Chapter Two

# Chapter Two

## ConcepTests for Section 2.1

For Problems 1–3, assume your car has a broken speedometer.

1. In order to find my average velocity of a trip from Tucson to Phoenix, I need

   (a) The total distance of the trip
   (b) The highway mile markers
   (c) The time spent traveling
   (d) How many stops I made during the trip
   (e) A friend with a stop watch
   (f) A working odometer
   (g) None of the above

   ANSWER:
   (a) and (c)
   COMMENT:
   The choices are intentionally vague. This is meant to provide discussion. Your students may select more than one item.

2. In order to find my velocity at the instant I hit a speed trap, I need

   (a) The total distance of the trip
   (b) The highway mile markers
   (c) The time spent traveling
   (d) How many stops I made during the trip
   (e) A friend with a stop watch
   (f) A working odometer
   (g) none of the above

   ANSWER:
   (e) and (f). After I pass the speed trap I can watch my odometer as it increases by $0.1$ miles while my friend (simultaneously) records the time it took to travel $0.1$ miles.
   COMMENT:
   Using (e) and (f) as the odometer increases by $0.1$ is a good estimate of the velocity at an instant. It may be beneficial to point out that if the odometer measured in hundredths of a mile, then you could compute an even better estimate of the instantaneous velocity.

3. Regarding the speed trap in Problem 2, when should your friend first start the stopwatch?

   (a) When the driver of an oncoming vehicle warns you of the speed trap ahead by flashing his/her bright headlights
   (b) When you spot the cop
   (c) Either scenario
   (d) Neither scenario

   ANSWER:
   (c). You can use an estimation of the average velocity before (or after) you hit the speed trap to estimate your actual velocity.
   COMMENT:
   The focus of this discussion should be on how $h$ can be either positive or negative in order to estimate the derivative.

4. At approximately which integer value of $x$ does the graph in Figure 2.1 have each of the following slopes?
   (a) $-2$
   (b) $-1$
   (c) $1$
   (d) $2$
   (e) $7$

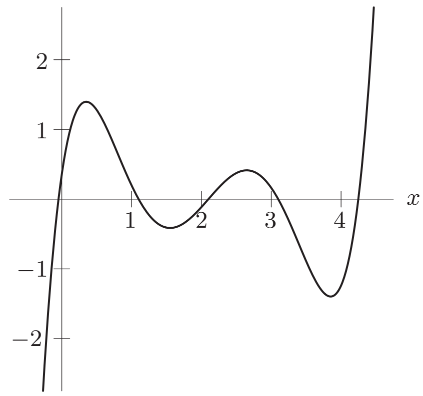



Figure 2.1

ANSWER:

(a) $x = 1$
(b) $x = 3$
(c) $x = 2$
(d) $x = 4$
(e) $x = 0$

COMMENT:

An enlarged version of this figure will make it easier to estimate slopes.

**Follow-up Question.** Put the slopes of the tangent lines occurring at $x = 0.5, 1.5, 2.5$, and $3.5$ in order from smallest to largest.

**Answer.** $x = 3.5$, $x = 0.5$, $x = 1.5$, $x = 2.5$

5. For the graph of $y = f(x)$ in Figure 2.2 arrange the following numbers in ascending order (i.e. smallest to largest).
   (a) Slope of the graph where $x = 0.2$
   (b) Slope of the graph where $x = 1.5$
   (c) Slope of the graph where $x = 1.9$
   (d) Slope of the line connecting the points on the graph where $x = 1.5$ and $x = 1.9$
   (e) The number 1

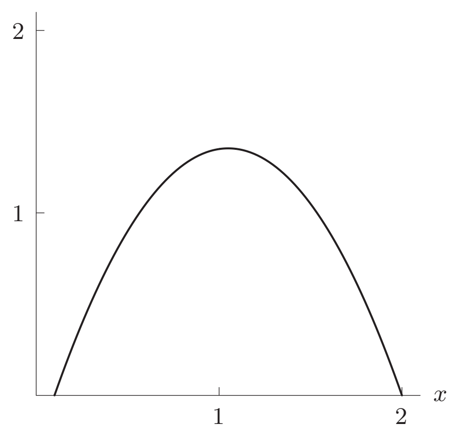



Figure 2.2

ANSWER:
(c), (d), (b), (e), (a)
COMMENT:
This is a good question for an elimination question in a classroom quiz session. One purpose for this question is to note the relationships between the slopes at the points $x = 1.5$ and $x = 1.9$ and the slope of the corresponding secant line.

For Problems 6–7, we want to find how the volume, $V$, of a balloon changes as it is filled with air. We know $V(r) = \frac{4}{3}\pi r^3$, where $r$ is the radius in inches and $V(r)$ is in cubic inches.

6. The expression

$$\frac{V(3) - V(1)}{3 - 1}$$

represents

(a) The average rate of change of the radius with respect to the volume when the radius changes from 1 inch to 3 inches.
(b) The average rate of change of the radius with respect to the volume when the volume changes from 1 cubic inch to 3 cubic inches.
(c) The average rate of change of the volume with respect to the radius when the radius changes from 1 inch to 3 inches.
(d) The average rate of change of the volume with respect to the radius when the volume changes from 1 cubic inch to 3 cubic inches.

ANSWER:
(c)
COMMENT:
This is a good way for students to see the formula and verbal description for average rate of change together.

7. Which of the following represents the rate at which the volume is changing when the radius is 1 inch?

(a) $\dfrac{V(1.01) - V(1)}{0.01} = 12.69 \text{ in}^3$
(b) $\dfrac{V(0.99) - V(1)}{-0.01} = 12.44 \text{ in}^3$
(c) $\lim\limits_{h\to 0} \left( \dfrac{V(1+h) - V(1)}{h} \right) \text{ in}^3$
(d) All of the above

ANSWER:
(d). Note that (c) is the exact rate of change while (a) and (b) approximate the rate of change.
COMMENT:
Students should be aware that (b) is equivalent to $\dfrac{V(1) - V(0.99)}{0.01}$.

8. For the function $g(x)$ shown in Figure 2.3, arrange the following numbers in increasing order.

   (a) $0$ (b) $g'(-2)$ (c) $g'(0)$ (d) $g'(1)$ (e) $g'(3)$

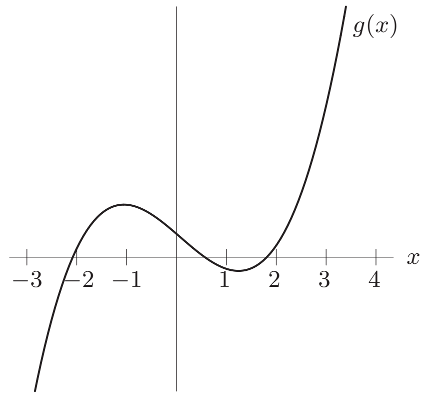



Figure 2.3

ANSWER:
(c), (d), (a), (b), (e)
COMMENT:
This can be used as an elimination question in a classroom quiz session.

9. Which of the following expressions represents the slope of a line drawn between the two points marked in Figure 2.4 ?

   (a) $m = \dfrac{F(a) + F(b)}{a + b}$ (b) $m = \dfrac{F(b) - F(a)}{b - a}$ (c) $m = \dfrac{a}{b}$

   (d) $m = \dfrac{F(a) - F(b)}{b - a}$ (e) $m = \dfrac{F(a) - F(b)}{a - b}$

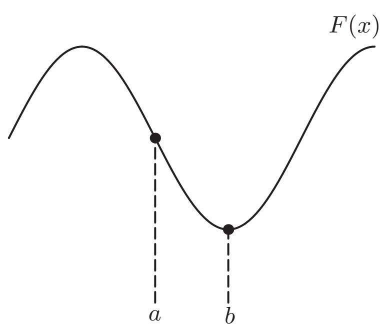



Figure 2.4

ANSWER:
(b) and (e). The coordinates of the two points shown are $(a, F(a))$ and $(b, F(b))$, so the slope of the line connecting them is $\dfrac{F(b) - F(a)}{b - a} = \dfrac{F(a) - F(b)}{a - b}$.
COMMENT:
You could repeat this question drawing the graph of a function that was increasing between $a$ and $b$.

10. Which of the following expressions represents the slope of a line drawn between the two points marked in Figure 2.5?

(a) $\dfrac{F(\Delta x) - F(x)}{\Delta x}$ (b) $\dfrac{F(x+\Delta x) - F(x)}{\Delta x}$ (c) $\dfrac{F(x+\Delta x) - F(x)}{x}$

(d) $\dfrac{F(x+\Delta x) - F(x)}{x + x - \Delta x}$ (e) $\dfrac{F(x+\Delta x) - F(x)}{x + \Delta x}$

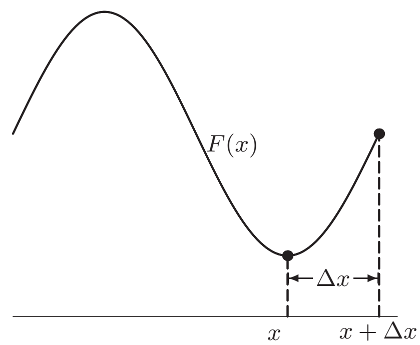



**Figure 2.5**

ANSWER:

(b). The coordinates of the two points are $(x, F(x))$ and $(x + \Delta x, F(x + \Delta x))$, so the slope of the line connecting them is

$$m = \frac{F(x+\Delta x) - F(x)}{x + \Delta x - x} = \frac{F(x+\Delta x) - F(x)}{\Delta x}.$$

COMMENT:

You could repeat this question drawing the graph of a function that was decreasing between $x$ and $x + \Delta x$.

## ConcepTests for Section 2.2

1. Which of the following graphs (a)–(d) could represent the slope at every point of the function graphed in Figure 2.6?

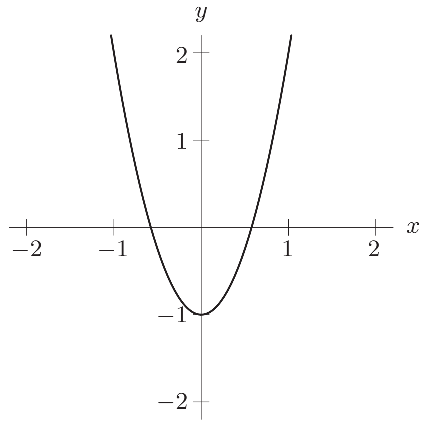
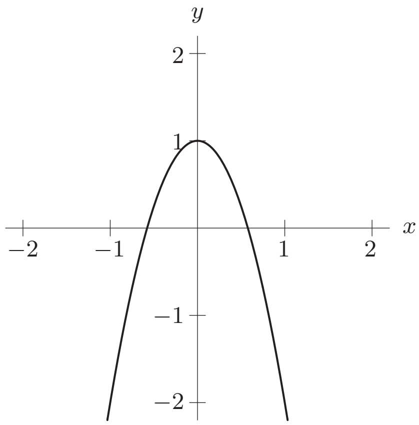
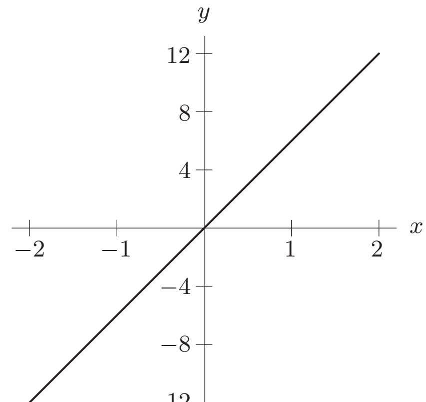



**Figure 2.6**

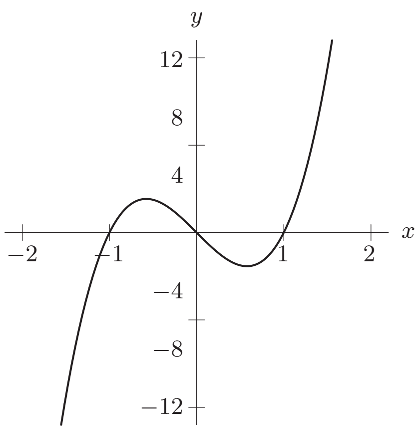
(a)



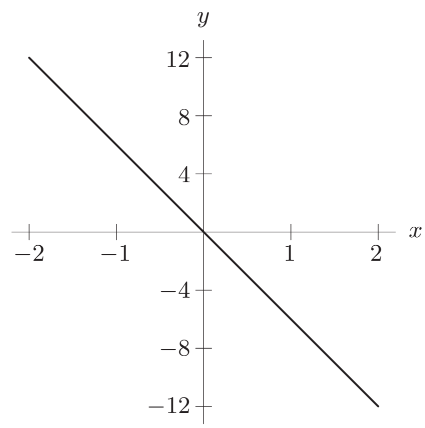

(b)




(c)




(d)




ANSWER:
(b). The function has negative slopes for $x < 0$, positive slopes for $x > 0$, and a zero slope for $x = 0$.
COMMENT:
You could have students explain why (a), (c), and (d) fail to be the correct answer.

2. Which of the following graphs (a)–(d) could represent the slope at every point of the function graphed in Figure 2.7?

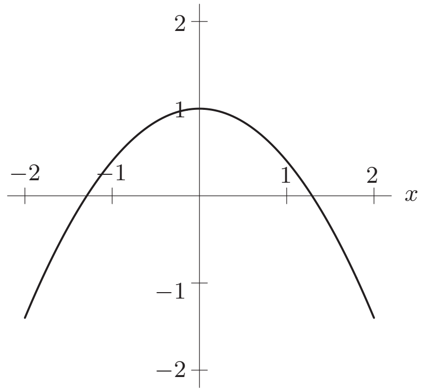
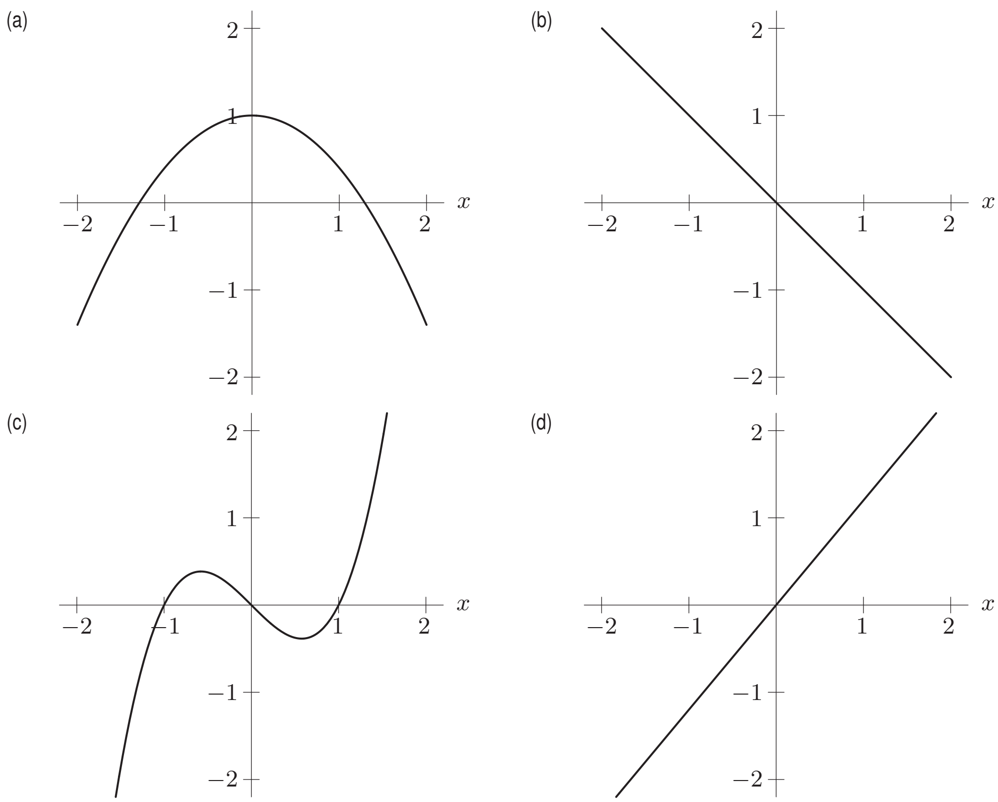



Figure 2.7




ANSWER:
(d)
COMMENT:

Your students should be analyzing where the extreme values of the original function occur and then whether the function is increasing or not. Have the students give specific reasons as to why the other three graphs could not represent the slope at every point of the function graphed in Figure 2.7.

3. Which of the following graphs (a)–(d) could represent the slope at every point of the function graphed in Figure 2.8?

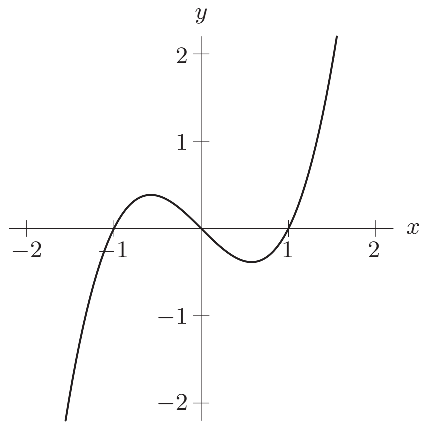
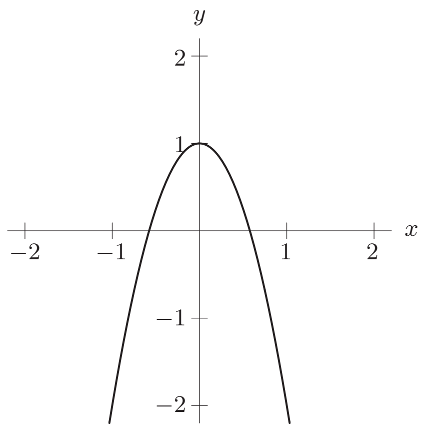
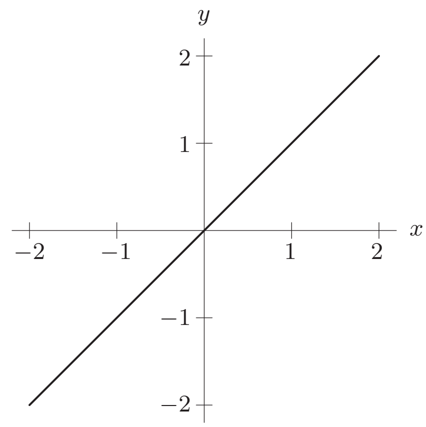



**Figure 2.8**

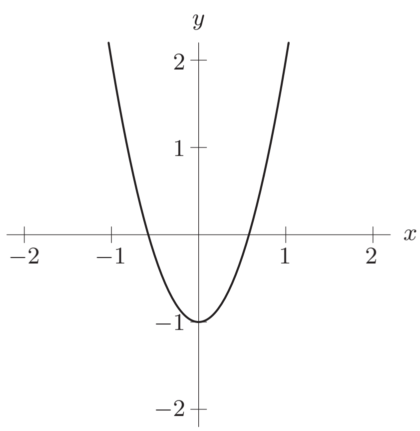
(a)



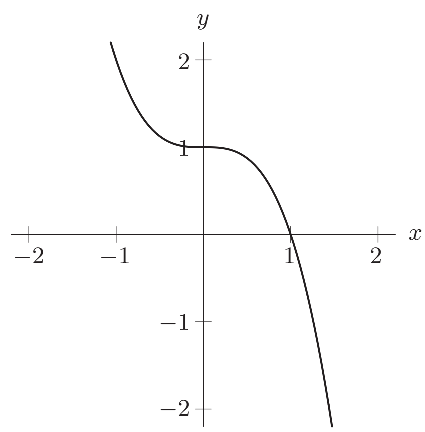

(b)




(c)




(d)




ANSWER:

(c). The function in Figure 2.8 has horizontal tangents at about $x = \pm 0.7$, is increasing for $x < -0.7$ and $x > 0.7$. Thus the graph of the slope of the function will be positive for $|x| > 0.7$, negative for $|x| < 0.7$, and zero for $x = \pm 0.7$.

COMMENT:

You could have students explain why (a), (b), and (d) fail to be the correct answer.

4. Which of the following graphs (a)–(d) could represent the slope at every point of the function graphed in Figure 2.9?

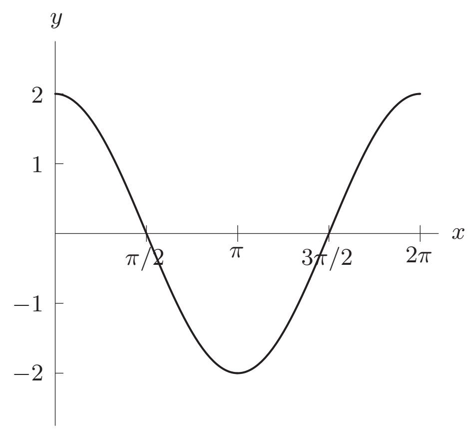
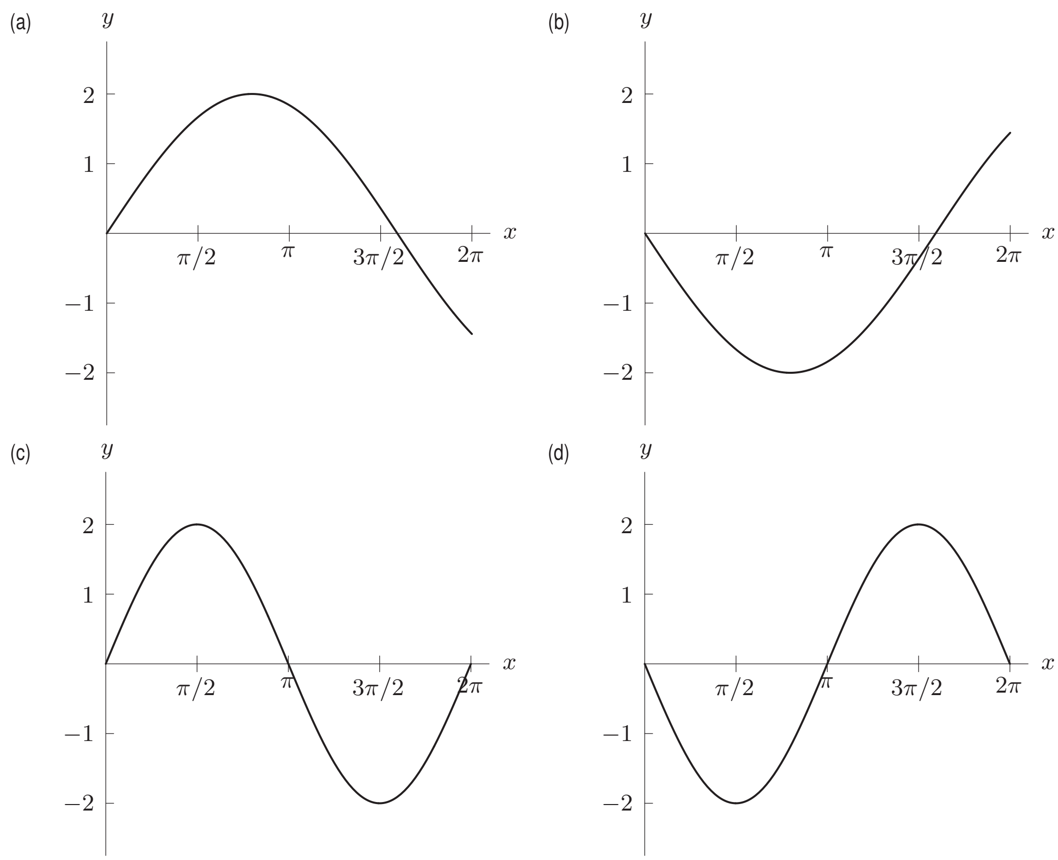



**Figure 2.9**




ANSWER:

(d). The original function has horizontal tangents at $0, \pi, 2\pi$, so the derivative will be zero there. This eliminates choices (a) and (b). The original function is also decreasing for $0 < x < \pi$, so its derivative must be negative in this region.

COMMENT:

You could have students explain why (a), (b), and (c) fail to be the correct answer.

5. Which of the following graphs (a)–(d) could represent the slope at every point of the function graphed in Figure 2.10?

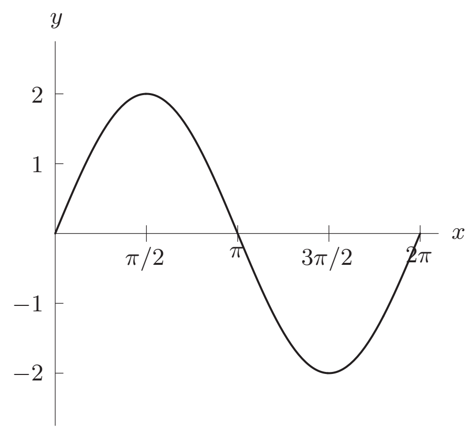
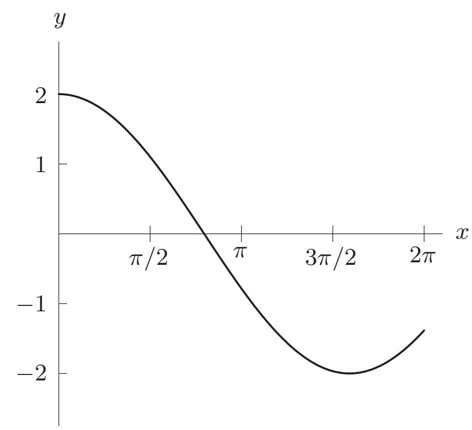
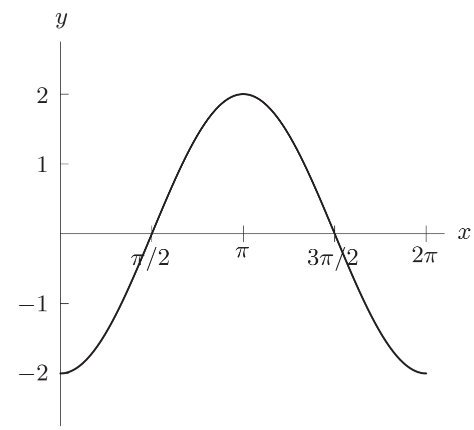



**Figure 2.10**

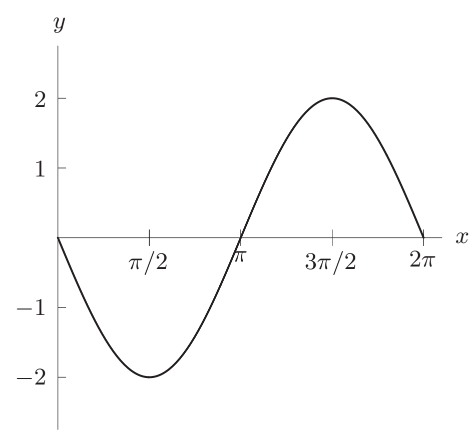
(a)



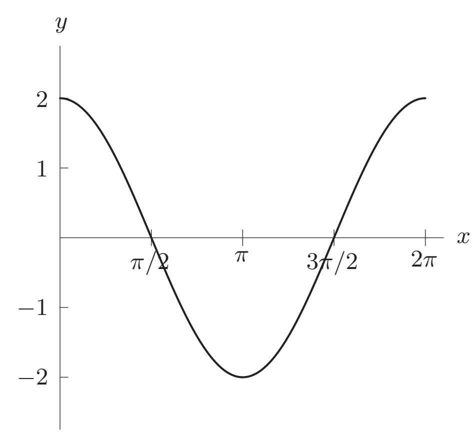

(b)




(c)




(d)




ANSWER:

(d). The original function has horizontal tangents at $x = \pi/2$ and $3\pi/2$, so the slope of the function is zero at those points. This eliminates (a) and (c). It is also increasing for $0 < x < \pi/2$, so its slope of the function is positive in this region.

COMMENT:

You could have students explain why (a), (b), and (c) fail to be the correct answer.

6. Which of the following graphs (a)–(d) could represent the slope at every point of the function graphed in Figure 2.11?

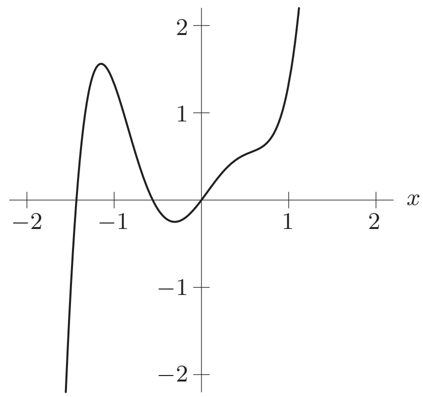
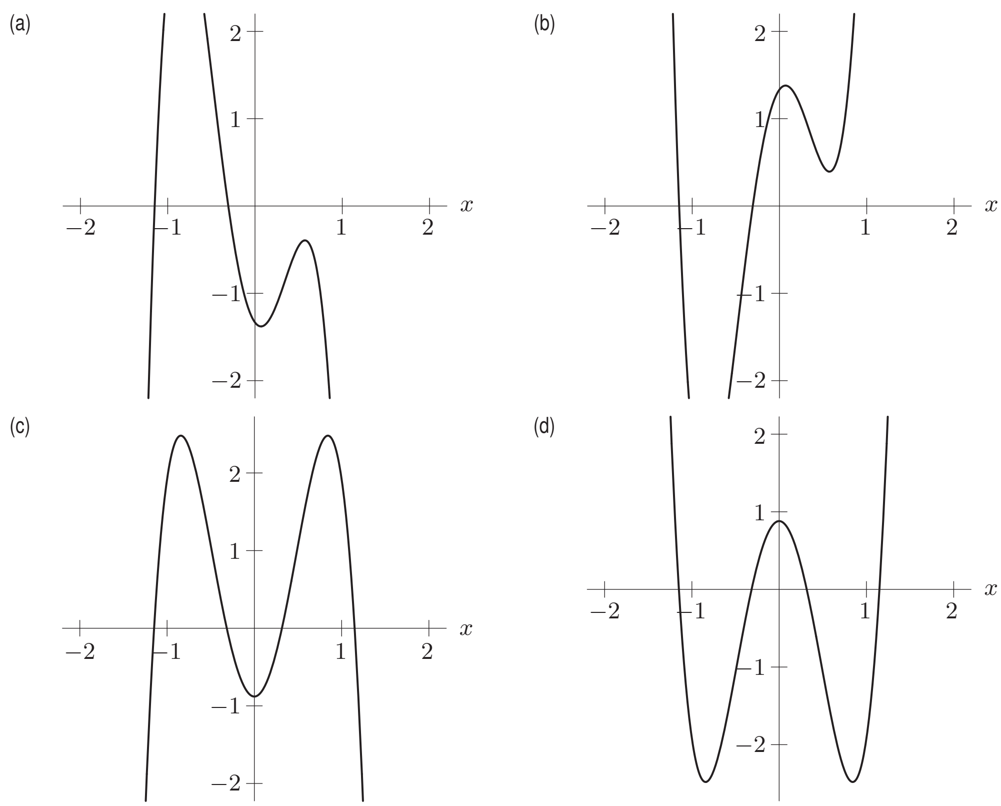



Figure 2.11




ANSWER:

(b)

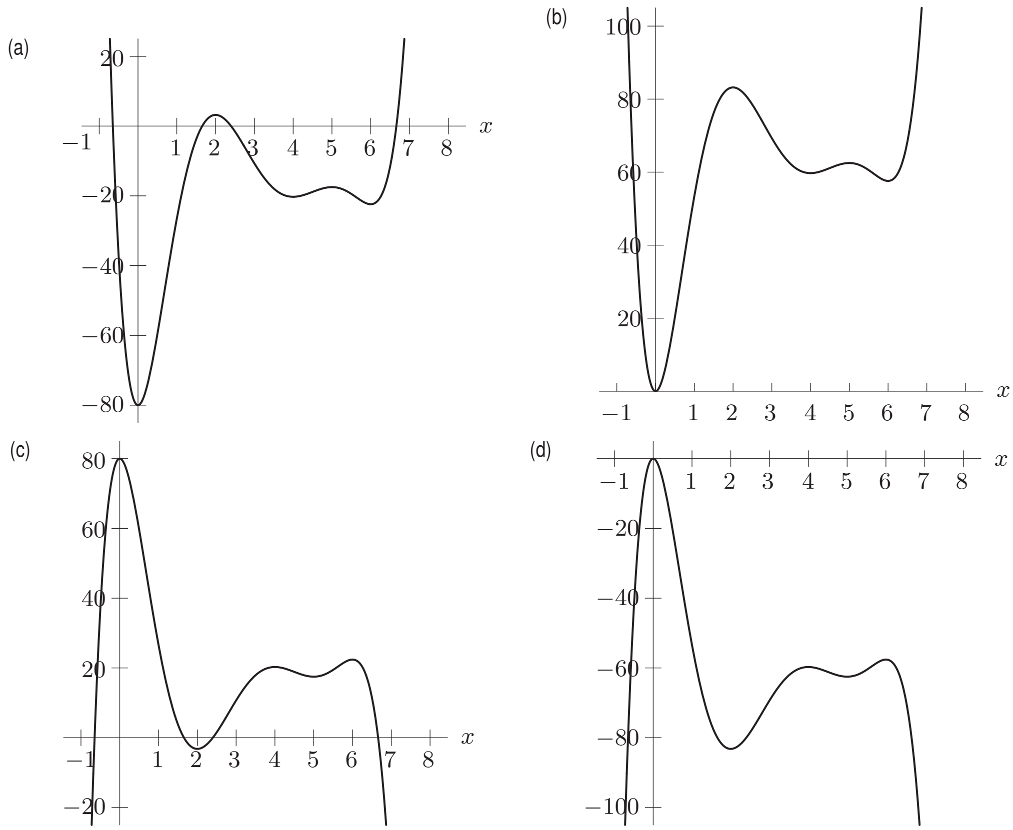
COMMENT:

Your students should analyze where the extreme values of the original function occur and where the function is increasing. Have the students give specific reasons as to why the other three graphs could not represent the slope at every point of the function graphed in Figure 2.11.

7. Suppose

$$f'(x) < 0, \text{ for } 0 < x < 2, \text{ for } 4 < x < 5, \text{ and for } 6 < x.$$
$$f'(x) > 0, \text{ for } x < 0, \text{ for } 2 < x < 4, \text{ and for } 5 < x < 6.$$

Which of the graphs (a)–(d) could be the graph of $f(x)$?




ANSWER:
(c) and (d)
COMMENT:
Your students should try to draw other graphs that meet the same requirements. Have students give reasons as to why the other two choices are invalid.

8. Which of the following graphs (a)–(d) could represent the function whose slope at every point is graphed in Figure 2.12?

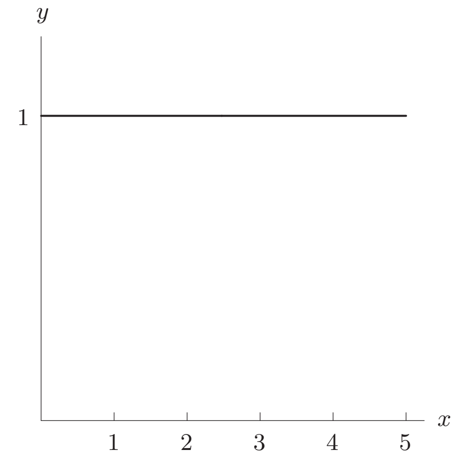
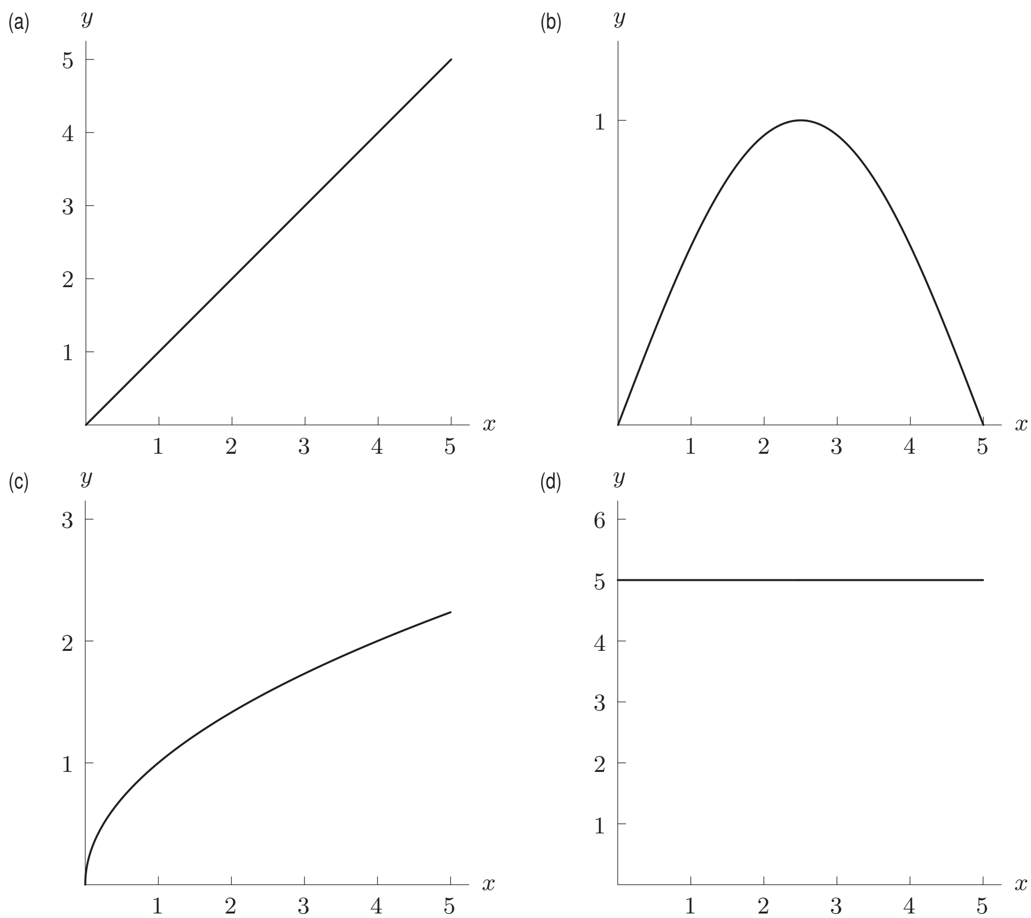



Figure 2.12




ANSWER:

(a). Because Figure 2.12 shows a constant value of 1, the original function will be a line with slope 1.

COMMENT:

This question (and the ones following) are meant to help students realize that they can think about the slope of a function from two viewpoints, either first using the function to find the slope at each point, or being given the slope at every point and then trying to recover the function. This should help them when they begin studying antiderivatives.

9. Which of the following graphs (a)–(d) could represent the function whose slope at every point is graphed in Figure 2.13?

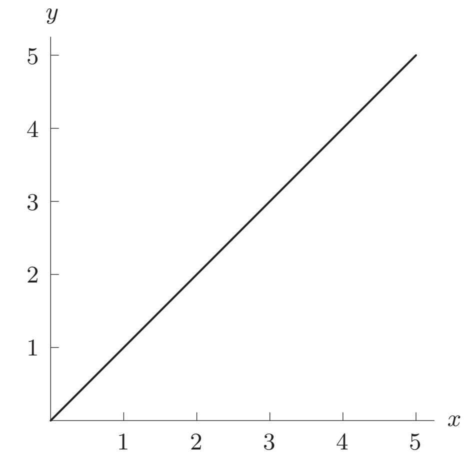
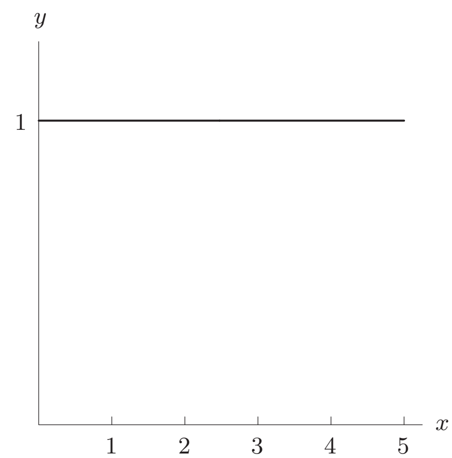
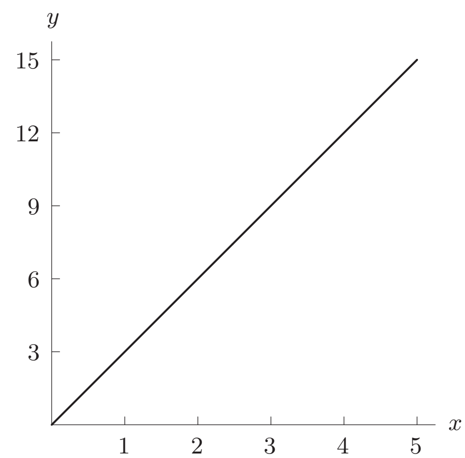



**Figure 2.13**

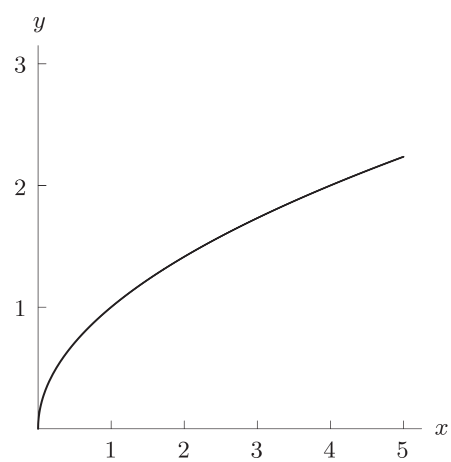
(a)



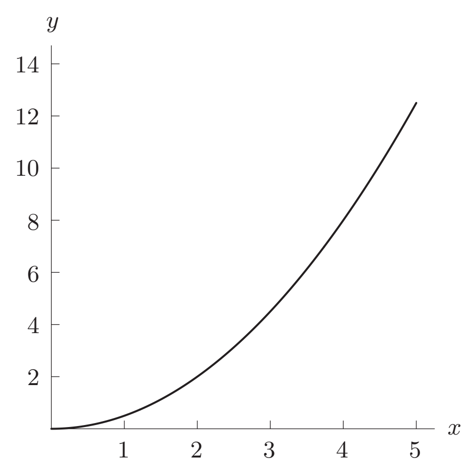

(b)




(c)




(d)




ANSWER:

(d). The graph of the slope is positive, and increasing from 0 to 5 as $x$ goes from 0 to 5, therefore the function must be increasing and concave up.

COMMENT:

See Comment for Problem 8.

10. Which of the following graphs (a)–(d) could represent the function whose slope at every point is graphed in Figure 2.14?

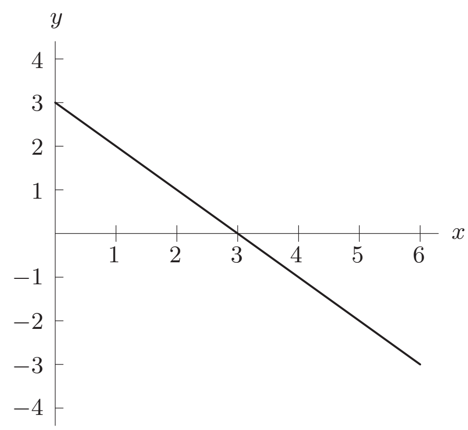
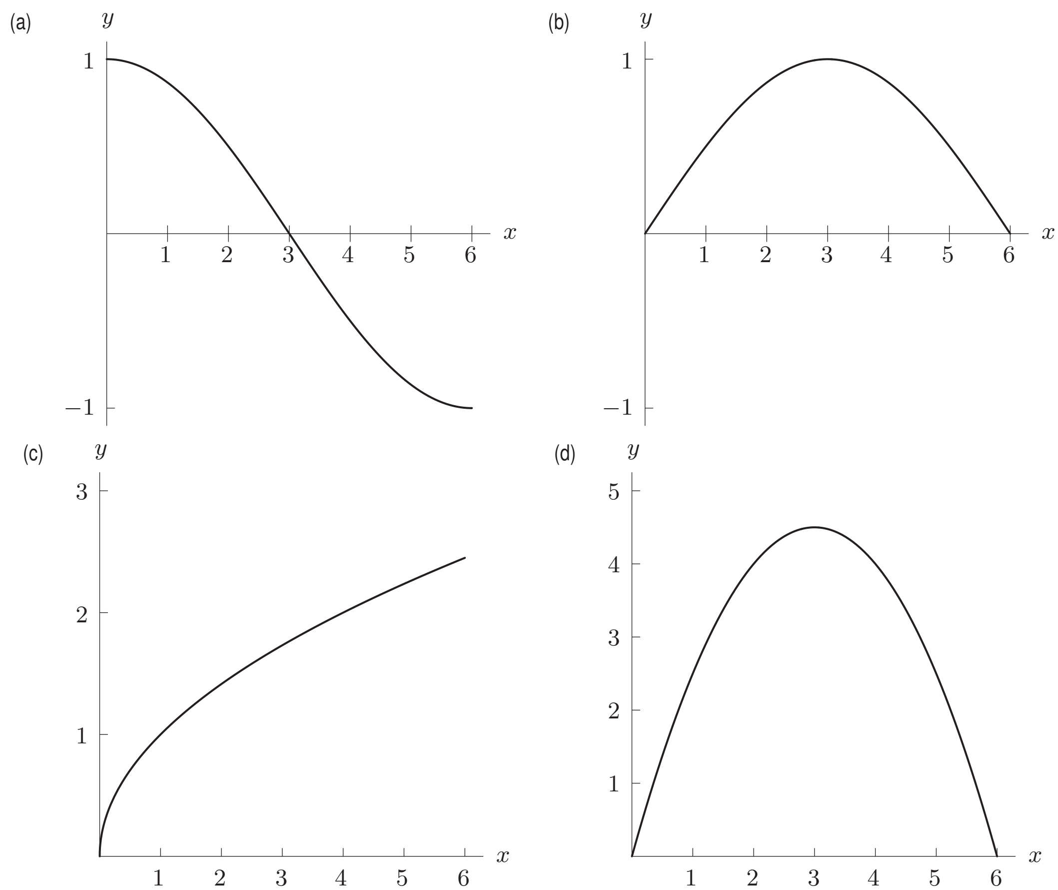



**Figure 2.14**




ANSWER:

(d). Since the graph of the slope is decreasing at every point, the function will be concave down. The graph of the function needs an initial slope of 3 and a horizontal tangent at $x = 3$.

COMMENT:

See Comment for Problem 8.

11. Which of the following graphs (a)–(d) could represent the function whose slope at every point is graphed in Figure 2.15?

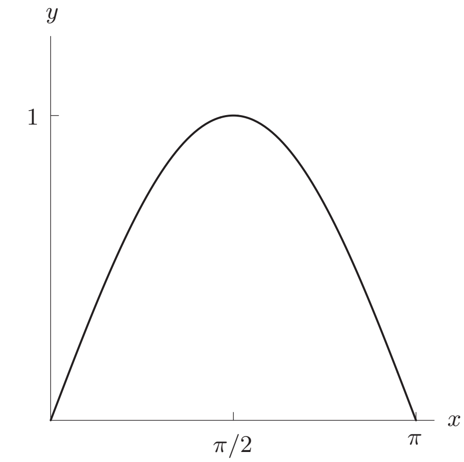
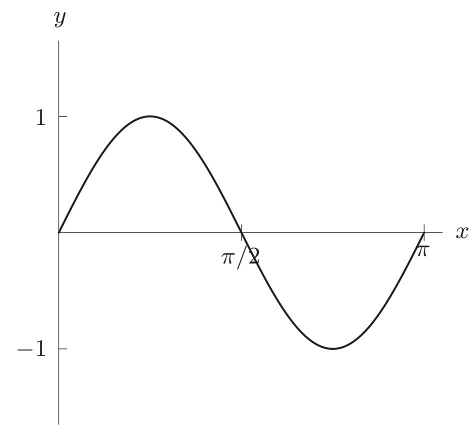
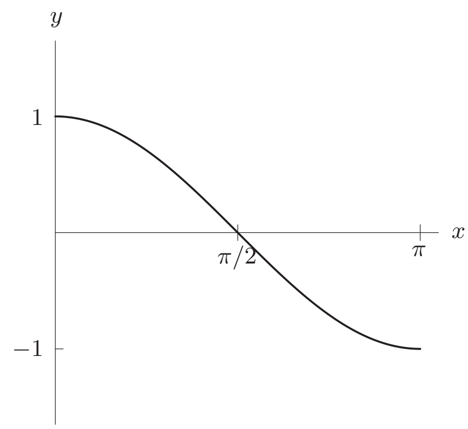



**Figure 2.15**

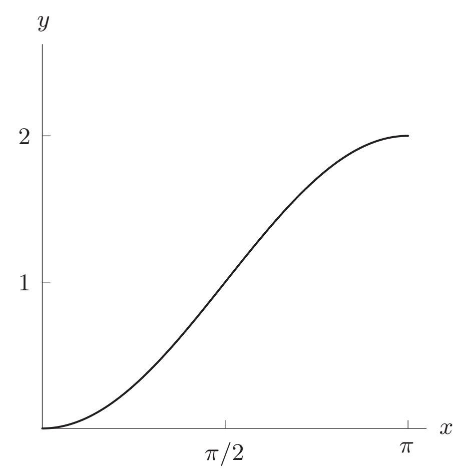
(a)



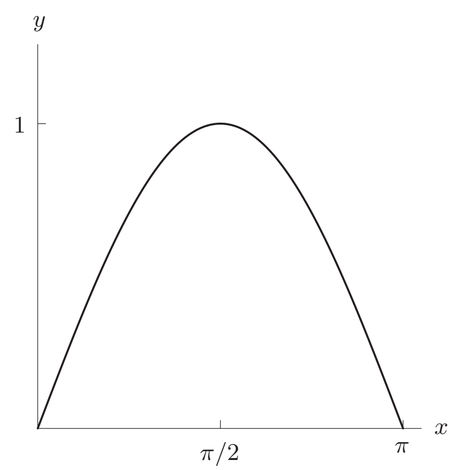

(b)




(c)




(d)




ANSWER:
(c). The graph of the slope is positive at every point, so the function must be increasing.
COMMENT:
See Comment for Problem 8.

12. Which of the following graphs (a)–(d) could represent the function whose slope at every point is graphed in Figure 2.16?

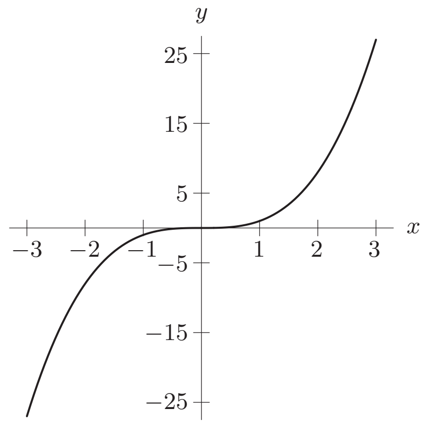
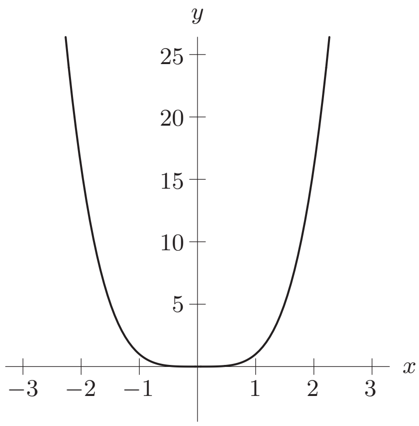
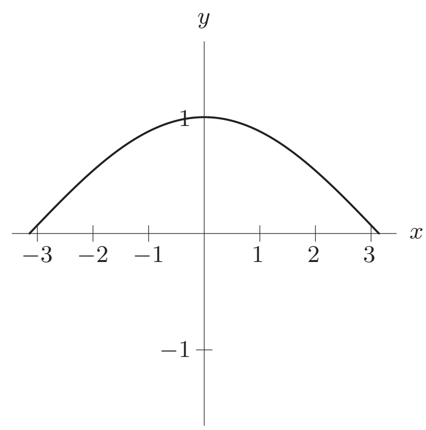



**Figure 2.16**

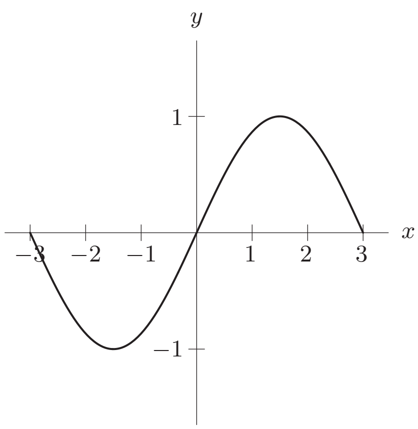
(a)



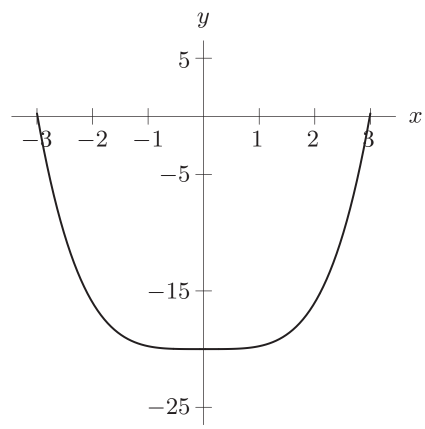

(b)




(c)




(d)




ANSWER:

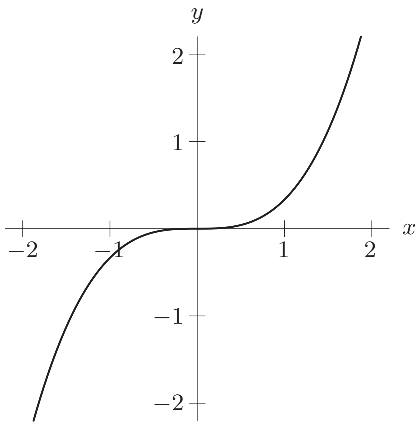
(a) and (d). The graph of the slope at every point increases from a negative value at $x = -3$, to a slope of zero at $x = 0$, to a positive value for $x > 0$.

COMMENT:

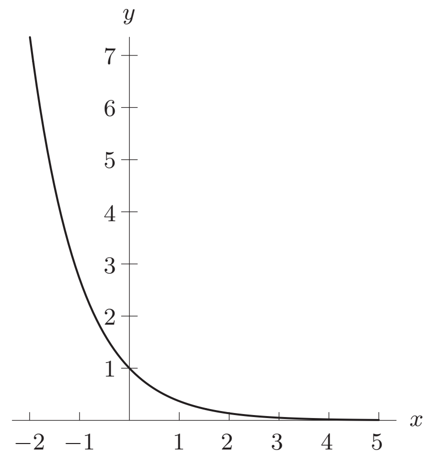
See Comment for Problem 8.

13. Which of the following is a graph of a function that is equal to its own derivative, that is, $f'(x) = f(x)$.

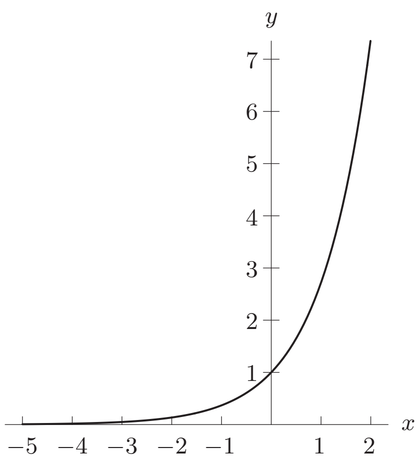
(a)



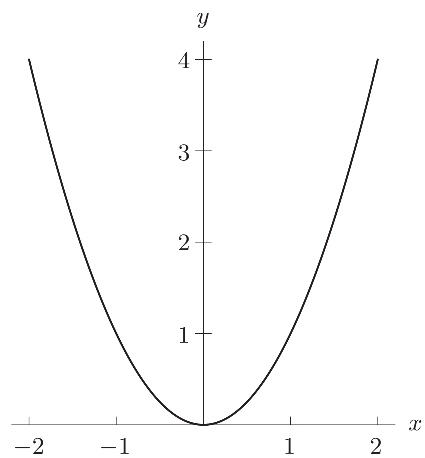

(b)




(c)




(d)



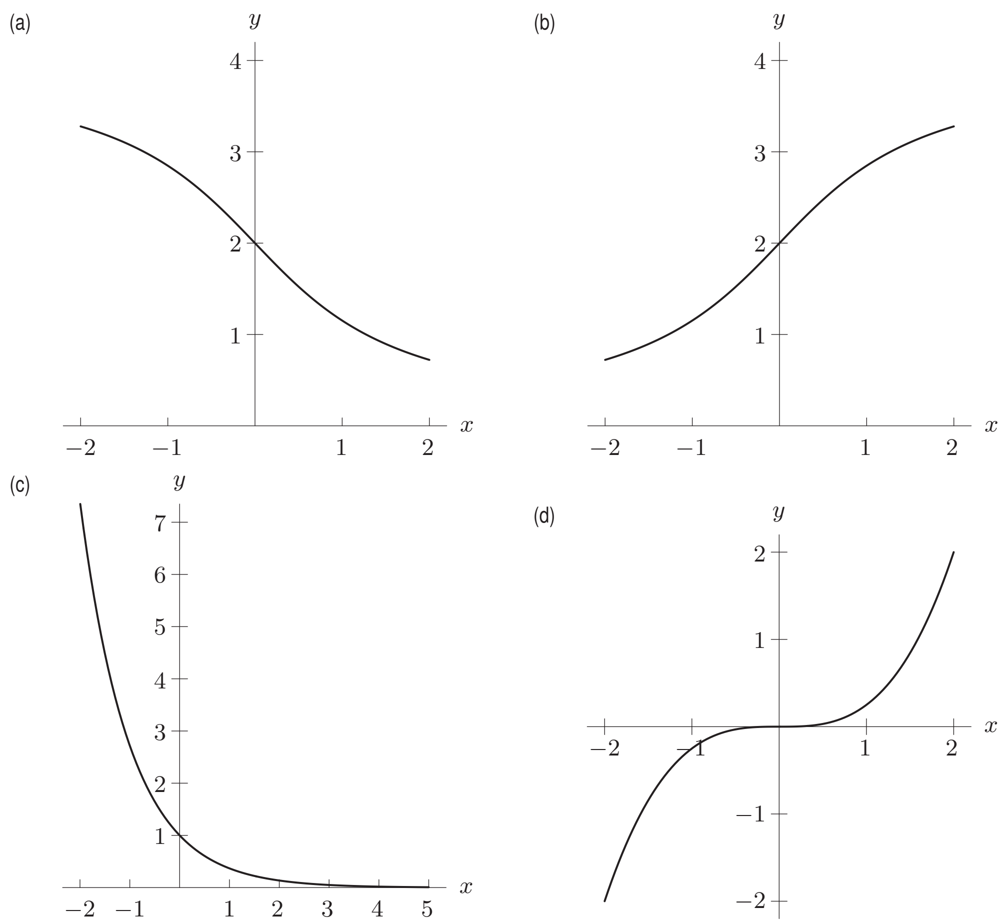

ANSWER:

(c). The slope of the curve in (a) is always positive while the function has both positive and negative values, so $f'(x) \neq f(x)$. The slope of the curve in (b) is always negative, while the function has only positive values, so $f'(x) \neq f(x)$. In (c), both the function and its slope are positive, and the slope at $(0, 1)$ appears to be 1. The function and slope increase together, so $f'(x) = f(x)$. The graph in (d) has only positive values while its slope has both positive and negative values, so $f'(x) \neq f(x)$.

COMMENT:

You could have students suggest formulas for the graphs in each choice.

14. Which of the following is a graph of a function that is equal to the negative of its own derivative, that is, $f(x) = -f'(x)$.




ANSWER:

(c). With $f(x) = -f'(x)$, the function will have positive slopes where it is negative and negative slopes where it is positive. This rules out (b) and (d). With $f(x) = -f'(x)$, as the function gets larger, the slope will be steeper. This rules out (a). The slope in (c) at $(0, 1)$ appears to be $-1$.

COMMENT:

You could have students suggest formulas for the graphs in each choice.

15. Which of the following graphs (a)–(d) could represent the slope at every point of the function graphed in Figure 2.17?

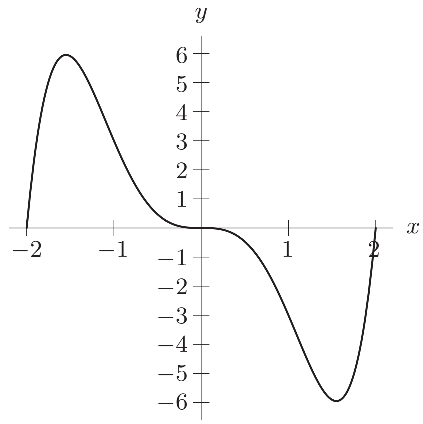
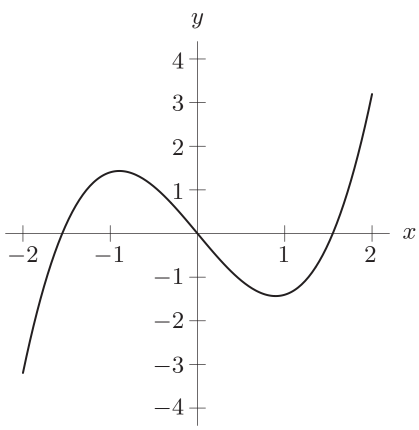
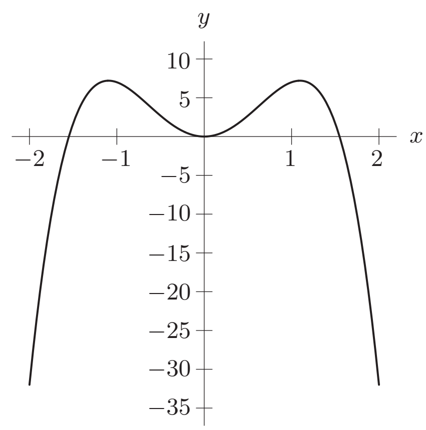


Figure 2.17

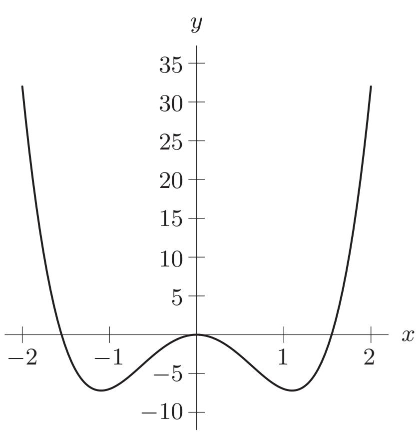
(a)



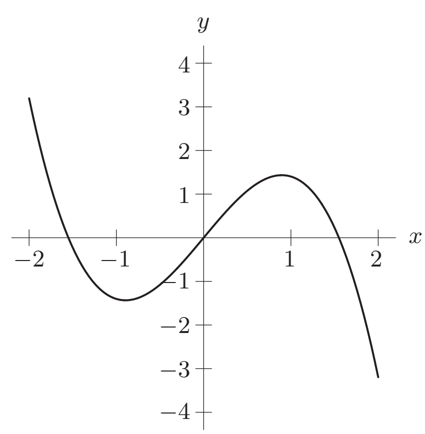
(b)



(c)



(d)



ANSWER:

(c). The graph of the function in Figure 2.17 has horizontal tangents at $x \approx 0, -1.6, 1.6$, is increasing for $x < -1.6$ and $x > 1.6$, and is decreasing for $-1.6 < x < 1.6$. Thus, the derivative will be positive for $x < -1.6$ and $x > 1.6$, negative for $0 < |x| < 1.6$, and zero for $x \approx 0, -1.6, 1.6$.

COMMENT:

You could have students explain why (a), (b), and (d) fail to be the correct answer.

## ConcepTests for Section 2.3

For Problems 1–3, the function $C(r)$ is the total cost of paying off a car loan borrowed at an interest rate of $r\%$ per year.

1. What are the units of $C'(r) = \dfrac{dC}{dr}$?

   (a) Year/$ (b) $/Year (c) $\$/(\%/\text{Year})$ (d) $(\%/\text{Year})/\$$

   ANSWER:
   (c)
   COMMENT:
   Remind students that many times looking at the units will help them solve the problem.

2. What is the practical meaning of $C'(5)$?

   (a) The rate of change of the total cost of the car loan is $C'(5)$.
   (b) If the interest rate increases by 1%, then the total cost of the loan increases by about $C'(5)$.
   (c) If the interest rate increases by 1%, then the total cost of the loan increases by about $C'(5)$ when the interest rate is 5%.
   (d) If the interest rate increases by 5%, then the total cost of the loan increases by about $C'(5)$.

   ANSWER:
   (c). $C'(5)$ requires the interest rate to be 5%.
   COMMENT:
   This problem is designed to help students make connections between real-world problems and mathematical concepts.

3. What is the sign of $C'(5)$?

   (a) Positive
   (b) Negative
   (c) Not enough information

   ANSWER:
   (a). If the interest rate increases, then the total car loan cost also increases, thus $C'(5)$ is positive.
   COMMENT:
   Students don't always realize how to make connections between real-world problems and mathematical concepts.

For Problems 4–5, you invest $1000 at an annual interest rate of $r\%$, compounded continuously. At the end of 10 years, you have a balance of $B$ dollars, where $B = g(r)$.

4. What is the financial interpretation of $g(5) = 1649$?

   (a) When $r$ is 5%, $B$ is $1649.
   (b) When the interest rate is 5, you have $1649.
   (c) When the interest rate is 5%, there is $1649.
   (d) When the interest rate is 5%, then in 10 years you have a balance of $1649.

   ANSWER:
   (d) contains the most information.
   COMMENT:
   Try to get your students to see the subtle differences between the four choices.

5. What is the financial interpretation of $g'(5) = 165$?

   (a) The balance in your account after 5 years is $165
   (b) The balance grows at a rate of $165 per % when $r = 5\%$.
   (c) If the interest rate increases from 5% to 6%, you would expect about $165 more in your account.
   (d) If the interest rate increases from 5% to 6% you would expect about $1814 in your account.

   ANSWER:
   (c) and (d) are equivalent. (d) uses the information from the previous problem.
   COMMENT:
   Students should start thinking of rate as an incremental change.

For Problems 6–7, assume $g(v)$ is the fuel efficiency, in miles per gallon, of a car going at a speed of $v$ miles per hour.

6. What are the units of $g'(v) = \dfrac{dg}{dv}$?
   (a) $(\text{miles})^2/(\text{gal})(\text{hour})$
   (b) hour/gal
   (c) gal/hour
   (d) $(\text{gal})(\text{hour})/(\text{miles})^2$
   (e) (miles/gallon)/(miles/hour)

   ANSWER:
   (b) (e)
   COMMENT:
   Ask students to think about the advantages and disadvantages of the two different forms of the answer.

7. What is the practical meaning of $g'(55) = -0.54$?
   (a) When the car is going 55 mph, the rate of change of the fuel efficiency decreases *to* approximately $0.54$ miles/gal.
   (b) When the car is going 55 mph, the rate of change of the fuel efficiency decreases *by* approximately $0.54$ miles/gal.
   (c) If the car speeds up from 55 mph to 56 mph, then the fuel efficiency is approximately $-0.54$ miles per gallon.
   (d) If the car speeds up from 55 mph to 56 mph, then the car becomes less fuel efficient by approximately 0.54 miles per gallon.

   ANSWER:
   (b) and (d) are equivalent, with (d) containing the most information. Notice that (a) and (c) are wrong.
   COMMENT:
   Students should strive to provide descriptions that are as complete as (d).

8. Let $P(t)$ be the population of California in year $t$. Then $P'(2005)$ represents:
   (a) The growth rate (in people per year) of the population.
   (b) The growth rate (in percent per year) of the population.
   (c) The approximate number of people by which the population increased in 2005.
   (d) The approximate percent increase in the population in 2005.
   (e) The average yearly rate of change in the population since $t = 0$.
   (f) The average yearly percent rate of change in the population since $t = 0$.

   ANSWER:
   (a). The derivative represents the rate of change of $P(t)$ in people per year. Answer (c) is approximately correct.
   COMMENT:
   Using $P$, have students write expressions for each of the other rates.

## ConcepTests for Section 2.4

1. The graph of $f(x)$ is shown in Figure 2.18. Which of the following are true for $f$ as shown in this window?

   (a) $f(x)$ is positive
   (b) $f(x)$ is increasing
   (c) $f'(x)$ is positive
   (d) $f'(x)$ is increasing
   (e) $f''(x)$ is non-negative

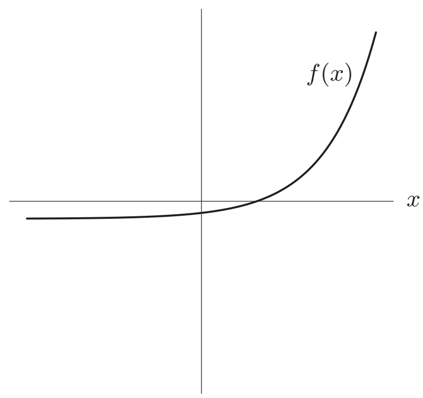



**Figure 2.18**

   ANSWER:
   (b), (c), (d), and (e)
   COMMENT:
   You could repeat this problem with other graphs.

2. If $f'(x)$ is positive, then $f''(x)$ is increasing.

   (a) True
   (b) False

   ANSWER:
   (b). $f'(x)$ positive means $f(x)$ is increasing. $f'(x) = x^4 - 8x^2 + 18$ provides a counterexample.
   COMMENT:
   Have students provide their own counterexample. You might also phrase this question in terms of concavity and give graphical counterexamples.

3. If $f'(x)$ is increasing, then $f(x)$ is increasing.

   (a) True
   (b) False

   ANSWER:
   (b). If $f'(x)$ is increasing, then the only acceptable conclusion is that $f(x)$ is concave up. For an example, consider $f'(x) = 2x$, then a possibility for $f(x)$ is $x^2$ which is not always increasing.
   COMMENT:
   Have students provide their own counterexample. You might also phrase this question in terms of concavity and give graphical counterexamples.

4. If $f''(x)$ is positive, then $f(x)$ is concave up.

   (a) True
   (b) False

   ANSWER:
   (a)
   COMMENT:
   You could ask what is true if $f''(x) < 0$.

5. If $f''(x)$ is positive, then $f'(x)$ is increasing.
   (a) True
   (b) False

   ANSWER:
   (a)
   COMMENT:
   You might note that $f''(x)$ is the rate of change of $f'(x)$.

6. If $f'(x)$ is increasing, then $f(x)$ is concave up.
   (a) True
   (b) False

   ANSWER:
   (a)
   COMMENT:
   You might note that $f'(x)$ increasing means $f''(x)$ is positive.

7. If the velocity of an object is constant, then its acceleration is zero.
   (a) True
   (b) False

   ANSWER:
   (a)
   COMMENT:
   **Follow-up Question.** If the velocity is zero at a specific instant in time, does the acceleration need to be zero at that same time also?

   **Answer.** No, a grapefruit that is tossed straight up in the air has a velocity of 0 ft/sec when the grapefruit reaches the highest point it will travel. However, at the point the acceleration of the grapefruit is that of gravity, which is not 0 ft/sec$^2$.

8. The value of the second derivative of the function shown in Figure 2.19 at the point $x = 1$ is

   (a) Positive (b) Negative

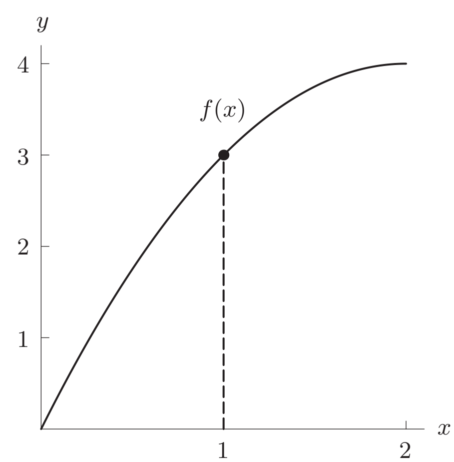



**Figure 2.19**

   ANSWER:
   (b). As $x$ increases, the slope of the tangent line decreases. Thus the second derivative is not positive.
   COMMENT:
   You could ask students if the magnitude of the second derivative of a function can be determined from the graph of the function. It cannot. For example, consider the function $f(x) = x^2$. It looks almost straight in places, i.e. no concavity, which would imply that the second derivative is zero. But, the value of the second derivative is always 2.

9. In Figure 2.20, the second derivative at points $a$, $b$, and $c$ is (respectively),

| | | |
|---|---|---|
| (a) $+, 0, -$ | (b) $-, 0, +$ | (c) $-, 0, -$ |
| (d) $+, 0, +$ | (e) $+, +, -$ | (f) $-, -, +$ |

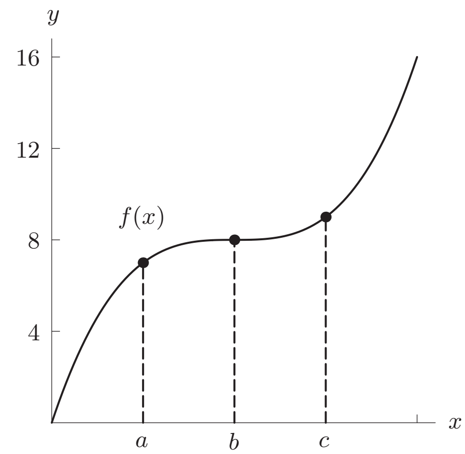



**Figure 2.20**

ANSWER:

(b). The graph is concave down at $a$, so $f''(a) \leq 0$ leaving (b), (c), and (f). The graph is concave up at $c$, so $f''(c) \geq 0$ leaving (b) and (f). The graph has an inflection point at $b$, so $f''(b) = 0$ leaving (b).

COMMENT:

See Problem 8.

10. In Figure 2.21, the second derivative at points $a$, $b$, and $c$ is (respectively),

| | | |
|---|---|---|
| (a) $+, 0, -$ | (b) $-, 0, +$ | (c) $-, 0, -$ |
| (d) $+, 0, +$ | (e) $0, +, 0$ | (f) $0, -, 0$ |

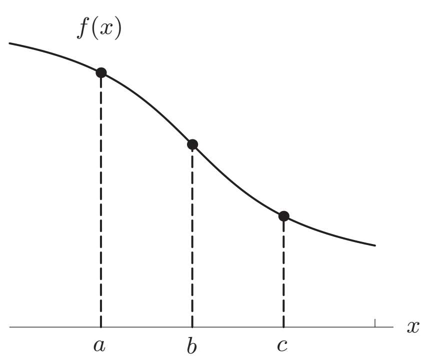



**Figure 2.21**

ANSWER:

(b). The graph is concave down at $a$, so $f''(a) \leq 0$ leaving (b), (c), (e), and (f). The graph is concave up at $c$, so $f''(c) \geq 0$ leaving (b), (e), and (f). The graph has an inflection point at $b$, so $f''(b) = 0$ leaving (b).

COMMENT:

See Problem 8.

11. In Figure 2.22, at $x = 0$ the signs of the function and the first and second derivatives, in order, are

(a) $+, 0, +$ (b) $+, 0, -$ (c) $-, +, -$
(d) $-, +, +$ (e) $+, -, +$ (f) $+, +, +$

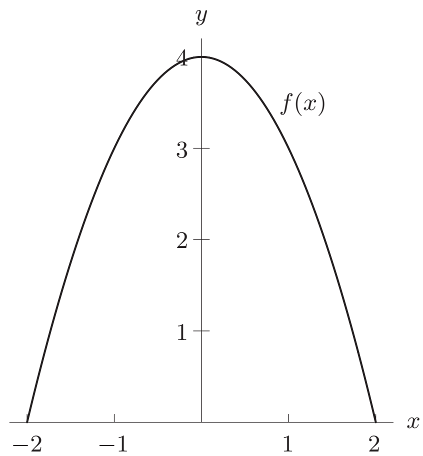



**Figure 2.22**

ANSWER:
(b). At $x = 0$ the graph is positive, has a horizontal tangent, and is concave down.
COMMENT:
See Problem 8.

12. In Figure 2.23, at $x = 0$ the signs of the function and the first and second derivatives, in order, are

(a) $-, +, +$ (b) $-, -, -$ (c) $-, +, -$
(d) $-, +, +$ (e) $+, -, +$ (f) $+, +, +$

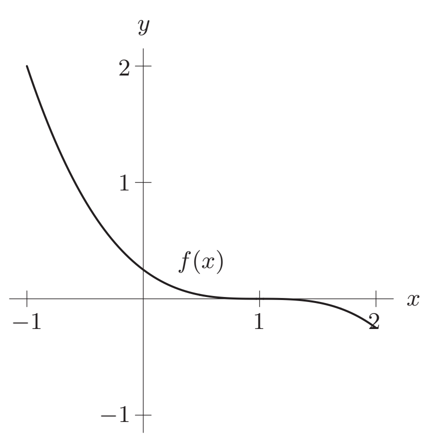



**Figure 2.23**

ANSWER:
(e). At $x = 0$ the graph is positive, decreasing, and concave up.
COMMENT:
See Problem 8.

13. In Figure 2.24, at $x = 0$ the signs of the function and the first and second derivatives, in order, are

(a) $+, 0, +$  (b) $-, 0, -$  (c) $+, 0, -$
(d) $-, +, 0$  (e) $+, -, 0$  (f) $+, +, +$

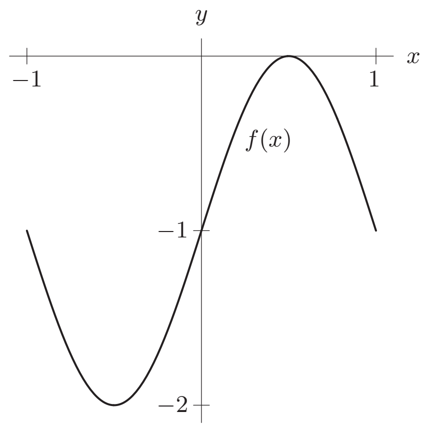



**Figure 2.24**

ANSWER:
(d). At $x = 0$ the graph is negative, increasing, and has an inflection point.
COMMENT:
See Problem 8.

14. Which of the following graphs (a)–(d) could represent the second derivative of the function in Figure 2.25 ?

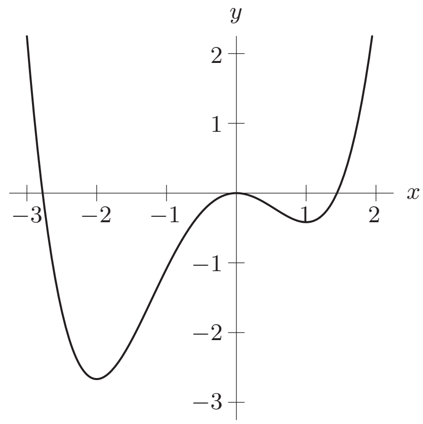
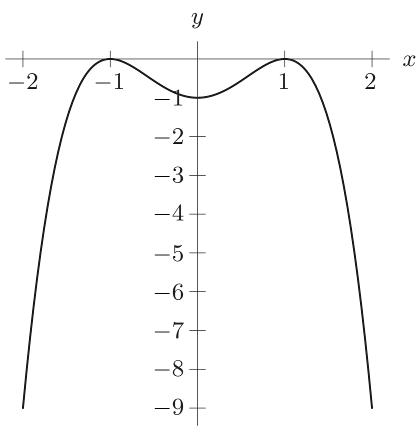
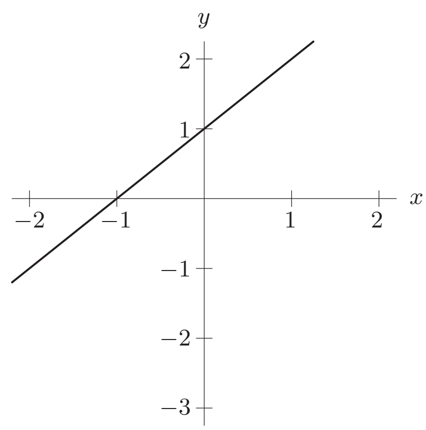



**Figure 2.25**

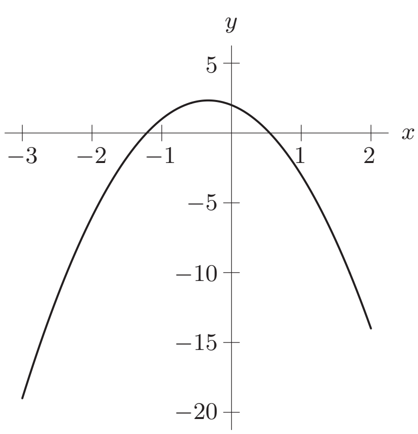
(a)



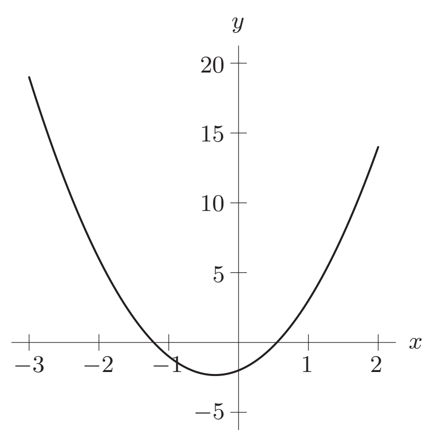

(b)




(c)




(d)




ANSWER:

(d). The graph in Figure 2.25 is concave up for $x < -1.2$ and $x > 0.5$ with inflection points at $x \approx -1.2$ and $0.5$. It is concave down elsewhere. So the second derivative is positive for $x < -1.2$ and $x > 0.5$, negative for $-1.2 < x < 0.5$, and zero at $x \approx -1.2$ and $0.5$.

COMMENT:

You could have students explain why (a), (b), and (c) fail to be the correct answer.

15. Which of the following graphs (a)–(d) could represent the second derivative of the function in Figure 2.26?

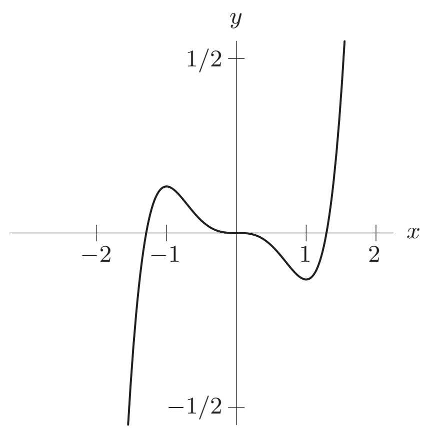
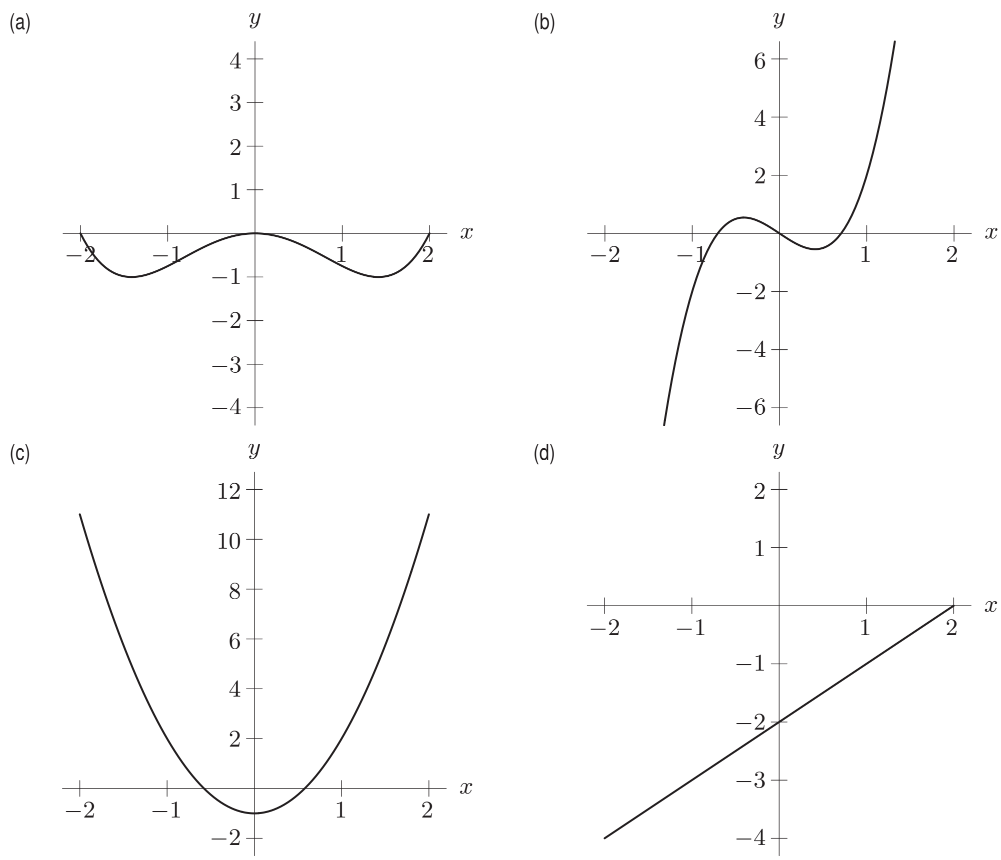



**Figure 2.26**




ANSWER:

(b). The graph in Figure 2.26 appears to be concave down for $-2 < x < -0.7$ and $0 < x < 0.7$. It is concave up elsewhere with inflection points at $x \approx -0.7, 0$, and $0.7$.

COMMENT:

You could have students explain why (a), (c), and (d) fail to be the correct answer.

16. Figure 2.27 shows position as a function of time for two sprinters running in parallel lanes. Which of the following is true?
   (a) At time $A$, both sprinters have the same velocity.
   (b) Both sprinters continually increase their velocity.
   (c) Both sprinters run at the same velocity at some time before $A$.
   (d) At some time before $A$, both sprinters have the same acceleration.

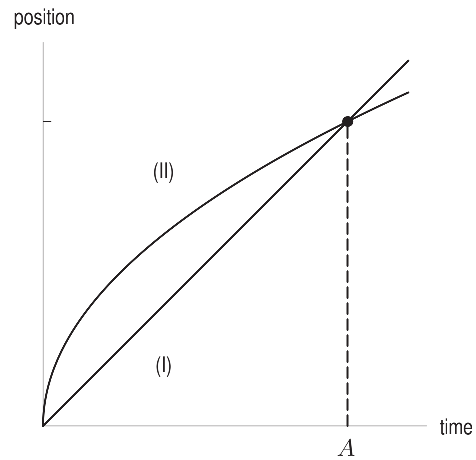



**Figure 2.27**

.

ANSWER:

(c). The sprinter whose position is given by (I) has a constant velocity, represented by the slope of the line. Since the slope of the curve (II) continually decreases, the velocity of the sprinter is continually decreasing. At $A$ both sprinters have the same position. The acceleration for sprinter (I) is zero, so the only true statement is (c). They have the same velocity when the slope of curve (II) is parallel with the line (I).

17. If an object's acceleration is negative, at that particular instant the object can be
   (a) Slowing down only
   (b) Speeding up only
   (c) Slowing down or momentarily stopped only
   (d) Slowing down, momentarily stopped, or speeding up

ANSWER:

(d). The acceleration of an object is the rate of change of its velocity with respect to time. If the acceleration is negative, its velocity is decreasing, but this tells us nothing about the value of the velocity.

COMMENT:

You could have students provide position graphs of an object with negative acceleration which satisfies (a), (b), and (c), respectively.

18. Figure 2.28 shows the graph of position versus time, $t$. Which of (a)–(d) represents a corresponding graph of acceleration as a function of time?

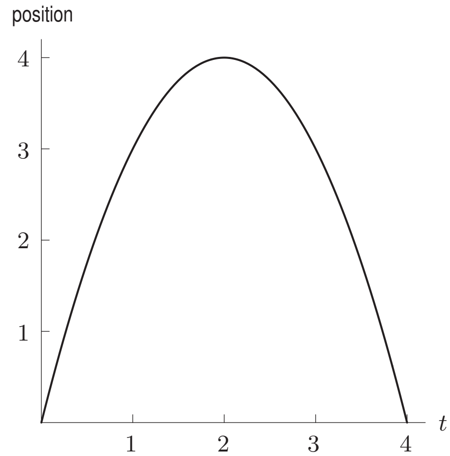
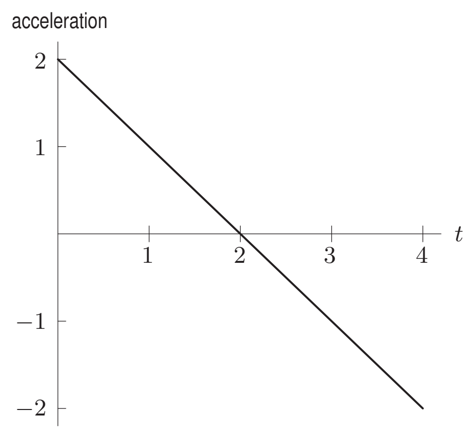
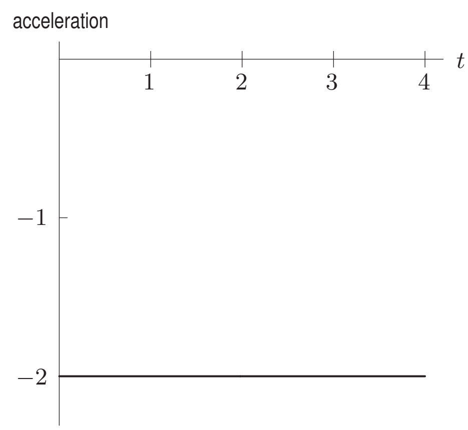



**Figure 2.28**

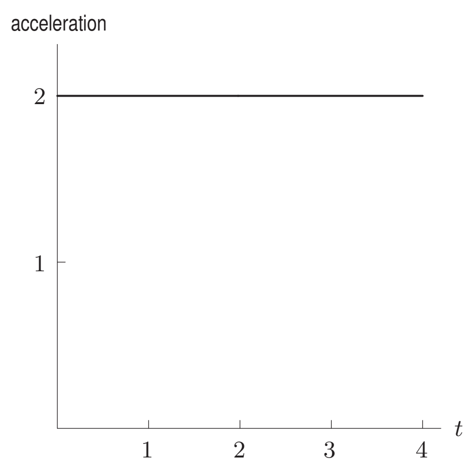
(a)



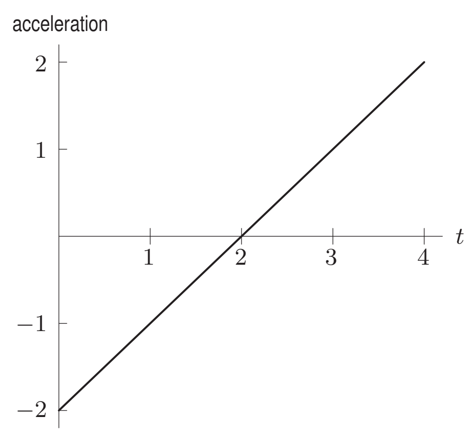

(b)




(c)




(d)




ANSWER:

(b). The position graph is concave down for $0 < t < 4$. Thus the acceleration is not positive for $0 < t < 4$.

COMMENT:

You could have students give specific points on the graphs of the other choices which have properties that are not consistent with Figure 2.28.

19. Figure 2.29 shows the graph of position versus time, $t$. Which of (a)–(d) represents a corresponding graph of acceleration as a function of time?

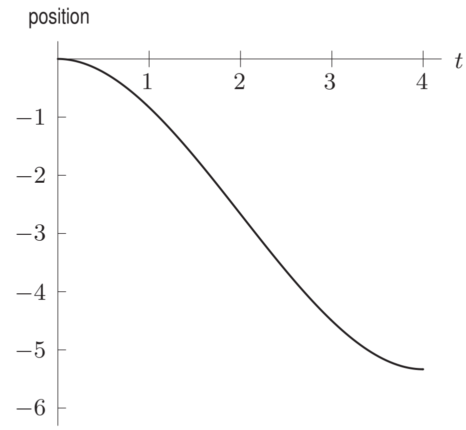
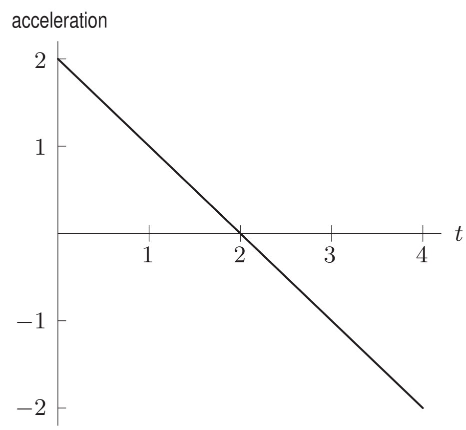
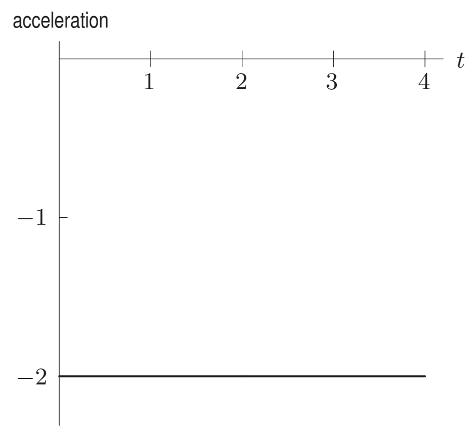



**Figure 2.29**

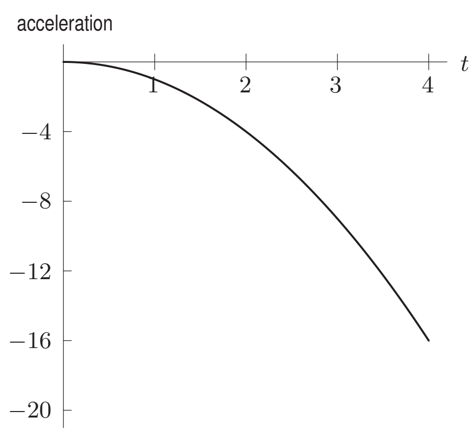
(a)



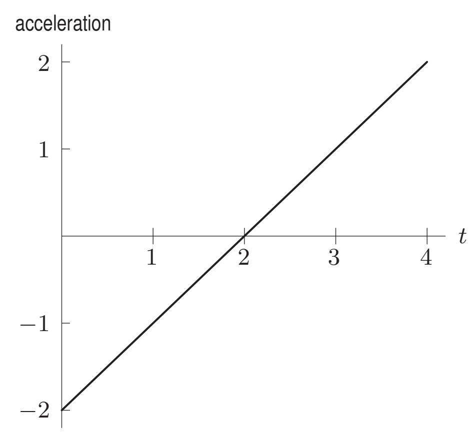

(b)




(c)




(d)




ANSWER:

(d). The graph appears to be concave down for $0 < t < 2$, concave up for $2 < t < 4$ with an inflection point at $t = 2$. Thus the acceleration is not positive for $0 < t < 2$, is not negative for $2 < t < 4$, and is zero at $t = 2$.

COMMENT:

You could have students give specific points on the graphs of the other choices which have properties that are not consistent with Figure 2.29.

20. Figure 2.30 represents acceleration as a function of time, $t$. Which of the following could represent the corresponding position versus time graph?

(a) (I) (b) (II) (c) (III)
(d) (I) and (II) (e) (I), (II), and (III) (f) None of these

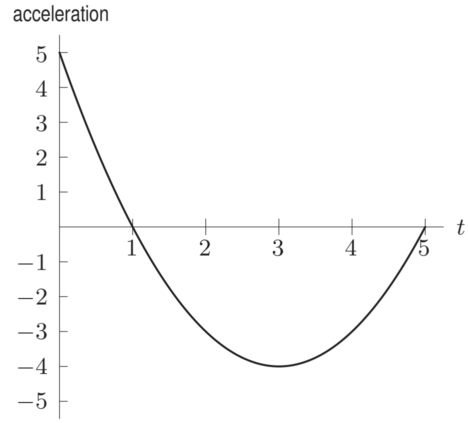
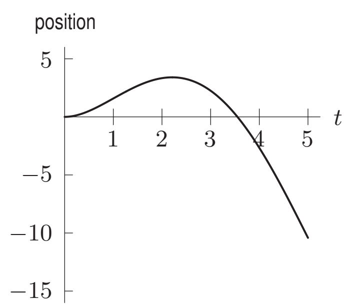
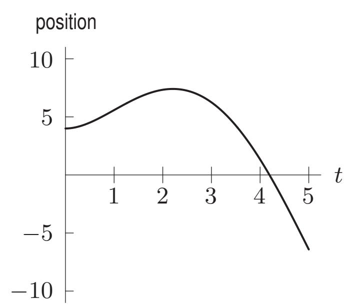



**Figure 2.30**

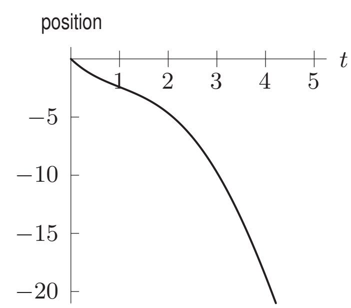
(I)




(II)




(III)




ANSWER:

(e). From Figure 2.30 we notice that the graph of the position function is concave up for $0 < t < 1$, is concave down for $1 < t < 5$, and has an inflection point when $t = 1$. Since the graphs shown in (I), (II), and (III) have these properties, then each could be a possible graph of the position function.

COMMENT:

You might point out that the graphs in (I) and (II) differ by a vertical translation.

21. At a specific instant in time, the radius of the universe was observed to be increasing. The second derivative of the radius with respect to time is known to to be always negative. Which of the following is true?

(a) The universe will keep expanding forever.
(b) At some point in the future the universe will stop expanding and begin contracting.
(c) With the given information either of these is a possibility.

ANSWER:

(c). A negative second derivative is possible for functions which are either increasing or decreasing.

COMMENT:

**Follow-up Question.** What if we knew that the radius was always increasing. Would that change the answer?
**Answer.** Yes, with this additional information we know that the first derivative is always positive. Therefore (a) is now correct. However, it does not say the radius grows without bound. It may approach an asymptotic value.

22. In *Star Trek: First Contact*, Worf almost gets knocked into space by the Borg. Assume he was knocked into space and his space suit was equipped with thrusters. Worf fires his thruster for 1 second which produces a constant acceleration in the positive direction. In the next second he turns off his thrusters. In the third second he fires his thrusters producing a constant negative acceleration. The acceleration as a function of time is given in Figure 2.31. Which of (a)–(d) represent his position versus time graph?

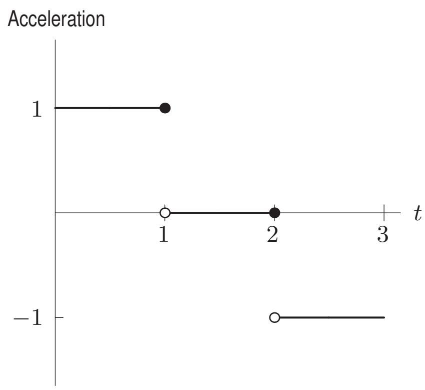
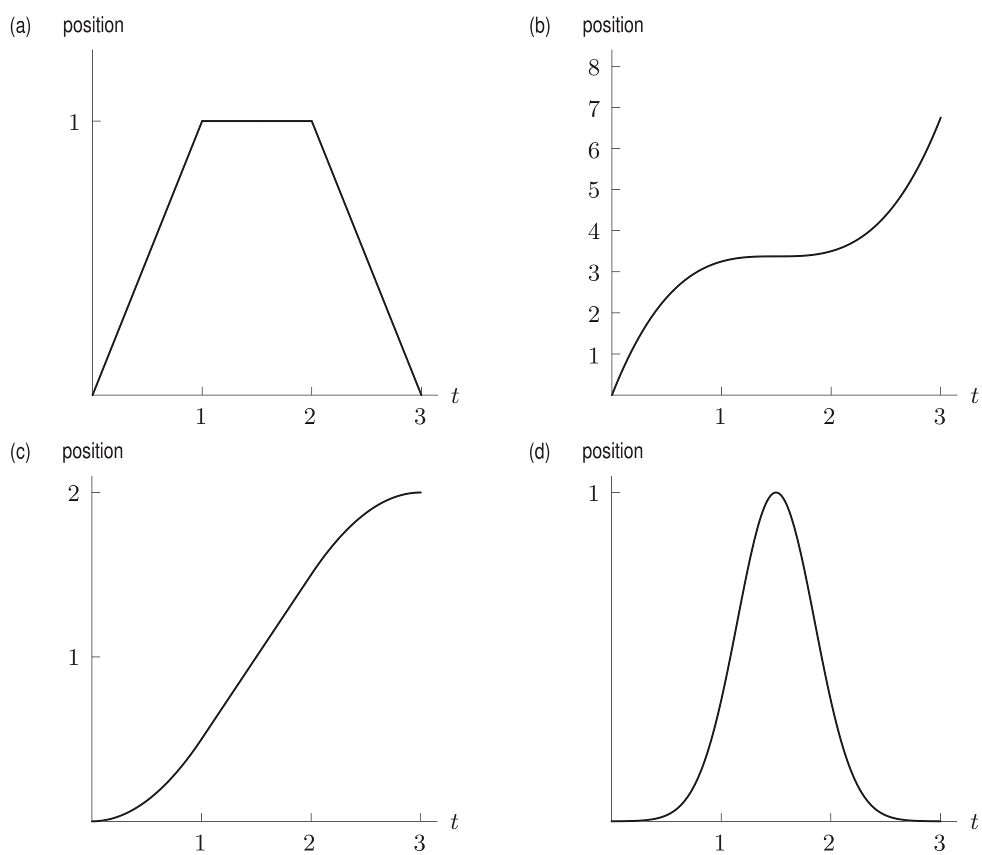



**Figure 2.31**



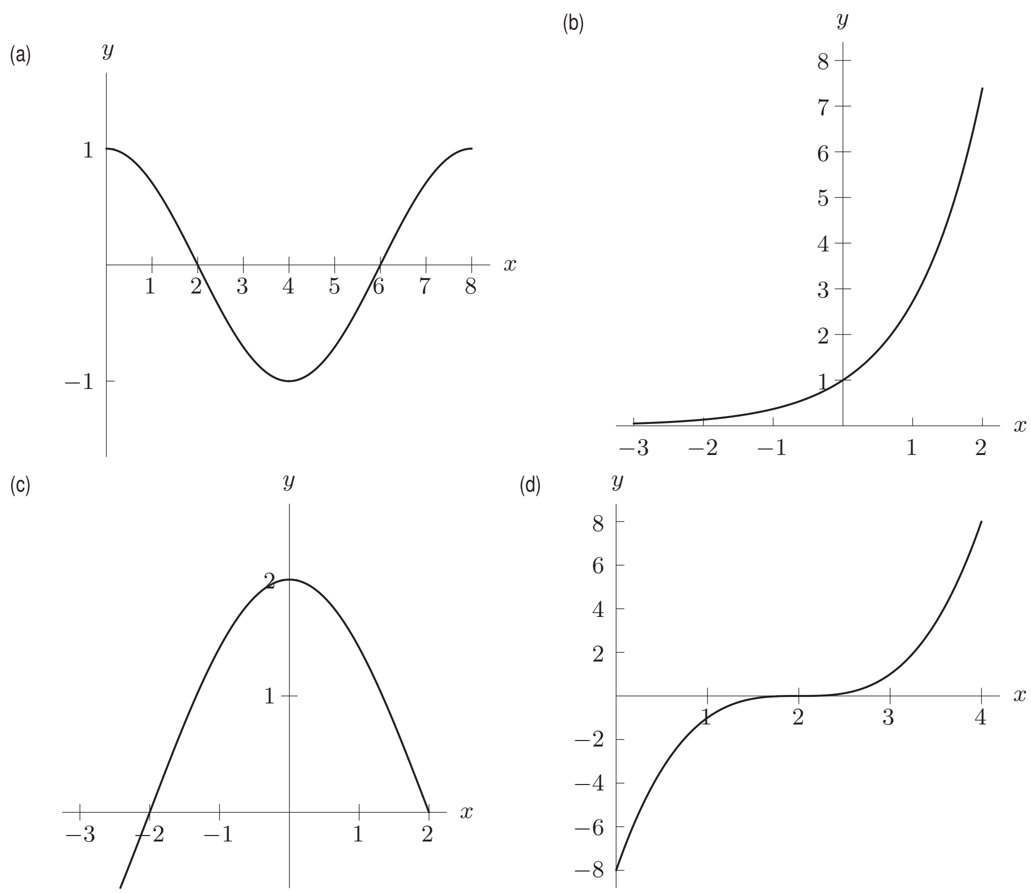

ANSWER:

(c). From the acceleration graph we see that the position graph will be concave up for $0 < t < 1$, concave down for $2 < t < 3$ and have a constant slope for $1 < t < 2$.

COMMENT:

You could have students give specific points on the graphs in the other choices which have properties not consistent with given acceleration graph.

23. Which of the following graphs satisfies the relationship $f''(x) = -f(x)$?




ANSWER:

(a). Functions that satisfy $f''(x) = -f(x)$ will be concave down where the function is positive and concave up where it is negative. Inflection points occur where the function is zero. The answer (c) would also be correct if we could tell that inflection points occurred at $x = \pm 2$.

COMMENT:

You could have students give specific points on the graphs in the other choices which have properties not consistent with the fact that $f''(x) = -f(x)$.

## ConcepTests for Section 2.5

1. The cost function, $C$, and revenue function, $R$, for a company are shown in Figure 2.32. If the company is currently producing 3000 units, which of the following is true?
   (a) The company is making a profit and should increase production.
   (b) The company is not making a profit and should increase production.
   (c) The company is making a profit and should decrease production.
   (d) The company is not making a profit and should decrease production.

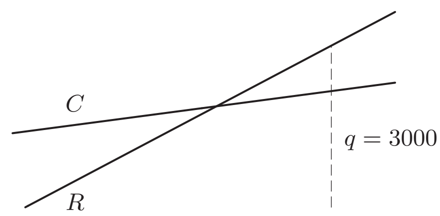



**Figure 2.32**

ANSWER:

(a) At $q = 3000$, we see that Revenue $R$ is above Cost $C$ so the company is making a profit. Since the marginal revenue (the slope of $R$) is greater than marginal cost (the slope of $C$), the company should increase production.

2. The cost function, $C$, and revenue function, $R$, for a company are shown in Figure 2.33. If the company is currently producing 2000 units, which of the following is true?
   (a) The company is making a profit and should increase production.
   (b) The company is not making a profit and should increase production.
   (c) The company is making a profit and should decrease production.
   (d) The company is not making a profit and should decrease production.

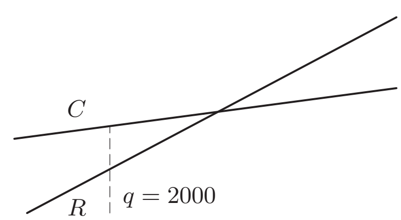



**Figure 2.33**

ANSWER:

(b) At $q = 2000$, we see that Revenue $R$ is below Cost $C$ so the company is losing money. Since the marginal revenue (the slope of $R$) is greater than marginal cost (the slope of $C$), the company will make a profit on the 2001$^{\text{st}}$ item and so should increase production.

Students might argue that since the company is losing money, it should decrease production (perhaps to zero!). This is a valid argument, and you should encourage a discussion about what additional information might be useful to make a good decision.

3. The cost $C$ and revenue $R$ functions for a certain company are shown in Figure 2.34. If the company is currently producing 750 units, which of the following is true?
   (a) The company is making a profit and should increase production.
   (b) The company is not making a profit and should increase production.
   (c) The company is making a profit and should decrease production.
   (d) The company is not making a profit and should decrease production.

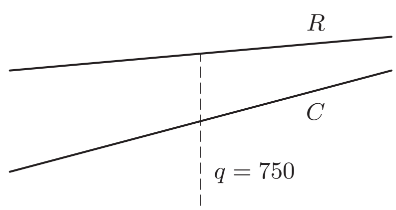



**Figure 2.34**

ANSWER:

(c) At $q = 750$, we see that Revenue $R$ is above Cost $C$ so the company is making money. Since the marginal revenue (the slope of $R$) is less than marginal cost (the slope of $C$), the company will lose money by making an additional unit. The company should decrease production.

## ConcepTests for Limits and the Definition of the Derivative

**1.** Possible criteria for continuity of a function at a point: *If the limit of the function exists at a point, the function is continuous at that point.* Which of the following examples fits the above criteria but is not continuous at $x = 0$?

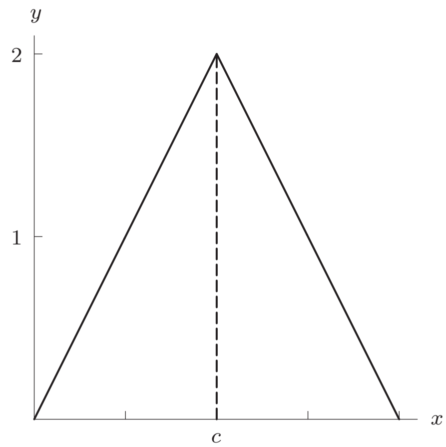
**(a)** $f(x) = x$. The limit of $f(x)$, as $x$ goes to 0, is 0 so this function is continuous at $x = 0$.
**(b)** $f(x) = x^2/x$. The limit of $f(x)$, as $x$ goes to 0, is 0 therefore $f(x)$ is continuous at $x = 0$.
**(c)** $f(x) = |x|/x$. The limit of $f(x)$, as $x$ goes to 0, is 1 therefore $f(x)$ is continuous at $x = 0$.
**(d)** None of these show a problem with this criteria.

ANSWER:
(b). $f(0)$ is not defined, so $f$ cannot be continuous at $x = 0$.
COMMENT:
You could ask students why the limit does not exist in choice (c).

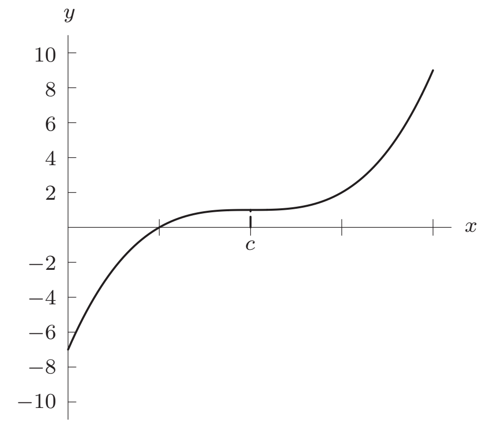
**2.** Definition of continuity of a function at a point: *If the limit of the function exists at a point and is equal to the function evaluated at that point, then the function is continuous at that point.* Which of the following is not continuous at $x = c$?

**(a)** (I) only
**(b)** (II) only
**(c)** (III) only
**(d)** (I) and (II)
**(e)** (I) and (III)
**(f)** (II) and (III)

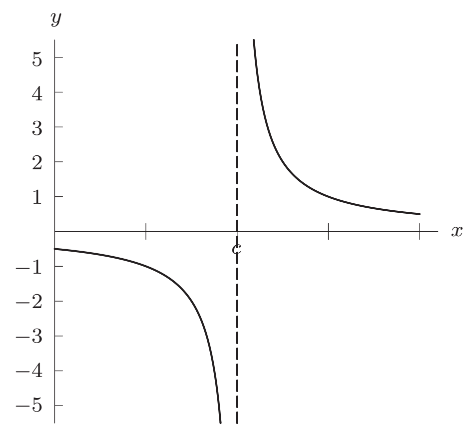
(I)




(II)




(III)




ANSWER:
(c)
COMMENT:
You could have students give reasons why the other choices are continuous functions.

**3.** If a function is continuous at a point, then it is differentiable at that point.

**(a)** True
**(b)** False

ANSWER:
(b). For example, the function $f(x) = |x|$ is continuous at $x = 0$, but it is not differentiable there.
COMMENT:
You might have students give other examples. ($f(x) = x^{1/3}$ is one.)