25(0.4)^5 \\ &= \frac{25(1-(0.4)^6)}{1-0.4} \\ &= 41.496 \text{ mg.}\end{aligned}$$

Notice that we could also have found the sum by adding the six terms.

(c) We find the quantity of drug in the long run by letting $n \to \infty$. Right after an injection, we have

$$\text{Quantity of drug in the long term } = 25 + 25(0.4) + 25(0.4)^2 + \cdots.$$

This is an infinite geometric series with $a = 25$ and $r = 0.4$. Since $-1 < r < 1$, the series converges and we have

$$\text{Sum } = \frac{25}{1-0.4} = 41.667 \text{ mg.}$$

20. If the person smokes a cigarette at 7 am, at 8 am, at 9 am, and every hour until 11 pm, the person has smoked 17 cigarettes. Sixteen hours have passed since the person smoked the cigarette at 7 am, and we have

$$\text{Quantity of nicotine at 11 pm } = 0.4 + 0.4(0.71) + 0.4(0.71)^2 + \cdots + 0.4(0.71)^{16}.$$

This is a finite geometric series with $a = 0.4$, $r = 0.71$, and $n = 17$. We use the formula for the sum of a finite geometric series:

$$\text{Quantity of nicotine at 11 pm } = \frac{0.4(1-(0.71)^{17})}{1-0.71} = 1.375 \text{ mg.}$$

21. **(a)** The average quantity in the body is $(65+15)/2 = 40$ mg.

(b) The average concentration for this patient (in milligrams of quinine per kilogram of body weight) is (40 mg)/(70 kg) = 0.57 mg/kg. This average concentration falls within the range that is both safe and effective.

(c) (i) Since this treatment produces an average of 40 mg of quinine in the body, a body weight W kg produces an average concentration below 0.4 mg/kg if

$$\frac{40}{W} < 0.4$$

so

$$W > 100.$$

The treatment is not effective for anyone weighing more than 100 kg (or about 220 pounds.)

(ii) A body weight W kg produces an average concentration above 3.0 mg/kg if

$$\frac{40}{W} > 3.0$$

so

$$W < 13.3.$$

The treatment is unsafe for anyone weighing less than 13.3 kg (or about 30 pounds.)

22. **(a)** The quantity of atenolol in the blood is given by $Q(t) = Q_0 e^{-kt}$, where $Q_0 = Q(0)$ and k is a constant. Since the half-life is 6.3 hours,

$$\frac{1}{2} = e^{-6.3k}, \quad k = -\frac{1}{6.3} \ln \frac{1}{2} \approx 0.11.$$

After 24 hours

$$Q = Q_0 e^{-k(24)} \approx Q_0 e^{-0.11(24)} \approx Q_0(0.07).$$

Thus, the percentage of the atenolol that remains after 24 hours $\approx 7\%$.

(b)

$$\begin{aligned}Q_0 &= 50 \\ Q_1 &= 50 + 50(0.07) \\ Q_2 &= 50 + 50(0.07) + 50(0.07)^2 \\ Q_3 &= 50 + 50(0.07) + 50(0.07)^2 + 50(0.07)^3 \\ &\vdots \\ Q_n &= 50 + 50(0.07) + 50(0.07)^2 + \cdots + 50(0.07)^n = \frac{50(1-(0.07)^{n+1})}{1-0.07}\end{aligned}$$

(c)

$$
\begin{aligned}
P_1 &= 50(0.07)\\
P_2 &= 50(0.07) + 50(0.07)^2\\
P_3 &= 50(0.07) + 50(0.07)^2 + 50(0.07)^3\\
P_4 &= 50(0.07) + 50(0.07)^2 + 50(0.07)^3 + 50(0.07)^4\\
&\vdots\\
P_n &= 50(0.07) + 50(0.07)^2 + 50(0.07)^3 + \cdots + 50(0.07)^n\\
&= 50(0.07)\left(1 + (0.07) + (0.07)^2 + \cdots + (0.07)^{n-1}\right) = \frac{0.07(50)(1-(0.07)^n)}{1-0.07}
\end{aligned}
$$

Solutions for Section 11.2

1. The 20^{th} deposit has just been made and has not yet earned any interest, the 19^{th} deposit has earned interest for one year and is worth $1000(1.085)$, the 18^{th} deposit has earned interest for two years and is worth $1000(1.085)^2$, and so on. The first deposit has earned interest for 19 years and is worth $1000(1.085)^{19}$. The total balance in the account right after the 20^{th} deposit is

$$\text{Balance } = 1000 + 1000(1.085) + 1000(1.085)^2 + \cdots + 1000(1.085)^{19}.$$

This is a finite geometric series with $a = 1000$, $r = 1.085$, and $n = 20$. Using the formula for the sum of a finite geometric series, we have

$$\text{Balance after the } 20^{\text{th}} \text{ deposit } = \frac{1000(1-(1.085)^{20})}{1-1.085} = \$48{,}377.01.$$

The twenty annual deposits of \$1000 have contributed a total of \$20,000 to this balance, and the remaining \$28,377.01 in the account comes from the interest earned.

2. Right after the 5^{th} deposit has been made, the 5^{th} deposit has not yet earned any interest, the 4^{th} deposit has earned interest for one year and is worth $2000(e^{0.06})$, the 3^{rd} deposit has earned interest for two years and is worth $2000(e^{0.06})^2$, and so on. The first deposit has earned interest for 4 years and is worth $2000(e^{0.06})^4$. The total balance in the account right after the 5^{th} deposit is

$$\text{Balance } = 2000 + 2000(e^{0.06}) + 2000(e^{0.06})^2 + 2000(e^{0.06})^3 + 2000(e^{0.06})^4.$$

This is a finite geometric series with $a = 2000$, $r = e^{0.06}$, and $n = 5$. Using the formula for the sum of a finite geometric series, we have

$$\text{Balance after the } 5^{\text{th}} \text{ deposit } = \frac{2000(1-(e^{0.06})^5)}{1-e^{0.06}} = \$11{,}315.60.$$

The balance right before the 5^{th} deposit of \$2000 is $11,315.61 - 2000 = \$9{,}315.60$.

3. The first payment of \$1000 is made now, so the present value of the first payment is 1000. The second payment is made one month from now, so the present value of the second payment is $1000(1.005^{-1})$, and so on. The present value of the 36^{th} payment, made 35 months from now, is $1000(1.005^{-1})^{35}$. Summing, we have

$$\text{Present value of the annuity } = 1000 + 1000(1.005^{-1}) + 1000(1.005^{-1})^2 + \cdots + 1000(1.005^{-1})^{35}.$$

This is a finite geometric series with $a = 1000$, $r = 1.005^{-1}$, and $n = 36$. Using the formula for the sum, we have

$$\text{Present value of the annuity } = \frac{1000(1-(1.005^{-1})^{36})}{1-1.005^{-1}} = \$33{,}035.37.$$

4. (a) The present value of the payment made one year from now is $50{,}000(1.072^{-1})$, and the present value of the payment made n years from now is $50{,}000(1.072^{-1})^n$. Since the first payment is made now, the 10^{th} payment is made 9 years from now. Summing, we have

$$\text{Present value } = 50000 + 50000(1.072^{-1}) + 50000(1.072^{-1})^2 + \cdots + 50000(1.072^{-1})^9.$$

This is a finite geometric series with $a = 50000$, $r = 1.072^{-1}$, and $n = 10$. Using the formula for the sum, we have

$$\text{Present value of the annuity } = \frac{50000(1-(1.072^{-1})^{10})}{1-1.072^{-1}} = \$373{,}008.06.$$

(b) If the payments are to continue indefinitely, the present value is the sum:

$$\text{Present value } = 50000 + 50000(1.072^{-1}) + 50000(1.072^{-1})^2 + 50000(1.072^{-1})^3 + \cdots.$$

This is an infinite geometric series with $a = 50000$ and $r = 1.072^{-1} = 0.9328358$. Since $-1 < r < 1$, this series converges and we use the formula for the sum of an infinite geometric series:

$$\text{Present value of the annuity } = \frac{50000}{1-1.072^{-1}} = \$744{,}444.44.$$

5. The amount that must be deposited now is the present value of this annuity. The present value of the first payment (made one year from now) is $5000(1.10^{-1})$, and the present value of the 20^{th} payment (made 20 years from now) is $5000(1.10^{-1})^{20}$. The total is

$$\text{Present value } = 5000(1.10^{-1}) + 5000(1.10^{-1})^2 + 5000(1.10^{-1})^3 + \cdots + 5000(1.10^{-1})^{20}.$$

Notice that the first payment is not made until a year from now so the first term of the series is $5000(1.10^{-1})$ rather than 5000. Since $5000(1.10^{-1})^n = 5000(1.10^{-1})(1.10^{-1})^{n-1}$, we can rewrite the series as

$$\text{Present value } = 5000(1.10^{-1}) + 5000(1.10^{-1})(1.10^{-1}) + 5000(1.10^{-1})(1.10^{-1})^2 + \cdots + 5000(1.10^{-1})(1.10^{-1})^{19}.$$

This is a finite geometric series with $a = 5000(1.10^{-1})$, $r = 1.10^{-1}$, and $n = 20$. Using the formula for the sum of a finite geometric series, we have

$$\text{Present value of the annuity } = \frac{5000(1.10^{-1})(1-(1.10^{-1})^{20})}{1-1.10^{-1}} = \$42{,}567.82.$$

6. We sum the geometric series

$$\begin{aligned}\text{Present value } &= 20{,}000 + \frac{20{,}000}{1.01} + \frac{20{,}000}{1.01^2} + \cdots \\ &= \frac{20{,}000}{1-1/1.01} = 2{,}020{,}000.\end{aligned}$$

Thus, the present value is 2.02 million dollars.

7. Since the account is paying out more than it is earning in interest, the balance will decrease and the account will eventually run out of money. When will this happen? If n payments are made, the present value of the payments is given by

$$\text{Present value } = 10000 + 10000(1.08^{-1}) + 10000(1.08^{-1})^2 + \cdots + 10000(1.08^{-1})^{n-1}.$$

We use the fact that the present value is \$100,000 and the formula for the sum of a finite geometric series with $a = 10000$ and $r = 1.08^{-1}$:

$$\text{Present value } = 100{,}000 = \frac{10000(1-(1.08^{-1})^n)}{1-1.08^{-1}}.$$

We simplify and then use logarithms to solve for n:

$$\begin{aligned}100000 &= \frac{10000(1-(1.08^{-1})^n)}{1-1.08^{-1}} \\ 10 &= \frac{1-(1.08^{-1})^n}{1-1.08^{-1}}\end{aligned}$$

$$\begin{aligned}10 &= \frac{1-(1.08^{-1})^n}{0.074074}\\ 0.74074 &= 1-(1.08^{-1})^n\\ 0.25926 &= (1.08^{-1})^n\\ 0.25926 &= (1.08)^{-n}\\ \ln(0.25926) &= -n\ln(1.08)\\ n &= -\frac{\ln(0.25926)}{\ln(1.08)} = 17.54.\end{aligned}$$

The account will be able to make about seventeen and a half payments before it runs out of money.

8. You earn 1 cent the first day, 2 cents the second day, $2^2 = 4$ the third day, $2^3 = 8$ the fourth day, and so on. On the n^{th} day, you earn 2^{n-1} cents. We have

$$\text{Total earnings for } n \text{ days } = 1 + 2 + 2^2 + 2^3 + \cdots + 2^{n-1}.$$

This is a finite geometric series with $a = 1$ and $r = 2$. We use the formula for the sum:

$$\text{Total earnings for } n \text{ days } = \frac{1-2^n}{1-2} = 2^n - 1.$$

(a) Using $n = 7$, we see that

$$\text{Total earnings for 7 days } = 2^7 - 1 = 127 = \$1.27.$$

(b) Using $n = 14$, we see that

$$\text{Total earnings for 14 days } = 2^{14} - 1 = 16383 = \$163.83.$$

(c) Using $n = 21$, we see that

$$\text{Total earnings for 21 days } = 2^{21} - 1 = 2097151 = \$20{,}971.51.$$

(d) Using $n = 28$, we see that

$$\text{Total earnings for 28 days } = 2^{28} - 1 = 268435455 = \$2{,}684{,}354.55.$$

9. Her salary is \$30,000 at the start of the first year, \$30,000(1.04) at the start of the second year, \$30,000$(1.04)^2$ at the start of the third year, and so on. At the start of the 11^{th} year, her salary is \$30,000$(1.04)^{10}$ $=$ \$44,407.33.
The employee's total earnings for the first ten years is the sum

$$\text{Total earnings } = 30000 + 30000(1.04) + 30000(1.04)^2 + \cdots + 30000(1.04)^9.$$

This is a finite geometric series with $a = 30000$, $r = 1.04$, and $n = 10$. We use the formula for the sum:

$$\text{Total earnings } = \frac{30000(1-(1.04)^{10})}{1-1.04} = \$360{,}183.21.$$

10. In any given year, there are 10,000 units in use that were manufactured that year, and 10,000(0.75) units in use that were manufactured the previous year (since 25% of them failed), and 10,000$(0.75)^2$ units in use that were manufactured two years ago, and so on. We have

$$\text{Total number of units in use } = 10{,}000 + 10{,}000(0.75) + 10{,}000(0.75)^2 + 10{,}000(0.75)^3 + \cdots.$$

This is an infinite geometric series with $a = 10{,}000$ and $r = 0.75$. Since $-1 < r < 1$, the series converges and we have

$$\text{Total number of units in use } = \frac{10{,}000}{1-0.75} = 40{,}000.$$

The market stabilization point for this product is 40,000 units.

11. Let k be the fraction of the \$1 bills in circulation that are removed each day. Then $r = 1 - k$ is the fraction of bills remaining in circulation the next day. On any day, there are 11 million new bills, and $11r$ million remaining from the day before, and $11r^2$ million remaining from the day before, and so on. In millions, the total number of \$1 bills in circulation is given by

$$N = 11 + 11r + 11r^2 + \cdots.$$

This is an infinite geometric series with $a = 11$ and $-1 < r < 1$ (since the fraction of bills remaining is positive and less than 1). The series converges, and its sum is given by

$$N = \frac{a}{1-r} = \frac{11}{1-r}.$$

Since we know the total number of \$1 bills in circulation is 9.2 billion, or 9200 million, we have

$$9200 = \frac{11}{1-r}.$$

Solving for r gives

$$\begin{aligned} 1 - r &= \frac{11}{9200} \\ r &= 1 - \frac{11}{9200} = 0.9988. \end{aligned}$$

Thus 99.88% of the notes remain in circulation the next day, so 0.12% are removed each day.

Notice that if 11 million new bills are introduced each day, then at steady state, the same number are removed each day. Thus, we can calculate r without using a series:

$$r = \frac{\text{11 million}}{\text{9.2 billion}} = 0.12\%.$$

12.

$$\begin{aligned} \text{Present value of first coupon} &= \frac{50}{1.06} \\ \text{Present value of second coupon} &= \frac{50}{(1.06)^2}, \text{ etc.} \end{aligned}$$

$$\begin{aligned} \text{Total present value} &= \underbrace{\frac{50}{1.06} + \frac{50}{(1.06)^2} + \cdots + \frac{50}{(1.06)^{10}}}_{\text{coupons}} + \underbrace{\frac{1000}{(1.06)^{10}}}_{\text{principal}} \\ &= \frac{50}{1.06}\left(1 + \frac{1}{1.06} + \cdots + \frac{1}{(1.06)^9}\right) + \frac{1000}{(1.06)^{10}} \\ &= \frac{50}{1.06}\left(\frac{1 - \left(\frac{1}{1.06}\right)^{10}}{1 - \frac{1}{1.06}}\right) + \frac{1000}{(1.06)^{10}} \\ &= 368.004 + 558.395 \\ &= \$926.40 \end{aligned}$$

13.

$$\begin{aligned} \text{Present value of first coupon} &= \frac{50}{1.04} \\ \text{Present value of second coupon} &= \frac{50}{(1.04)^2}, \text{etc.} \end{aligned}$$

$$\text{Total present value} = \underbrace{\frac{50}{1.04} + \frac{50}{(1.04)^2} + \cdots + \frac{50}{(1.04)^{10}}}_{\text{coupons}} + \underbrace{\frac{1000}{(1.04)^{10}}}_{\text{principal}}$$

$$= \frac{50}{1.04}\left(1 + \frac{1}{1.04} + \cdots + \frac{1}{(1.04)^9}\right) + \frac{1000}{(1.04)^{10}}$$
$$= \frac{50}{1.04}\left(\frac{1 - \left(\frac{1}{1.04}\right)^{10}}{1 - \frac{1}{1.04}}\right) + \frac{1000}{(1.04)^{10}}$$
$$= 405.545 + 675.564$$
$$= \$1081.11$$

14. (a)

$$\text{Present value of first coupon} = \frac{50}{1.05}$$
$$\text{Present value of second coupon} = \frac{50}{(1.05)^2}, \text{etc.}$$

$$\text{Total present value} = \underbrace{\frac{50}{1.05} + \frac{50}{(1.05)^2} + \cdots + \frac{50}{(1.05)^{10}}}_{\text{coupons}} + \underbrace{\frac{1000}{(1.05)^{10}}}_{\text{principal}}$$
$$= \frac{50}{1.05}\left(1 + \frac{1}{1.05} + \cdots + \frac{1}{(1.05)^9}\right) + \frac{1000}{(1.05)^{10}}$$
$$= \frac{50}{1.05}\left(\frac{1 - \left(\frac{1}{1.05}\right)^{10}}{1 - \frac{1}{1.05}}\right) + \frac{1000}{(1.05)^{10}}$$
$$= 386.087 + 613.913$$
$$= \$1000$$

(b) When the interest rate is 5%, the present value equals the principal.

(c) When the interest rate is more than 5%, the present value is smaller than it is when interest is 5% and must therefore be less than the principal. Since the bond will sell for around its present value, it will sell for less than the principal; hence the description *trading at discount*.

(d) When the interest rate is less than 5%, the present value is more than the principal. Hence the bond will be selling for more than the principal, and is described as *trading at a premium*.

15. (a)

$$\text{Total amount of money deposited} = 100 + 92 + 84.64 + \cdots$$
$$= 100 + 100(0.92) + 100(0.92)^2 + \cdots$$
$$= \frac{100}{1 - 0.92} = 1250 \quad \text{dollars}$$

(b) Credit multiplier $= 1250/100 = 12.50$
The 12.50 is the factor by which the bank has increased its deposits, from \$100 to \$1250.

16. (a) The people who receive the 40 billion dollars in tax rebates spend 80% of it, for an initial amount spent of $40(0.80) =$ 32 billion dollars. The people who receive this 32 billion dollars spend 80% of it, for an additional amount spent of 32(0.80), and so on. We have

$$\text{Total amount spent} = 32 + 32(0.80) + 32(0.80)^2 + 32(0.80)^3 + \cdots.$$

Notice that the initial amount spent is not the original tax rebate of 40 billion dollars, but 80% of 40 billion dollars, or 32 billion dollars. The total amount spent is an infinite geometric series with $a = 32$ and $r = 0.80$. Since $-1 < r < 1$, the series converges and we have

$$\text{Total amount spent} = \frac{32}{1 - 0.80} = 160 \text{ billion dollars.}$$

(b) If everyone who receives money spends 90% of it, then the initial amount spent is $40(0.90) = 36$ billion dollars. The people who receive this 36 billion dollars spend 90% of it, and so on. We have

$$\text{Total amount spent} = 36 + 36(0.90) + 36(0.90)^2 + 36(0.90)^3 + \cdots.$$

This is an infinite geometric series with $a = 36$ and $r = 0.90$. Since $-1 < r < 1$, the series converges and we have

$$\text{Total amount spent} = \frac{36}{1-0.90} = 360 \text{ billion dollars.}$$

Notice that an increase in the spending rate from 80% to 90% causes a dramatic increase in the total effect on spending.

17. **(a)** The people who receive the 100 billion dollars in tax rebates spend 80% of it, for an initial amount spent of $100(0.80) = 80$ billion dollars. The people who receive this 80 billion dollars spend 80% of it, for an additional amount spent of 80(0.80), and so on. We have

$$\text{Total amount spent} = 80 + 80(0.80) + 80(0.80)^2 + 80(0.80)^3 + \cdots.$$

Notice that the initial amount spent is not the original tax rebate of 100 billion dollars, but 80% of 100 billion dollars, or 80 billion dollars. The total amount spent is an infinite geometric series with $a = 80$ and $r = 0.80$. Since $-1 < r < 1$, the series converges and we have

$$\text{Total amount spent} = \frac{80}{1-0.80} = 400 \text{ billion dollars.}$$

(b) If everyone who receives money spends 90% of it, then the initial amount spent is $100(0.90) = 90$ billion dollars. The people who receive this 90 billion dollars spend 90% of it, and so on. We have

$$\text{Total amount spent} = 90 + 90(0.90) + 90(0.90)^2 + 90(0.90)^3 + \cdots.$$

This is an infinite geometric series with $a = 90$ and $r = 0.90$. Since $-1 < r < 1$, the series converges and we have

$$\text{Total amount spent} = \frac{90}{1-0.90} = 900 \text{ billion dollars.}$$

Notice that an increase in the spending rate from 80% to 90% causes a dramatic increase in the total effect on spending.

18. If we assume that everyone spends a proportion r of what they get, then the person who gets the dollar spends r dollars and the person who gets that money spends $r \cdot r = r^2$ dollars, and so on. Thus

$$\text{Total additional spending} = r + r^2 + r^3 + \cdots.$$

This is an infinite geometric series with first term $a = r$ and constant multiplier r. Using the formula for the sum of the series and the fact that the total additional spending from the \$1 tax cut is \$3, we have:

$$\begin{aligned} \frac{a}{1-r} &= 3 \\ \frac{r}{1-r} &= 3 \\ r &= 3(1-r) \\ 4r &= 3 \\ r &= 0.75. \end{aligned}$$

The financial advisers are assuming that, on average, people spend 75% of what they receive.

19. **(a)** The first person deposits N dollars, and the second person deposits $(1-r)N$ dollars. After the second deposit, the total value of the two deposits is $N + (1-r)N$ dollars.

(b) The third person deposits $1-r$ times the amount of the second deposit, that is, $(1-r)^2 N$ dollars. The total value of the first three deposits is $N + (1-r)N + (1-r)^2 N$ dollars.

(c) Each person deposits $1-r$ times the amount of the preceding deposit. If the process continues forever, the total value of all deposits is the geometric series

$$\text{Total value} = N + (1-r)N + (1-r)^2 N + (1-r)^3 N + \cdots.$$

Since $0 < r < 1$, we have $0 < 1-r < 1$, so the series converges. We have

$$\text{Total value of all deposits} = \frac{N}{1-(1-r)} = \frac{N}{r} \text{ dollars.}$$

Solutions for Section 11.3

1. In 2008, the total quantity of oil consumed was 29.3 billion barrels; each subsequent year, the quantity is multiplied by 0.945. Thus, in 2009, we have $29.3(0.945) = 27.7$ billion barrels; in 2010 we have $29.3(0.945)^2 = 26.2$ billion barrels; and so on.

Year	2008	2009	2010	2011	2012	2013	2014	2015	2016	2017
Oil	29.3	27.7	26.2	24.7	23.4	22.1	20.9	19.7	18.6	17.6

In 2017, the total quantity of oil consumed expected to be $29.3(0.945)^9$. Thus, we sum the geometric series:

$$\begin{aligned}\text{Total consumption over decade} &= 29.3 + 29.3(0.945) + 29.3(0.945)^2 + \cdots + 29.3(0.945)^9 \\ &= \frac{29.3(1-(0.945)^{10})}{1-0.945} = 230.159 \text{ billion barrels.}\end{aligned}$$

2. If consumption changes by a factor of r each year (so we use $r = 0.945$ in part (a) and $r = 1.025$ in part (b)), then consumption in 2009 is predicted to be $29.3r$ billion barrels, in 2010 it is predicted to be $29.3r^2$ billion barrels, in 2011 it is predicted to be $29.3r^3$ billion barrels, and so on. Thus, if Q_n is total consumption in n years, in billions of barrels,

$$Q_n = 29.3r + 29.3r^2 + 29.3r^3 + \cdot + 29.3r^n.$$

Using the formula for the sum of a finite geometric series, we have

$$Q_n = 29.3r\frac{1-r^n}{1-r}.$$

Since the total reserves are 1950 billion barrels, in each case we try to find n making $Q_n = 1950$.

(a) Using $r = 0.945$, we try to solve

$$\begin{aligned}29.3(0.945)\frac{1-(0.945)^n}{1-0.945} &= 1950\\ 504.43(1-(0.945)^n) &= 1950\\ 1-(0.945)^n &= \frac{1950}{504.43} = 3.87\\ (0.945)^n &= -2.87.\end{aligned}$$

This equation has no solution. This tells us that if consumption continues to decrease at 5% a year, the oil supply never runs out.

We can also observe that since $r = 0.945 < 1$, the infinite series

$$29.3(0.945) + 29.3(0.945)^2 + 29.3(0.945)^3 + \cdots$$

converges to

$$29.3(0.945)\frac{1}{1-0.945} = 503.43.$$

Since 503.43 is less than 1950, the reserves, the oil does not run out under this scenario.

(b) Using $r = 1.025$, we have

$$\begin{aligned}29.3(1.025)\frac{1-(1.025)^n}{1-1.025} &= 1950\\ 1201((1.025)^n - 1) &= 1950\\ (1.025)^n - 1 &= \frac{1950}{1201} = 1.6236\\ (1.025)^n &= 2.6236\\ n\ln(1.025) &= \ln(2.6236)\\ n &= \frac{\ln(2.6236)}{\ln(1.025)} = 39.1 \text{ years.}\end{aligned}$$

3. After receiving the n^{th} injection, the quantity in the body is 50 mg from the n^{th} injection, $50(0.60)$ from the injection the previous day, $50(0.60)^2$ from the injection two days before, and so on. The quantity remaining from the first injection (which has been in the body for $n-1$ days) is $50(0.60)^{n-1}$. We have

$$\text{Quantity after } n^{\text{th}} \text{ injection } = 50 + 50(0.60) + 50(0.60)^2 + \cdots + 50(0.60)^{n-1}.$$

(a) The quantity of drug in the body after the 3^{rd} injection is

$$\text{Quantity after } 3^{\text{rd}} \text{ injection} = 50 + 50(0.60) + 50(0.60)^2 = 98 \text{ mg}.$$

Alternately, we could find the sum using the formula for a finite geometric series with $a = 50$, $r = 0.60$, and $n = 3$:

$$\text{Quantity after } 3^{\text{rd}} \text{ injection } = \frac{50(1-(0.60)^3)}{1-0.60} = 98 \text{ mg}.$$

(b) Similarly, we have

$$\text{Quantity after } 7^{\text{th}} \text{ injection} = 50 + 50(0.60) + 50(0.60)^2 + \cdots + 50(0.60)^6.$$

We use the formula for the sum of a finite geometric series with $a = 50$, $r = 0.60$, and $n = 7$:

$$\text{Quantity after } 7^{\text{th}} \text{ injection} = \frac{50(1-(0.60)^7)}{1-0.60} = 121.5 \text{ mg}.$$

(c) The steady state quantity is the long-run quantity of drug in the body if injections are continued indefinitely. In the long run,

$$\text{Quantity right after injection} = 50 + 50(0.60) + 50(0.60)^2 + \cdots.$$

This is an infinite geometric series with $a = 50$ and $r = 0.60$. Since $-1 < r < 1$, the series converges. Its sum is

$$\text{Quantity right after injection} = \frac{50}{1-0.60} = 125 \text{ mg}.$$

4. Since the quantity of ampicillin excreted during the time interval between tablets is 200 mg, we have

$$\begin{aligned} \text{Quantity of ampicillin excreted} &= \text{Original quantity} - \text{Final quantity} \\ 200 &= Q - (0.12)Q. \end{aligned}$$

Solving for Q gives, as before,

$$Q = \frac{200}{1-0.12} \approx 227.27 \text{ mg}.$$

5. **(a)** The steady state quantity is the quantity of drug in the body if tablets are taken daily for the long run. Right after a tablet is taken, in the long run we have

$$\text{Quantity after tablet } = 120 + 120(0.70) + 120(0.70)^2 + \cdots.$$

This is an infinite geometric series with $a = 120$ and $r = 0.70$. Since $-1 < r < 1$, the series converges and we use the formula for the sum of an infinite geometric series: In the long run,

$$\text{Quantity right after tablet } = \frac{120}{1-0.70} = 400 \text{ mg}.$$

(b) Right after a tablet is taken, at the steady state there are 400 mg of the drug in the body. In one day, 30% of this quantity, or $400(0.30) = 120$ mg, is excreted. This is equal to the quantity that is ingested in one tablet.

6. **(a)** After a single dose of 50 mg of fluoxetine, the quantity, Q, in the body decays exponentially, so $Q = 50b^t$. After 3 days, half the original dose remains. We use this information to solve for b:

$$\begin{aligned} 25 &= 50b^3 \\ 0.5 &= b^3 \\ b &= (0.5)^{1/3} = 0.7937. \end{aligned}$$

The fraction of a dose remaining after one day is 0.7937.

(b) The quantity of fluoxetine remaining right after taking the 7^{th} dose is the sum of a finite geometric series with $a = 50$, $r = 0.7937$, and $n = 7$.

$$\begin{aligned}\text{Quantity after } 7^{\text{th}} \text{ dose} &= 50 + 50(0.7937) + 50(0.7937)^2 + \cdots + 50(0.7937)^6\\ &= \frac{50(1-(0.7937)^7)}{1-0.7937} = 194.27 \text{ mg.}\end{aligned}$$

(c) In the long run, the quantity of fluoxetine in the body right after taking a dose is the sum of an infinite geometric series with $a = 50$ and $r = 0.7937$. Since $-1 < r < 1$, the series converges. Its sum is

$$\begin{aligned}\text{Quantity after dose} &= 50 + 50(0.7937) + 50(0.7937)^2 + \cdots.\\ &= \frac{50}{1-0.7937} = 242.37 \text{ mg.}\end{aligned}$$

7. (a) The quantity of morphine in the body right after taking the 6^{th} tablet is

$$\text{Quantity after } 6^{\text{th}} \text{ tablet} = 30 + 30(0.25) + 30(0.25)^2 + 30(0.25)^3 + 30(0.25)^4 + 30(0.25)^5.$$

This is a finite geometric series with $a = 30$, $r = 0.25$, and $n = 6$. We have

$$\text{Quantity after } 6^{\text{th}} \text{ tablet} = \frac{30(1-(0.25)^6)}{1-0.25} = 39.99 \text{ mg.}$$

Since the 6^{th} tablet contributes 30 mg toward this total, the quantity of morphine in the body right before the 6^{th} tablet is $39.99 - 30 = 9.99$ mg.

(b) The steady state quantity is the long-run quantity of drug in the body if the tablets are taken every 4 hours indefinitely. In the long run, we have

$$\text{Quantity after tablet} = 30 + 30(0.25) + 30(0.25)^2 + \cdots.$$

This is an infinite geometric series with $a = 30$ and $r = 0.25$. Since $-1 < r < 1$, the series converges and we use the formula for the sum of an infinite geometric series:

$$\text{Quantity after tablet} = \frac{30}{1-0.25} = 40 \text{ mg.}$$

Right before a tablet is taken, the quantity in the body is 30 mg less, so

$$\text{Quantity before tablet} = 40 - 30 = 10 \text{ mg.}$$

8. (a) We use the half-life to find the fraction of the drug remaining after one week. After a single dose of 100 mg of the drug, the quantity, Q, in the body decays exponentially so $Q = 100b^t$. We use the fact that the half-life is 18 weeks to solve for b:

$$\begin{aligned}50 &= 100b^{18}\\ 0.5 &= b^{18}\\ b &= (0.5)^{1/18}.\end{aligned}$$

The fraction remaining after one week is $(0.5)^{1/18}$, or 0.9622238.

At the steady state, the quantity right after a dose is the sum of an infinite geometric series with $a = 100$ and $r = (0.5)^{1/18}$. Since $-1 < r = (0.5)^{1/18} = 0.9622238 < 1$, the series converges and we use the formula for the sum of an infinite geometric series:

$$\begin{aligned}\text{Long-run quantity} &= 100 + 100((0.5)^{1/18}) + 100((0.5)^{1/18})^2 + \cdots\\ &= \frac{100}{1-(0.5)^{1/18}}\\ &= 2647.17 \text{ mg.}\end{aligned}$$

(b) Since at the steady state, the quantity is 2647.17 mg, the quantity of the drug in the body eventually passes 2000 mg. How many weeks does this take? The quantity of drug in the body after the n^{th} dose (at the start of the n^{th} week) is a finite geometric series with $a = 100$ and $r = (0.5)^{1/18}$. We have

$$\begin{aligned}\text{Quantity after dose in } n^{\text{th}} \text{ week } &= 100 + 100((0.5)^{1/18}) + 100((0.5)^{1/18})^2 + \cdots + 100((0.5)^{1/18})^{n-1} \\ &= \frac{100(1-((0.5)^{1/18})^n)}{1-(0.5)^{1/18}}.\end{aligned}$$

We want to find the value of n for which this quantity is 2000. We simplify and use logarithms to solve for n.

$$\begin{aligned}2000 &= \frac{100(1-((0.5)^{1/18})^n)}{1-(0.5)^{1/18}} \\ 20 &= \frac{1-((0.5)^{1/18})^n}{1-(0.5)^{1/18}} \\ 0.755523 &= 1-((0.5)^{1/18})^n \\ 0.244477 &= (0.5)^{n/18} \\ \ln(0.244477) &= \frac{n}{18}\ln(0.5) \\ n &= \frac{18\ln(0.244477)}{\ln(0.5)} = 36.58 \text{ weeks.}\end{aligned}$$

The drug first becomes effective in the 37^{th} week.

9. Since nicotine leaves the body at a continuous rate of 34.65%, the fraction remaining after T hours is $e^{-0.3465T}$.

(a) The fraction of nicotine remaining after one hour is $e^{-0.3465} = 0.707$. The steady state is the sum of the infinite geometric series with $a = 1.2$ and $r = 0.707$. Since $-1 < r < 1$, the series converges and we have

$$\text{Long-run quantity after cigarette } = 1.2 + 1.2(e^{-0.3465}) + 1.2(e^{-0.3465})^2 + \cdots = \frac{1.2}{1-e^{-0.3465}} = 4.10 \text{ mg.}$$

(b) The fraction of nicotine remaining after 0.5 hour is $e^{-0.3465(0.5)} = e^{-0.17325} = 0.841$. The steady state is the sum of the infinite geometric series with $a = 1.2$ and $r = 0.841$. Since $-1 < r < 1$, the series converges and we have

$$\text{Long-run quantity after cigarette } = 1.2 + 1.2(e^{-0.17325}) + 1.2(e^{-0.17325})^2 + \cdots = \frac{1.2}{1-e^{-0.17325}} = 7.54 \text{ mg.}$$

(c) Since 15 minutes = 0.25 hour, the fraction of nicotine remaining after 15 minutes is $e^{-0.3465(0.25)} = e^{-0.086625} = 0.917$. The steady state is the sum of the infinite geometric series with $a = 1.2$ and $r = 0.917$. Since $-1 < r < 1$, the series converges and we have

$$\text{Long-run quantity after cigarette } = 1.2 + 1.2(e^{-0.086625}) + 1.2(e^{-0.086625})^2 + \cdots = \frac{1.2}{1-e^{-0.086625}} = 14.46 \text{ mg.}$$

(d) Since 6 minutes = 0.1 hour, the fraction of nicotine remaining after 6 minutes is $e^{-0.3465(0.1)} = e^{-0.03465} = 0.966$. The steady state is the sum of the infinite geometric series with $a = 1.2$ and $r = 0.966$. Since $-1 < r < 1$, the series converges and we have

$$\text{Long-run quantity after cigarette } = 1.2 + 1.2(e^{-0.03465}) + 1.2(e^{-0.03465})^2 + \cdots = \frac{1.2}{1-e^{-0.03465}} = 35.24 \text{ mg.}$$

(e) Since 3 minutes = 0.05 hour, the fraction of nicotine remaining after 3 minutes is $e^{-0.3465(0.05)} = e^{-0.017325} = 0.983$. The steady state is the sum of the infinite geometric series with $a = 1.2$ and $r = 0.983$. Since $-1 < r < 1$, the series converges and we have

$$\text{Long-run quantity after cigarette } = 1.2 + 1.2(e^{-0.017325}) + 1.2(e^{-0.017325})^2 + \cdots = \frac{1.2}{1-e^{-0.017325}} = 69.87 \text{ mg.}$$

Notice that under these circumstances (smoking a high-nicotine cigarette every 3 minutes), the quantity of nicotine in the body does reach the lethal level. In fact, the lethal level is reached in less than 6 hours.

10. Since the toxin is metabolized at a continuous rate of 0.5% per day, the quantity remaining one day after consuming a single quantity of 8 micrograms is $8e^{-0.005}$. The person is consuming 8 micrograms every day, so the total accumulated toxin the person has in the body right after consuming the toxin is the sum of 8 (from the quantity just consumed) + $8(e^{-0.005})$ (from the quantity consumed the previous day) + $8(e^{-0.005})^2$ (from the quantity consumed two days ago), and so on. The total accumulation in the body is the sum of an infinite geometric series with $a = 8$ and $r = e^{-0.005}$. Since $-1 < r = 0.9950124 < 1$, the series converges to the sum:

$$\begin{aligned}\text{Total accumulation right after lunch } &= 8 + 8(e^{-0.005}) + 8(e^{-0.005})^2 + \cdots \\ &= \frac{8}{1 - e^{-0.005}} = 1604.0 \text{ micrograms.}\end{aligned}$$

Since the person consumes 8 micrograms each day, the total accumulation of the toxin right before lunch is $1604 - 8 = 1596$ micrograms.

11. If Q_n denotes the total amount of the mineral mined and used in the first n years after 2008 (that is, 2009, 2010, $\ldots$), we have

$$Q_n = 5000 + 5000(1.08) + 5000(1.08)^2 + \cdots + 5000(1.08)^{n-1}.$$

This is a finite geometric series with $a = 5000$ and $r = 1.08$, so we have

$$Q_n = \frac{a(1-r^n)}{1-r} = \frac{5000(1-(1.08)^n)}{1-1.08} = 62500((1.08)^n - 1).$$

We want the value of n for which the total consumption, Q_n, reaches 350,000 m^3, the total amount available. We can estimate n numerically (using trial and error) or graphically, or we can find n analytically:

$$\begin{aligned}62{,}500((1.08)^n - 1) &= 350{,}000 \\ (1.08)^n - 1 &= 5.6 \\ (1.08)^n &= 6.6.\end{aligned}$$

Taking logarithms and using $\ln(A^p) = p \ln A$, we have

$$\begin{aligned}n \ln(1.08) &= \ln(6.6) \\ n &= \frac{\ln(6.6)}{\ln(1.08)} = 24.5 \text{ years.}\end{aligned}$$

Thus, if consumption of this mineral continues to increase at 8% a year, all reserves will be exhausted in 25 years. If, however, the relative rate of increase changes, the length of time until the reserve runs out may be very different.

12. Consumption of the mineral this year is 1500 kg, consumption next year is predicted to be $1500(1.04)$, consumption the following year is predicted to be $1500(1.04)^2$ and so on. Total consumption during the next n years is given by

$$\text{Total consumption for } n \text{ years } = 1500 + 1500(1.04) + 1500(1.04)^2 + \cdots + 1500(1.04)^{n-1}.$$

This is a finite geometric series with $a = 1500$ and $r = 1.04$. We have

$$\text{Total consumption for } n \text{ years } = \frac{1500(1-(1.04)^n)}{1-1.04}.$$

We wish to find the value of n making total consumption equal to $120{,}000$.

$$\begin{aligned}120{,}000 &= \frac{1500(1-(1.04)^n)}{1-1.04} \\ 80 &= \frac{1-(1.04)^n}{1-1.04} \\ -3.2 &= 1 - (1.04)^n \\ 4.2 &= (1.04)^n \\ \ln(4.2) &= n \ln(1.04) \\ n &= \frac{\ln(4.2)}{\ln(1.04)} = 36.59 \text{ years.}\end{aligned}$$

Assuming the percent rate of increase remains constant, the reserves of this mineral will run out in 36 or 37 years.

13. Assuming consumption increases by 2% per year, consumption in 2008 was expected to be $3(1.02)$ trillion m^3. In 2009, we expect $3(1.02)^2$ to be consumed, and in 2010, we expect $3(1.02)^3$, and so on. In trillion m^3, Q_n, the total consumption in n years after 2007, is given by

$$Q_n = 3(1.02) + 3(1.02)^2 + 3(1.02)^3 + \cdots + 3(1.02)^n.$$

Using the formula for the sum of a finite geometric series, we have

$$Q_n = 3(1.02)\frac{1-(1.02)^n}{1-1.02} = 153((1.02)^n - 1).$$

To find when reserves will be exhausted, we set $Q_n = 180$ trillion.

$$\begin{aligned} 153((1.02)^n - 1) &= 180 \\ (1.02)^n - 1 &= \frac{180}{153} = 1.176 \\ (1.02)^n &= 2.176 \\ n\ln(1.02) &= \ln(2.176) \\ n &= \frac{\ln(2.176)}{\ln(1.02)} = 39.3 \text{ years.} \end{aligned}$$

Thus, natural gas reserves are predicted to be exhausted in $2007 + 39 = 2046$.

14. **(a)** In this case, consumption remains 3 trillion m^3 per year. Since reserves are 180 trillion m^3, the reserves are exhausted in n years, where

$$\begin{aligned} 3n &= 180 \\ n &= \frac{180}{3} = 60 \text{ years.} \end{aligned}$$

(b) Assuming consumption increases each year by a factor of 1.05, consumption in 2008 is predicted to be $3(1.05)$ trillion m^3; in 2009, it is predicted to be $3(1.05)^2$; and in 2010, it is predicted to be $3(1.05)^3$, and so on. Representing total usage in n years by Q_n trillion m^3, we have

$$Q_n = 3(1.05) + 3(1.05)^2 + 3(1.05)^3 + \cdots + 3(1.05)^n.$$

Using the formula for the sum of a finite geometric series, we have

$$Q_n = 3(1.05)\frac{1-(1.05)^n}{1-1.05} = 63((1.05)^n - 1).$$

To find how long reserves will last, we set $Q_n = 180$ and solve for n:

$$\begin{aligned} 63((1.05)^n - 1) &= 180 \\ (1.05)^n - 1 &= \frac{180}{63} = 2.857 \\ (1.05)^n &= 3.857 \\ n\ln(1.05) &= \ln(3.857) \\ n &= \frac{\ln(3.857)}{\ln(1.05)} = 27.7 \text{ years.} \end{aligned}$$

15. Consumption of the mineral this year is 5000 m^3, consumption next year is predicted to be $5000(1.04)$, consumption the following year is predicted to be $5000(1.04)^2$ and so on. Total consumption during the next n years is given by

$$\text{Total consumption for } n \text{ years } = 5000 + 5000(1.04) + 5000(1.04)^2 + \cdots + 5000(1.04)^{n-1}.$$

This is a finite geometric series with $a = 5000$ and $r = 1.04$. We have

$$\text{Total consumption for } n \text{ years } = \frac{5000(1-(1.04)^n)}{1-1.04}.$$

We wish to find the value of n making total consumption equal to 350,000.

$$\begin{aligned}
350{,}000 &= \frac{5000(1-(1.04)^n)}{1-1.04}\\
70 &= \frac{1-(1.04)^n}{1-1.04}\\
-2.8 &= 1-(1.04)^n\\
3.8 &= (1.04)^n\\
\ln(3.8) &= n\ln(1.04)\\
n &= \frac{\ln(3.8)}{\ln(1.04)} = 34.04 \text{ years.}
\end{aligned}$$

If the percent rate of increase stays constant at 4% per year, the total reserve of this mineral will run out in about 34 years.

16. If the mineral is used at a constant rate of 5000 m^3 per year, the total reserves of 350,000 m^3 will be used up in

$$\frac{350{,}000}{5000} = 70 \text{ years.}$$

17. If usage decreases by 4% each year, then consumption of the mineral this year is 5000 m^3, consumption next year is predicted to be $5000(0.96)$, consumption the following year is predicted to be $5000(0.96)^2$ and so on. Total consumption during the next n years is given by

$$\text{Total consumption for } n \text{ years } = 5000 + 5000(0.96) + 5000(0.96)^2 + \cdots + 5000(0.96)^{n-1}.$$

This is a finite geometric series with $a = 5000$ and $r = 0.96$. We have

$$\text{Total consumption for } n \text{ years } = \frac{5000(1-(0.96)^n)}{1-0.96}.$$

If we try to find the value of n making total consumption equal to 350,000, we see that there are no such values of n. Why? If we consider consumption of this mineral forever under these circumstances, we have the infinite geometric series:

$$\text{Total consumption forever } = 5000 + 5000(0.96) + 5000(0.96)^2 + \cdots.$$

Since $-1 < r = 0.96 < 1$, this infinite series converges and we have

$$\text{Total consumption } = \frac{5000}{1-0.96} = 125{,}000 \text{ m}^3.$$

If usage of this mineral decreases by 4% per year, we can use the mineral forever and the total reserve never runs out.

18. **(a)** Since doses are given at time intervals equal to the half-life, the fraction of one dose remaining when the next dose is given is 0.5. At the steady state, the quantity right after a dose is given is the sum of an infinite geometric series with $a = D$ (the size of one dose) and $r = 0.5$. Since $-1 < r < 1$, the series converges to the sum:

$$\begin{aligned}
\text{Steady state quantity after dose } &= D + D(0.5) + D(0.5)^2 + \cdots\\
&= \frac{D}{1-0.5} = 2D.
\end{aligned}$$

Under these conditions, at the steady state, the quantity is twice the quantity of a single dose.

(b) Since at the steady state, the quantity is twice the quantity of a single dose, if we want the steady state quantity to be 300 mg, each dose should be 150 mg.

19. Right after a dose,

$$\text{Steady state quantity } = S = D + Dr + Dr^2 + Dr^3 + \cdots = \frac{D}{1-r}.$$

At the steady state, right after a dose is given, S units of the drug are in the body. Over one time interval (the dosage interval), a fraction r of the drug remains, so the fraction $(1-r)$ of the drug is excreted. Thus,

$$\text{Quantity excreted over one time interval } = S\cdot(1-r) = \frac{D}{1-r}\cdot(1-r) = D = \text{Quantity of one dose.}$$

20. The quantity of cephalexin in the body is given by $Q(t) = Q_0 e^{-kt}$, where $Q_0 = Q(0)$ and k is a constant. Since the half-life is 0.9 hours,

$$\frac{1}{2} = e^{-0.9k}, \quad k = -\frac{1}{0.9}\ln\frac{1}{2} \approx 0.8.$$

(a) After 6 hours

$$Q = Q_0 e^{-k(6)} \approx Q_0 e^{-0.8(6)} = Q_0(0.01).$$

Thus, the percentage of the cephalexin that remains after 6 hours $\approx 1\%$.

(b)

$$\begin{aligned} Q_1 &= 250 \\ Q_2 &= 250 + 250(0.01) \\ Q_3 &= 250 + 250(0.01) + 250(0.01)^2 \\ Q_4 &= 250 + 250(0.01) + 250(0.01)^2 + 250(0.01)^3 \end{aligned}$$

(c)

$$\begin{aligned} Q_3 &= \frac{250(1-(0.01)^3)}{1-0.01} \\ &\approx 252.5 \\ Q_4 &= \frac{250(1-(0.01)^4)}{1-0.01} \\ &\approx 252.5 \end{aligned}$$

Thus, by the time a patient has taken three cephalexin tablets, the quantity of drug in the body has leveled off to 252.5 mg.

(d) Looking at the answers to part (b) shows that

$$\begin{aligned} Q_n &= 250 + 250(0.01) + 250(0.01)^2 + \cdots + 250(0.01)^{n-1} \\ &= \frac{250(1-(0.01)^n)}{1-0.01}. \end{aligned}$$

(e) In the long run, $n \to \infty$. So,

$$Q = \lim_{n\to\infty} Q_n = \frac{250}{1-0.01} = 252.5.$$

Solutions for Chapter 11 Review

1. The sum can be rewritten as

$$2 + 2(2) + 2(2^2) + \cdots + 2(2^9).$$

This is a finite geometric series with $a = 2$, $r = 2$, and $n = 10$. We have

$$\text{Sum} = \frac{2(1-(2)^{10})}{1-2} = 2046.$$

2. We use the formula for the sum of a finite geometric series with $a = 20$, $r = 1.4$, and $n = 9$. We have

$$\text{Sum} = \frac{20(1-(1.4)^9)}{1-1.4} = 983.05.$$

3. This is an infinite geometric series with $a = 1000$ and $r = 1.08$. Since $r > 1$, the series diverges and the sum does not exist.

4. We use the formula for the sum of a finite geometric series with $a = 500$, $r = 0.6$, and $n = 16$. We have

$$\text{Sum} = \frac{500(1-(0.6)^{16})}{1-0.6} = 1249.65.$$

5. This is an infinite geometric series with $a = 30$ and $r = 0.85$. Since $-1 < r < 1$, the series converges and we have

$$\text{Sum} = \frac{30}{1 - 0.85} = 200.$$

6. This is an infinite geometric series with $a = 25$ and $r = 0.2$. Since $-1 < r < 1$, the series converges and we have

$$\text{Sum} = \frac{25}{1 - 0.2} = 31.25.$$

7. We use the formula for the sum of a finite geometric series with $a = 1$, $r = 1/2$, and $n = 9$. We have

$$\text{Sum} = \frac{1 - (0.5)^9}{1 - 0.5} = 1.9961.$$

8. This is an infinite geometric series with $a = 1$ and $r = 1/3$. Since $-1 < r < 1$, the series converges and we have

$$\text{Sum} = \frac{1}{1 - 1/3} = 1.5.$$

9. (a) (i) On the night of December 31, 1999:

First deposit will have grown to $2(1.04)^7$ million dollars.
Second deposit will have grown to $2(1.04)^6$ million dollars.
...
Most recent deposit (Jan.1, 1999) will have grown to $2(1.04)$ million dollars.

Thus

$$\begin{aligned}\text{Total amount} &= 2(1.04)^7 + 2(1.04)^6 + \cdots + 2(1.04)\\ &= 2(1.04)\underbrace{(1 + 1.04 + \cdots + (1.04)^6)}_{\text{finite geometric series}}\\ &= 2(1.04)\left(\frac{1 - (1.04)^7}{1 - 1.04}\right)\\ &= 16.43 \text{ million dollars.}\end{aligned}$$

(ii) Notice that if 10 payments were made, there are 9 years between the first and the last. On the day of the last payment:

First deposit will have grown to $2(1.04)^9$ million dollars.
Second deposit will have grown to $2(1.04)^8$ million dollars.
...
Last deposit will be 2 million dollars.

Therefore

$$\begin{aligned}\text{Total amount} &= 2(1.04)^9 + 2(1.04)^8 + \cdots + 2\\ &= 2\underbrace{(1 + 1.04 + (1.04)^2 + \cdots + (1.04)^9)}_{\text{finite geometric series}}\\ &= 2\left(\frac{1 - (1.04)^{10}}{1 - 1.04}\right)\\ &= 24.01 \text{ million dollars.}\end{aligned}$$

(b) In part (a) (ii) we found the future value of the contract 9 years in the future. Thus

$$\text{Present Value} = \frac{24.01}{(1.04)^9} = 16.87 \text{ million dollars.}$$

Alternatively, we can calculate the present value of each of the payments separately:

$$\begin{aligned}\text{Present Value} &= 2 + \frac{2}{1.04} + \frac{2}{(1.04)^2} + \cdots + \frac{2}{(1.04)^9} \\ &= 2\left(\frac{1-(1/1.04)^{10}}{1-1/1.04}\right) = 16.87 \text{ million dollars.}\end{aligned}$$

Notice that the present value of the contract ($16.87 million) is considerably less than the face value of the contract, $20 million.

10. Right after a dose is given, the quantity of the drug in the body is the quantity from that dose (100 mg) plus the amounts remaining from all previous doses. In the long run

$$\text{Quantity of drug in the body after a dose } = 100 + 100(0.82) + 100(0.82)^2 + \cdots.$$

This is an infinite geometric series with $a = 100$ and $r = 0.82$. Since $-1 < r < 1$, the series converges. Using the formula for the sum:

$$\begin{aligned}\text{Long-run quantity, right after a dose } &= 100 + 100(0.82) + 100(0.82)^2 + \cdots \\ &= \frac{100}{1-0.82} = 555.556 \text{ mg.}\end{aligned}$$

Since each dose is 100 mg, the long-run quantity right before a dose is 100 mg less than the long-run quantity right after a dose. We have

$$\text{Long-run quantity, right before a dose } = 555.556 - 100 = 455.556 \text{ mg.}$$

11. The cost of owning the car is the sum of the original cost, $15,000, plus the repairs, in dollars:

$$\begin{aligned}\text{Repairs } &= 500 + 500(1.2) + 500(1.2)^2 + \cdots + 500(1.2)^9 \\ &= \frac{500(1-(1.2)^{10})}{1-1.2} = 12{,}979.34.\end{aligned}$$

Thus, the total cost of the car is $15,000 + $12,979.34 = $27,979.34.

12. **(a)** The amount that must be deposited now is the present value of the annuity to fund this scholarship. The present value of the award made one year from now is $10{,}000(1.06^{-1})$, and the present value of the award made n years from now is $10{,}000(1.06^{-1})^n$. Since the awards start immediately, the 20^{th} award is made 19 years from now. We have

$$\text{Present value } = 10000 + 10000(1.06^{-1}) + 10000(1.06^{-1})^2 + \cdots + 10000(1.06^{-1})^{19}.$$

This is a finite geometric series with $a = 10000$, $r = 1.06^{-1}$, and $n = 20$. Using the formula for the sum, we have

$$\text{Present value of the annuity } = \frac{10000(1-(1.06^{-1})^{20})}{1-1.06^{-1}} = \$121{,}581.16.$$

The donor must deposit $121,581.16 now in order to fund this scholarship for twenty years.

(b) If the awards are to continue indefinitely, the present value is the sum of infinitely many terms:

$$\text{Present value } = 10000 + 10000(1.06^{-1}) + 10000(1.06^{-1})^2 + 10000(1.06^{-1})^3 + \cdots.$$

This is an infinite geometric series with $a = 10000$ and $r = 1.06^{-1} = 0.9433962$. Since $-1 < r < 1$, this series converges to the sum:

$$\text{Present value of the annuity } = \frac{10000}{1-1.06^{-1}} = \$176{,}666.67.$$

The donor must deposit $176,666.67 now in order to fund this scholarship forever. We can check that this present value makes sense. After the first payment, the amount in the account is $166,666.67. This amount will fund the awards forever because the interest earned on the account in one year exactly matches the money paid out of the account to fund the scholarship award: $166,666.67(0.06) = $10,000.

13. The amount of additional income generated directly by people spending their extra money is $\$100(0.8) = \$80$ million. This additional money in turn is spent, generating another $(\$100(0.8))\,(0.8) = \$100(0.8)^2$ million. This continues indefinitely, resulting in

$$\text{Total additional income} = 100(0.8) + 100(0.8)^2 + 100(0.8)^3 + \cdots = \frac{100(0.8)}{1 - 0.8} = \$400 \text{ million}$$

14. **(a)** The people who receive the 5 billion dollars in tax rebates spend 80% of it, for an initial amount spent of $5(0.80) = 4$ billion dollars. The people who receive this 4 billion dollars spend 80% of it, for an additional amount spent of 4(0.80), and so on. We have

$$\text{Total amount spent } = 4 + 4(0.80) + 4(0.80)^2 + 4(0.80)^3 + \cdots.$$

Notice that the initial amount spent is not the original tax rebate of 5 billion dollars, but 80% of 5 billion dollars, or 4 billion dollars. The total amount spent is an infinite geometric series with $a = 4$ and $r = 0.80$. Since $-1 < r < 1$, the series converges and we have

$$\text{Total amount spent } = \frac{4}{1 - 0.80} = 20 \text{ billion dollars.}$$

(b) If everyone who receives money spends 90% of it, then the initial amount spent is $5(0.90) = 4.5$ billion dollars. The people who receive this 4.5 billion dollars spend 90% of it, and so on. We have

$$\text{Total amount spent } = 4.5 + 4.5(0.90) + 4.5(0.90)^2 + 4.5(0.90)^3 + \cdots.$$

This is an infinite geometric series with $a = 4.5$ and $r = 0.90$. Since $-1 < r < 1$, the series converges and we have

$$\text{Total amount spent } = \frac{4.5}{1 - 0.90} = 45 \text{ billion dollars.}$$

Notice that an increase in the spending rate from 80% to 90% causes a dramatic increase in the total effect on spending.

15. **(a)** The people who receive the N dollars in tax rebates spend $N \cdot k$ of it. The people who receive this money spend $N \cdot k^2$ of it, and so on. We have

$$\text{Total amount spent } = Nk + Nk^2 + Nk^3 + \cdots.$$

The total amount spent is an infinite geometric series with $a = Nk$ and $r = k$. Since $0 < k < 1$, the series converges. We use the formula for the sum to see

$$\text{Total amount spent } = \frac{Nk}{1 - k} = N\left(\frac{k}{1 - k}\right).$$

(b) We substitute $k = 0.85$ into the formula from part (a):

$$\text{Total amount spent } = N\left(\frac{k}{1 - k}\right) = N\left(\frac{0.85}{1 - 0.85}\right) = 5.667N.$$

The total additional spending is more than 5 times the size of the original tax rebate.

16. A person should expect to pay the present value of the bond on the day it is bought.

$$\text{Present value of first payment } = \frac{10}{1.04}$$
$$\text{Present value of second payment } = \frac{10}{(1.04)^2}, \text{ etc.}$$

Therefore,

$$\text{Total present value } = \frac{10}{1.04} + \frac{10}{(1.04)^2} + \frac{10}{(1.04)^3} + \cdots.$$

This is a geometric series with $a = \dfrac{10}{1.04}$ and $x = \dfrac{1}{1.04}$, so

$$\text{Total present value } = \frac{\frac{10}{1.04}}{1 - \frac{1}{1.04}} = \pounds 250.$$

17. **(a)** $0.232323\ldots = 0.23 + 0.23(0.01) + 0.23(0.01)^2 + \cdots$ which is a geometric series with $a = 0.23$ and $r = 0.01$.

(b) The sum is $\dfrac{0.23}{1-0.01} = \dfrac{0.23}{0.99} = \dfrac{23}{99}$.

18. **(a)** Let h_n be the height of the n^{th} bounce after the ball hits the floor for the n^{th} time. Then from Figure 11.1,

$$\begin{aligned} h_0 &= \text{height before first bounce } = 10 \text{ feet},\\ h_1 &= \text{height after first bounce } = 10\left(\frac{3}{4}\right) \text{ feet},\\ h_2 &= \text{height after second bounce } = 10\left(\frac{3}{4}\right)^2 \text{ feet}. \end{aligned}$$

Generalizing gives

$$h_n = 10\left(\frac{3}{4}\right)^n.$$

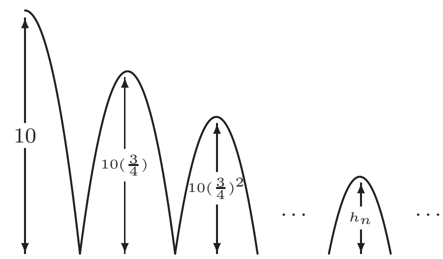

Figure 11.1

(b) When the ball hits the floor for the first time, the total distance it has traveled is just $D_1 = 10$ feet. (Notice that this is the same as $h_0 = 10$.) Then the ball bounces back to a height of $h_1 = 10\left(\frac{3}{4}\right)$, comes down and hits the floor for the second time. See Figure 11.1. The total distance it has traveled is

$$D_2 = h_0 + 2h_1 = 10 + 2\cdot 10\left(\frac{3}{4}\right) = 25 \text{ feet}.$$

Then the ball bounces back to a height of $h_2 = 10\left(\frac{3}{4}\right)^2$, comes down and hits the floor for the third time. It has traveled

$$D_3 = h_0 + 2h_1 + 2h_2 = 10 + 2\cdot 10\left(\frac{3}{4}\right) + 2\cdot 10\left(\frac{3}{4}\right)^2 = 25 + 2\cdot 10\left(\frac{3}{4}\right)^2 = 36.25 \text{ feet}.$$

Similarly,

$$\begin{aligned} D_4 &= h_0 + 2h_1 + 2h_2 + 2h_3\\ &= 10 + 2\cdot 10\left(\frac{3}{4}\right) + 2\cdot 10\left(\frac{3}{4}\right)^2 + 2\cdot 10\left(\frac{3}{4}\right)^3\\ &= 36.25 + 2\cdot 10\left(\frac{3}{4}\right)^3\\ &\approx 44.69 \text{ feet}. \end{aligned}$$

(c) When the ball hits the floor for the n^{th} time, its last bounce was of height h_{n-1}. Thus, by the method used in part (b), we get

$$\begin{aligned} D_n &= h_0 + 2h_1 + 2h_2 + 2h_3 + \cdots + 2h_{n-1}\\ &= 10 + \underbrace{2\cdot 10\left(\frac{3}{4}\right) + 2\cdot 10\left(\frac{3}{4}\right)^2 + 2\cdot 10\left(\frac{3}{4}\right)^3 + \cdots + 2\cdot 10\left(\frac{3}{4}\right)^{n-1}}_{\text{finite geometric series}}\\ &= 10 + 2\cdot 10\cdot\left(\frac{3}{4}\right)\left(1 + \left(\frac{3}{4}\right) + \left(\frac{3}{4}\right)^2 + \cdots + \left(\frac{3}{4}\right)^{n-2}\right) \end{aligned}$$

$$= 10 + 15\left(\frac{1-\left(\frac{3}{4}\right)^{n-1}}{1-\left(\frac{3}{4}\right)}\right)$$
$$= 10 + 60\left(1-\left(\frac{3}{4}\right)^{n-1}\right).$$

19.

$$\text{Total present value, in dollars} = 1000 + 1000e^{-0.04} + 1000e^{-0.04(2)} + 1000e^{-0.04(3)} + \cdots$$
$$= 1000 + 1000(e^{-0.04}) + 1000(e^{-0.04})^2 + 1000(e^{-0.04})^3 + \cdots$$

This is an infinite geometric series with $a = 1000$ and $x = e^{(-0.04)}$, and sum

$$\text{Total present value, in dollars} = \frac{1000}{1-e^{-0.04}} = 25{,}503.$$

20. **(a)** In any given year, the number of units manufactured is 1000, the number of units in use that were manufactured the previous year is $1000(0.80)$ (since 20% of them failed), the number of units that were manufactured two years ago is $1000(0.80)^2$, and so on. We have

$$\text{Total number of units in use} = 1000 + 1000(0.80) + 1000(0.80)^2 + 1000(0.80)^3 + \cdots.$$

This is an infinite geometric series with $a = 1000$ and $r = 0.80$. Since $-1 < r < 1$, this series converges to a finite sum. We have

$$\text{Total number of units in use} = \frac{a}{1-r} = \frac{1000}{1-0.80} = 5000 \text{ units.}$$

This sum, 5000 units, is the market stabilization point for this product.

(b) The total number in use after $n = 5$ production cycles is

$$S_5 = 1000 + 1000(0.80) + 1000(0.80)^2 + 1000(0.80)^3 + 1000(0.80)^4$$
$$= \frac{1000(1-(0.80)^5)}{1-0.80}$$
$$= 3362 \text{ units.}$$

Similarly, we find the other values in Table 11.1

Table 11.1 *Market's approach to the stabilization point*

n	5	10	15	20
S_n	3362	4463	4824	4942

CHECK YOUR UNDERSTANDING

1. True, since there is a constant ratio of 2 between successive terms.
2. False. It has 11 terms.
3. False. It is not a geometric series since the ratio of consecutive terms is not constant.
4. True. There are 6 terms and consecutive terms all have the same ratio of -2.
5. True, since this is a geometric series with 11 terms, with first term 1 and constant ratio $1/3$.
6. False. The sum is $3(1-2^{21})/(1-2)$.
7. False. The series diverges because the constant ratio is $3 \geq 1$.
8. False. The first term of the series is $1/2$ so the sum is $(1/2)/(1-(1/2))$.
9. True. The first term is $1/3$, so the sum is $(1/3)/(1-(1/3)) = (1/3)/(2/3) = 1/2$.
10. True. The first term is $5(1/2)$ so the sum is $(5/2)/(1-1/2)) = 5$.

11. True, as specified in the text.

12. True, as specified in the text.

13. True, since without any interest, we would need to deposit $6000 \cdot 10 = 60{,}000$ dollars in the annuity today to pay 6000 dollars for 10 years. With the addition of interest, we can deposit less than \$60,000.

14. False. For example, an initial deposit of \$1000 could pay \$30 a year forever at an annual interest rate of 3%.

15. False. The fourth deposit would be at the beginning of the fourth year; the value given is for the end of the fourth year.

16. True. The amount after the tenth deposit would be $3000(1.02)^9 + 3000(1.02)^8 + \cdots + 3000 = 3000\frac{1-(1.02)^{10}}{1-1.02}$.

17. True. The account will earn $735{,}000(0.05) = 36,750$ dollars a year so it can generate \$35,000 annual payments in perpetuity.

18. False. The present value will be less than \$2000, not more. It should be $2000(1.03)^{-5}$ dollars.

19. True, since the first payment has present value 600, the second payment one year from now has present value $600(1.04)^{-1}$ and the third payment two years from now has present value $600(1.04)^{-2}$.

20. True, since the present value of the series of ten payments is geometric, with first term 600 and constant ratio $(1.04)^{-1}$.

21. False. Since the person is metabolizing the drug throughout the day, the person will have less than the full 100 mg in the body.

22. False. Steady state does not mean that the level stays constant, only that the level oscillates between a maximum level right after a dose is taken and a minimum level right before a dose is taken.

23. False. The total amount in the long run is $3+3(0.95)+3(0.95)^2+\cdots = 3\frac{1}{1-0.95} = 60$. Alternatively, if the steady level is L, then the amount eliminated in one day would be $0.05L$ and should equal the amount consumed 3. Thus $0.05L = 3$ so $L = 60$.

24. True. The total oil consumption for the next 5 years is given by the geometric series $10(1.02) + \cdots 10(1.02)^5$, which has first term $10(1.02)$ and constant ratio 1.02.

25. False. The level goes up right after each new dose.

26. True. If the amount eliminated were less than D, then the level would increase each day. Similarly, if the amount eliminated were greater than D, then the level would decrease each day.

27. False. If the drug is administered intravenously, the amount consumed is continuous, so a differential equation is a better model than geometric series.

28. True, since nine hours is 3 half-lives, so the amount drops by a factor of $(1/2)^3 = 1/8$.

29. False; the correct value for long-term quantity is $50/(1 - e^{-0.05})$.

30. True, since we are looking for an equilibrium solution to the differential equation $dQ/dt = 50 - 0.05Q$, which can be found by solving $0 = 50 - 0.05Q$, yielding $Q = 1000$.

PROJECTS FOR CHAPTER ELEVEN

1. (a) Each person has 2 parents plus 2^2 grandparents plus 2^3 great-grandparents, and so on. The number of ancestors going back 3 generations is $2 + 2^2 + 2^3$. We have

$$\text{Number of ancestors going back } n \text{ generations } = 2 + 2^2 + 2^3 + \cdots + 2^n.$$

(b) We estimate that a generation is about 25 years. (Many different answers are also possible.) Using 25 years for a generation, there are 4 generations in 100 years, 20 generations in 500 years, 40 generations in 1000 years, and 80 generations in 2000 years. (Others answers are also possible.)

(c) The number of ancestors going back n generations is a finite geometric series with $a = 2$ and $r = 2$ and final term $2(2^{n-1})$. We use the formula for the sum of a finite geometric series.

$$\begin{aligned}\text{Number of ancestors in 100 years } &= 2 + 2^2 + 2^3 + 2^4 \\ &= \frac{2(1-2^4)}{1-2} \\ &= 30.\end{aligned}$$

$$\begin{aligned}\text{Number of ancestors in 500 years } &= 2 + 2^2 + 2^3 + \cdots + 2^{20}\\ &= \frac{2(1-2^{20})}{1-2}\\ &= 2,097,150.\end{aligned}$$

$$\begin{aligned}\text{Number of ancestors in 1000 years } &= 2 + 2^2 + 2^3 + \cdots + 2^{40}\\ &= \frac{2(1-2^{40})}{1-2}\\ &= \text{ about } 2.2 \times 10^{12}.\end{aligned}$$

$$\begin{aligned}\text{Number of ancestors in 2000 years } &= 2 + 2^2 + 2^3 + \cdots + 2^{80}\\ &= \frac{2(1-2^{80})}{1-2}\\ &= \text{ about } 2.4 \times 10^{24}.\end{aligned}$$

There are many other possible answers.

(d) If there were no common ancestors, the number of ancestors 80 generations back (about 2000 years ago) would be $2^{80} = 1.2 \times 10^{24}$ which is far greater than the 2×10^8 people actually living at that time. In fact, the number of ancestors 40 generations back is $2^{40} = 1.1 \times 10^{12}$, which is bigger than the current population of the world. It is clear that every person on earth has had many common ancestors.

2. (a) We have

$$\begin{aligned}f(1) &= kf(0) - h = kC - h,\\ f(2) &= kf(1) - h = k(kC-h) - h = k^2C - kh - h,\\ f(3) &= kf(2) - h = k(k^2C - kh - h) - h = k^3C - k^2h - kh - h.\end{aligned}$$

(b) We have

$$f(1) = kC - h,$$

and

$$f(2) = k^2C - kh - h = k^2C - (1+k)h,$$

and

$$f(3) = k^3C - k^2h - kh - h = k^3C - (1+k+k^2)h.$$

Looking at the pattern, we guess that

$$f(n) = k^nC - (1 + k + k^2 + \cdots + k^{n-1})h.$$

(c) Using the formula for the sum of a finite geometric sequence with $a = 1$ and $r = k$, we have

$$\begin{aligned}f(n) &= k^nC - (1 + k + k^2 + \cdots + k^{n-1})h\\ &= k^nC - \left(\frac{1-k^n}{1-k}\right)h.\end{aligned}$$

3. (a) (i) p^2

(ii) There are two ways to do this. One way is to compute your opponent's probability of winning two in a row, which is $(1-p)^2$. Then the probability that neither of you win the next points is:

$$\begin{aligned}1 - (&\text{Probability you win next two} + \text{Probability opponent wins next two})\\ &= 1 - (p^2 + (1-p)^2)\\ &= 1 - (p^2 + 1 - 2p + p^2)\\ &= 2p^2 - 2p\\ &= 2p(1-p).\end{aligned}$$

The other way to compute this is to observe either you win the first point and lose the second or vice versa. Both have probability $p(1-p)$, so the probability you split the points is $2p(1-p)$.

(iii)

$$\begin{aligned}\text{Probability} &= (\text{Probability of splitting next two}) \cdot (\text{Probability of winning two after that}) \\ &= 2p(1-p)p^2\end{aligned}$$

(iv)

$$\begin{aligned}\text{Probability} &= (\text{Probability of winning next two}) + (\text{Probability of splitting next two,} \\ &\qquad \text{winning two after that}) \\ &= p^2 + 2p(1-p)p^2\end{aligned}$$

(v) The probability is:

$$\begin{aligned}w &= (\text{Probability of winning first two}) \\ &\quad + (\text{Probability of splitting first two})\cdot(\text{Probability of winning next two}) \\ &\quad + (\text{Prob. of split. first two})\cdot(\text{Prob. of split. next two})\cdot(\text{Prob. of winning next two}) \\ &\quad + \cdots \\ &= p^2 + 2p(1-p)p^2 + (2p(1-p))^2\, p^2 + \cdots .\end{aligned}$$

This is an infinite geometric series with a first term of p^2 and a ratio of $2p(1-p)$. Therefore the probability of winning is

$$w = \frac{p^2}{1-2p(1-p)}.$$

(vi) For $p = 0.5$, $w = \frac{(0.5)^2}{1-2(0.5)(1-(0.5))} = 0.5$. This is what we would expect. If you and your opponent are equally likely to score the next point, you and your opponent are equally likely to win the next game.

For $p = 0.6$, $w = \frac{(0.6)^2}{1-2(0.6)(0.4)} = 0.69$. Here your probability of winning the next point has been magnified to a probability 0.69 of winning the game. Thus it gives the better player an advantage to have to win by two points, rather than the "sudden death" of winning by just one point. This makes sense: when you have to win by two, the stronger player always gets a second chance to overcome the weaker player's winning the first point on a "fluke."

For $p = 0.7$, $w = \frac{(0.7)^2}{1-2(0.7)(0.3)} = 0.84$. Again, the stronger player's probability of winning is magnified.

For $p = 0.4$, $w = \frac{(0.4)^2}{1-2(0.4)(0.6)} = 0.31$. We already computed that for $p = 0.6$, $w = 0.69$. Thus the value for w when $p = 0.4$, should be the same as the probability of your opponent winning for $p = 0.6$, namely $1 - 0.69 = 0.31$.

(b) (i)

$$\begin{aligned}S &= (\text{Prob. you score first point}) \\ &\qquad +(\text{Prob. you lose first point, your opponent loses the next,} \\ &\qquad\quad \text{you win the next}) \\ &\qquad +(\text{Prob. you lose a point, opponent loses, you lose,} \\ &\qquad\quad \text{opponent loses, you win}) \\ &\qquad +\cdots \\ &= (\text{Prob. you score first point}) \\ &\qquad +(\text{Prob. you lose})\cdot(\text{Prob. opponent loses})\cdot(\text{Prob. you win}) \\ &\qquad +(\text{Prob. you lose})\cdot(\text{Prob. opponent loses})\cdot(\text{Prob. you lose}) \\ &\qquad\quad \cdot(\text{Prob. opponent loses})\cdot(\text{Prob. you win}) + \cdots \\ &= p + (1-p)(1-q)p + ((1-p)(1-q))^2\, p + \cdots \\ &= \frac{p}{1-(1-p)(1-q)}\end{aligned}$$

(ii) Since S is your probability of winning the next point, we can use the formula computed in part (v) of (a) for winning two points in a row, thereby winning the game:

$$w = \frac{S^2}{1 - 2S(1 - S)}.$$

- When $p = 0.5$ and $q = 0.5$,

$$S = \frac{0.5}{1 - (0.5)(0.5)} = 0.67.$$

  Therefore

$$w = \frac{S^2}{1 - 2S(1 - S)} = \frac{(0.67)^2}{1 - 2(0.67)(1 - 0.67)} = 0.80.$$

- When $p = 0.6$ and $q = 0.5$,

$$S = \frac{0.6}{1 - (0.4)(0.5)} = 0.75 \quad \text{and} \quad w = \frac{(0.75)^2}{1 - 2(0.75)(1 - 0.75)} = 0.9.$$