Chapter Three

Chapter Three

ConcepTests for Section 3.1

1. If $f'(x) = g'(x)$, then $f(x) = g(x)$.
   (a) True
   (b) False

   ANSWER:
   (b). If $f'(x) = g'(x)$, then $f(x) = g(x) + C$, where C is some constant.
   COMMENT:
   You might point out that the graphs of f and g differ by a vertical shift. A student's instinct is to answer **True**. Have students suggest their own counterexamples.

2. If $y = \pi^5$, then $y' = 5\pi^4$.
   (a) True
   (b) False

   ANSWER:
   (b). Since π^5 is a constant, then $y' = 0$.
   COMMENT:
   This question seems remarkably obvious in class. However, on an exam students tend to miss this question.

3. If $y = (x+1)(x+2)(x+3)(x+4)$, then $\dfrac{d^5y}{dx^5} = 0$.
   (a) True
   (b) False

   ANSWER:
   (a). y is a polynomial of degree four.
   COMMENT:
   Make sure your students realize that they don't need to take a derivative to answer this question.

4. The graph of a function f is given in Figure 3.1. If f is a polynomial of degree 3, then the value of $f'''(0)$ is
   (a) Positive
   (b) Negative
   (c) Zero

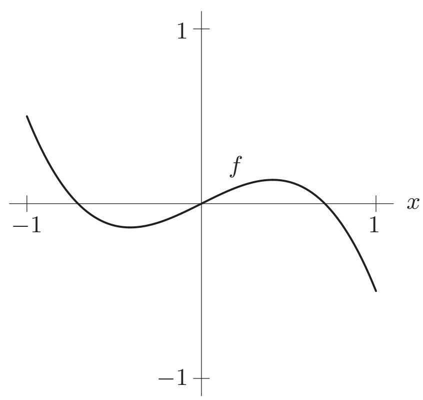


Figure 3.1

   ANSWER:
   (b). Because the graph of this polynomial of degree 3 is negative for large values of x, the coefficient of x^3 will be negative. (Recall the third derivative of a polynomial of degree 3 is a constant.)
   COMMENT:
   You could ask students why $f(x)$ could not become positive for $x > 1$.

5. The graph of a function f is given in Figure 3.2. If f is a polynomial of degree 3, then $f'''(0)$ is
   (a) Positive
   (b) Negative
   (c) Zero

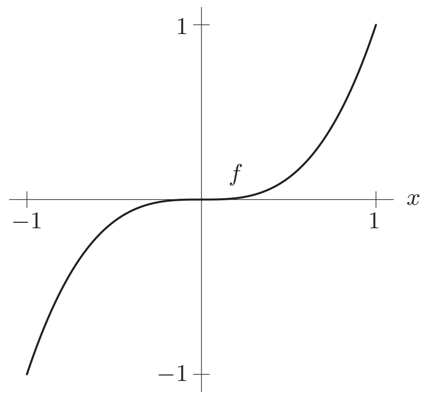


Figure 3.2

ANSWER:

(a). Because the graph of this polynomial of degree 3 is positive for large values of x, the coefficient of x^3 will be positive. (Recall the third derivative of a polynomial of degree 3 is a constant.)

COMMENT:

You could ask students if there could be other inflection points for f.

6. The graph of a function f is given in Figure 3.3. If f is a polynomial of degree 3, then the values of $f'(0)$, $f''(0)$, and $f'''(0)$ are (respectively)
   (a) $0, 0, +$
   (b) $0, 0, -$
   (c) $0, +, -$
   (d) $0, -, -$
   (e) $+, -, +$
   (f) $0, +, +$

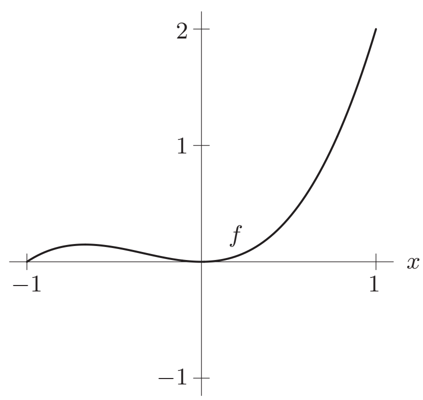


Figure 3.3

ANSWER:

(f). There is a horizontal tangent at the origin, so $f'(0) = 0$. The graph shows that f has horizontal intercepts at -1 and 0, with a double root at 0. Thus f has the form $f(x) = k(x + 1)x^2$. Because $f(x) > 0$ for $x > 0$, then $k > 0$. So $f'(x) = k(3x^2 + 2x)$, $f''(x) = k(6x + 2)$, and $f'''(x) = 6k$.

COMMENT:

You could ask students why a double root at zero means that f has a factor of x^2.

7. The graph of a function f is given in Figure 3.4. If f is a polynomial of degree 3, then the values of $f'(0)$, $f''(0)$, and $f'''(0)$ are (respectively)

(a) $+, 0, +$	(b) $-, 0, -$	(c) $+, 0, -$
(d) $+, -, -$	(e) $+, -, +$	(f) $+, +, +$

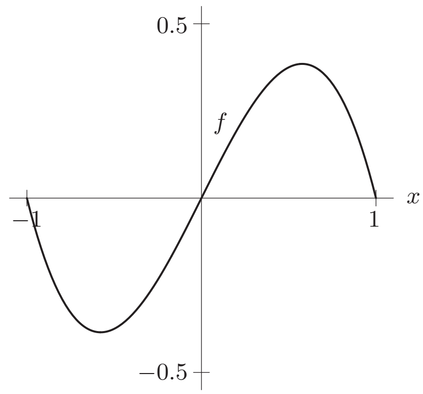


Figure 3.4

ANSWER:

(c). The graph shows that f has horizontal intercepts at $x = -1, 0$, and 1. Thus f has the form $f(x) = kx(x+1)(x-1)$. Because $f(x) > 0$ for $0 < x < 1$, we have $k < 0$. Then $f'(x) = k(3x^2 - 1)$, $f''(x) = k(6x)$, and $f'''(x) = 6k$.

COMMENT:

You could ask why there could not be another horizontal intercept outside this viewing window.

8. The graph of a function f is given in Figure 3.5. If f is a polynomial of degree 3, then the values of $f'(0)$, $f''(0)$, and $f'''(0)$ are (respectively)

(a) $-, +, +$	(b) $-, -, -$	(c) $-, +, -$
(d) $-, +, +$	(e) $+, -, +$	(f) $+, +, +$

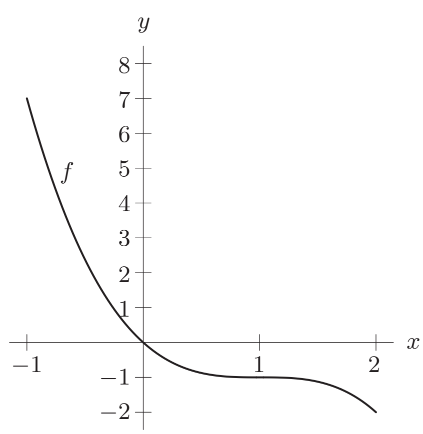


Figure 3.5

ANSWER:

(c). At $x = 0$, the graph is decreasing and concave up, so $f'(0) < 0$ and $f''(0) \geq 0$. Because the graph becomes more negative as x increases beyond 1, the sign of the coefficient of x^3 (and thus the sign of $f'''(0)$) is negative. (Recall the third derivative of a polynomial of degree 3 is a constant.)

COMMENT:

You could ask why the graph of this function could not become positive for larger values of x.

9. The graph of a function f is given in Figure 3.6. If f is a polynomial of degree 3, then the values of $f'(0)$, $f''(0)$, and $f'''(0)$ are (respectively)

(a) $-,-,+$	(b) $-,0,-$	(c) $-,+,-$
(d) $-,+,+$	(e) $+,-,+$	(f) $+,+,+$

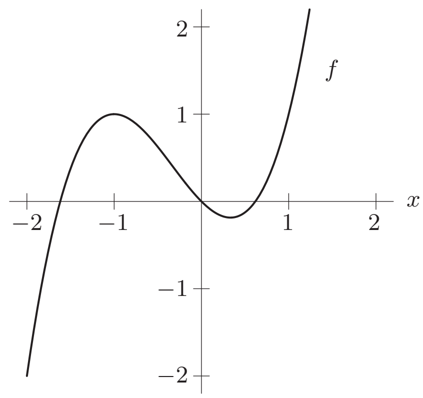


Figure 3.6

ANSWER:

(d). At $x = 0$, the graph is decreasing and concave up, so $f'(0) < 0$ and $f''(0) \geq 0$. Because the graph is positive as x increases beyond 0.5, the sign of the coefficient of x^3 (and thus the sign of $f'''(0)$) is positive. (Recall the third derivative of a polynomial of degree 3 is a constant.)

COMMENT:

You could ask why the graph of this function could not become negative for larger values of x.

ConcepTests for Section 3.2

1. Figure 3.7 shows the graph of $f(x) = a^x$ and its derivative $f'(x)$. Which of the following are possible values for the parameter a?

(a) $a = 1$
(b) $a = 2$
(c) $a = 3$
(d) $a = 4$
(e) None of the above

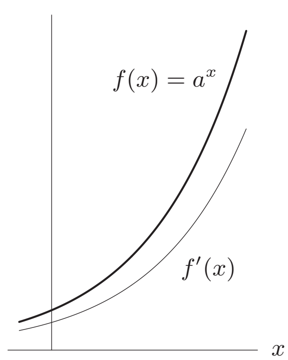


Figure 3.7

ANSWER:

(b) The only possible value of a of the options listed is $a = 2$. The value $a = 1$ is not possible, since the graph of $f(x) = 1^x = 1$ is a horizontal line. Since the derivative of $f(x) = a^x$ lies below the function, we know that $a < e = 2.71828...$ since the graph of the derivative of $f(x) = e^x$ lies exactly on the graph of $f(x) = e^x$. The only values of a that give a graph similar to the one shown (with the derivative below the function) are values of a between 1 and e.

Alternately, we notice that $\ln a$ must be less than 1 for the graph of the derivative $f'(x) = \ln a \cdot a^x$ to lie below the graph of $f(x) = a^x$. Since $\ln 3 > 1$ and $\ln 4 > 1$, the only possible option is $a = 2$.

ConcepTests for Section 3.3

1. Given the graphs of the functions $f(x)$ and $g(x)$ in Figures 3.8 and 3.9, which of (a)–(d) is a graph of $f(g(x))$?

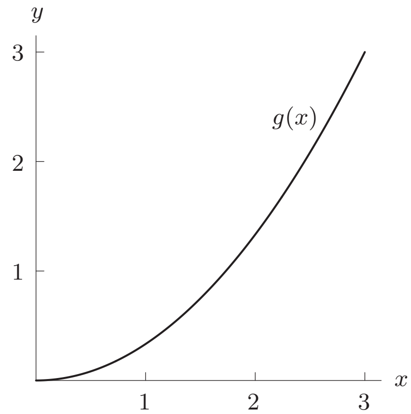


Figure 3.8

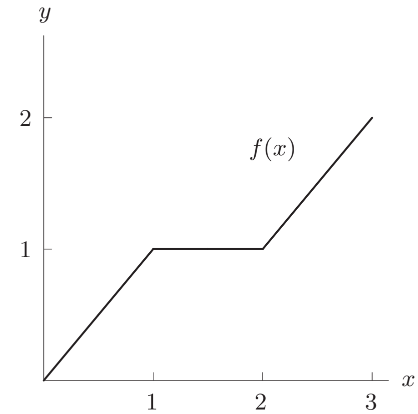
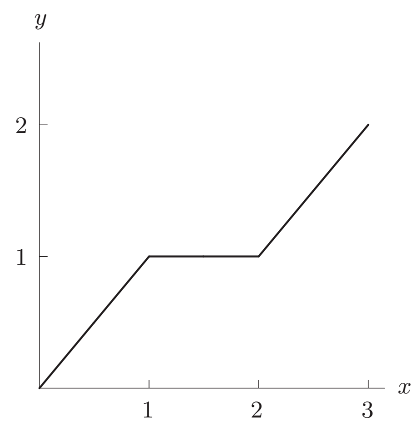
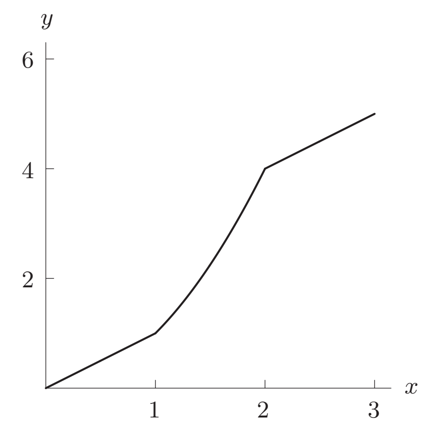


Figure 3.9

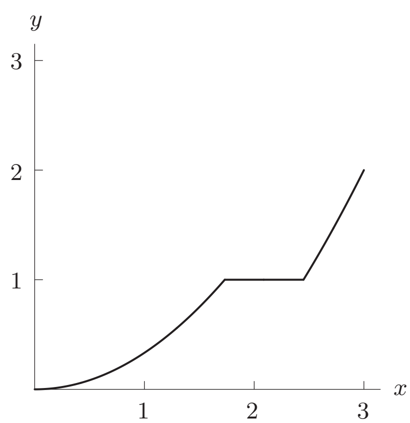
(a)



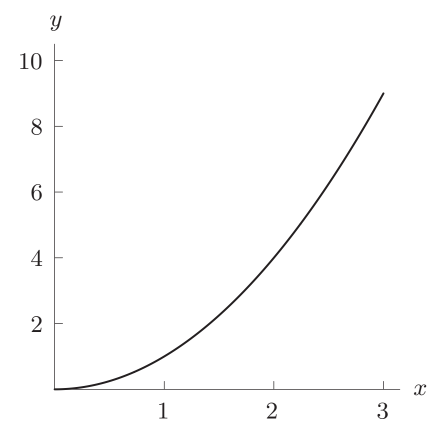
(b)



(c)



(d)



ANSWER:

(c). Because $(f(g(x)))' = f'(g(x))g'(x)$, we see $f(g(x))$ has a horizontal tangent whenever $g'(x) = 0$ or $f'(g(x)) = 0$. Now, $f'(g(x)) = 0$ for $1 < g(x) < 2$ and this approximately corresponds to $1.7 < x < 2.5$.

COMMENT:

You could have students give specific places in the other choices where there was a conflict with information given in the graphs of f and g.

2. Given the graphs of the functions $f(x)$ and $g(x)$ in Figures 3.10 and 3.11, which of (a)–(d) is a graph of $f(g(x))$?

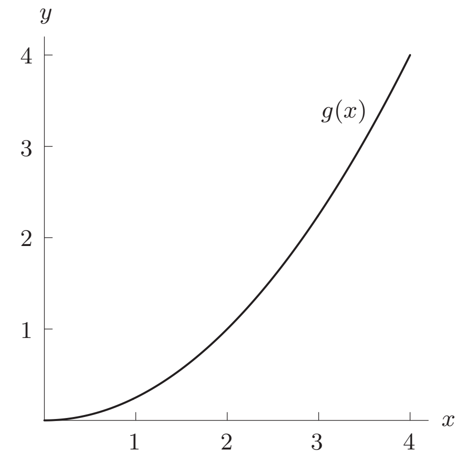


Figure 3.10

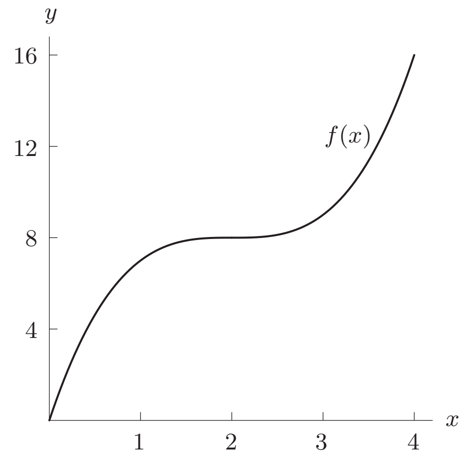
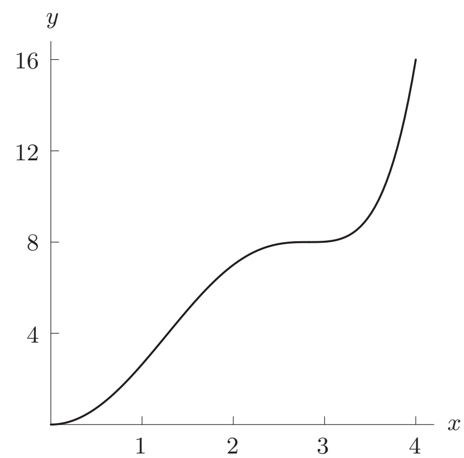
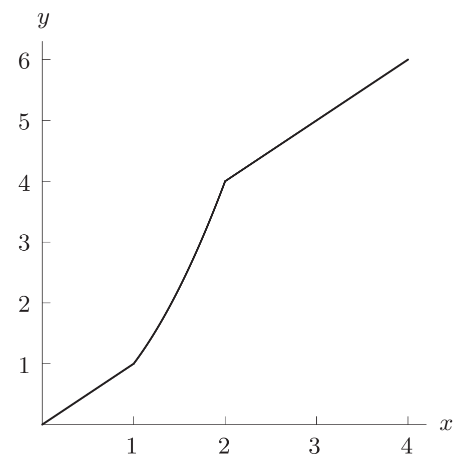


Figure 3.11

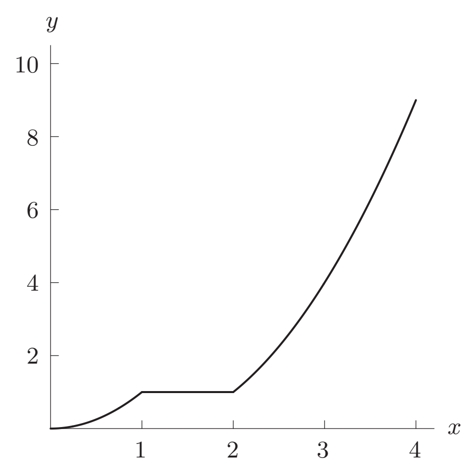
(a)



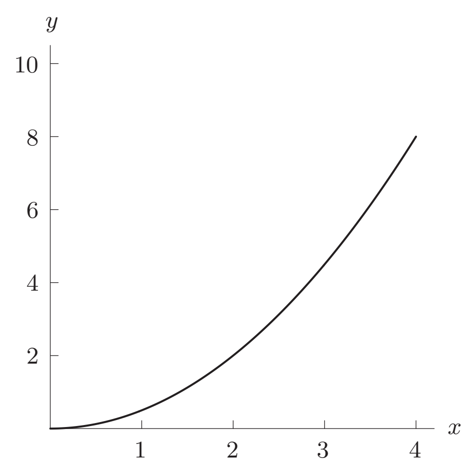
(b)



(c)



(d)



ANSWER:

(a). Because $(f(g(x)))' = f'(g(x))g'(x)$, we see $f(g(x))$ has a horizontal tangent whenever $g'(x) = 0$ or $f'(g(x)) = 0$. Now $f'(x) = 0$ only when $x = 2$, so the composite function only has horizontal tangents when $g'(x) = 0$ or when $g(x) = 2$.

COMMENT:

Students may want to check points, which is tedious. This is an opportunity to show the power of reasoning based on the chain rule.

3. Given the graphs of the function $g(x)$ and $f(x)$ in Figures 3.12 and 3.13 , which of (a)–(d) represents $f(g(x))$?

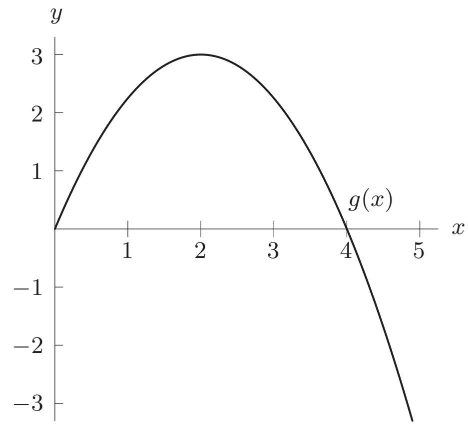


Figure 3.12

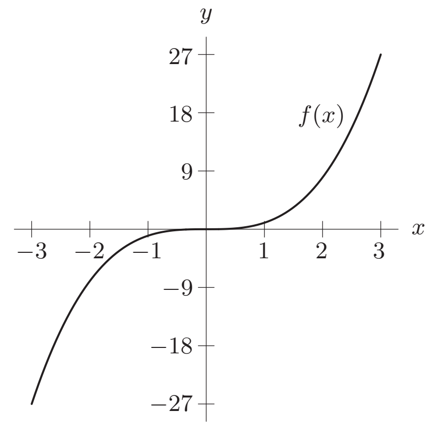
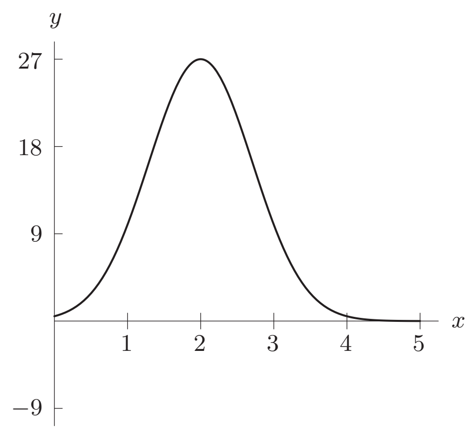
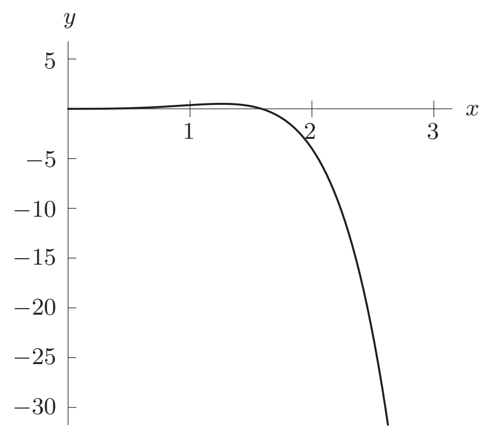


Figure 3.13

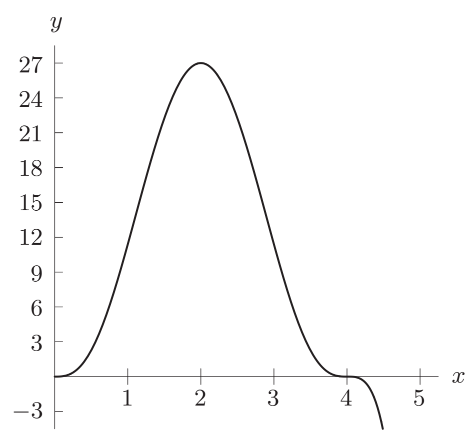
(a)



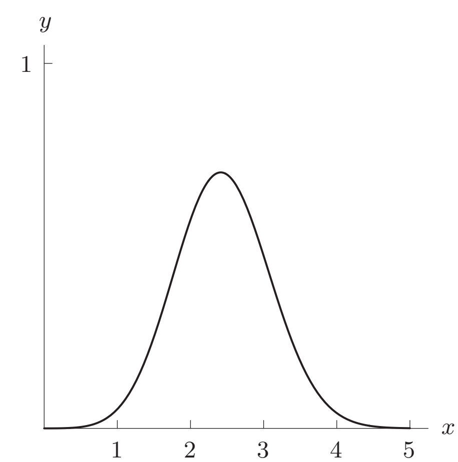
(b)



(c)



(d)



ANSWER:

(c). Because $(f(g(x)))' = f'(g(x))g'(x)$, we see $f(g(x))$ has a horizontal tangent whenever $g'(x) = 0$ or $g(x) = 0$. This happens when $x = 0, 2,$ and 4. $f(g(x))$ is also negative for $x > 4$. Alternatively, $f(g(4)) = f(0) = 0$ identifies answer (c).

COMMENT:

You could have students give specific places in the other choices where there was a conflict with information given in the graphs of f and g.

ConcepTests for Section 3.4

1. To differentiate the function
$$\frac{(x^2+5)^3}{\sqrt{3x+1}},$$
begin by using
(a) The product rule
(b) The quotient rule
(c) The chain rule
(d) None of the above

ANSWER:
(b) The "outside function" here is a quotient of two inside functions, so we begin with the quotient rule.
COMMENT:
If your students can argue a way to rewrite the expression so that one of the other rules is more appropriate, that would be great! You might ask them if there is any way to rewrite the expression so that the chain rule is more appropriate, for example, or the product rule. All are possible, depending on whether we rewrite the expression as
$$\sqrt{\frac{(x^2+5)^6}{3x+1}}$$
for the chain rule, or
$$(x^2+1)^3 \cdot (3x+1)^{-1/2}$$
for the product rule, or a variety of other possible ways.

2. To differentiate the function
$$\sqrt{\frac{2x+1}{5x-2}},$$
begin by using
(a) The product rule
(b) The quotient rule
(c) The chain rule
(d) None of the above

ANSWER:
(c) The "outside function" here is the square root of a quotient, so the outside function is to raise something to the $1/2$ power and we would begin with the chain rule.
COMMENT:
If your students can argue a way to rewrite the expression so that one of the other rules is more appropriate, that would be great! You might ask them if there is any way to rewrite the expression so that the quotient rule is more appropriate, for example, or the product rule. All are possible, depending on whether we rewrite the expression as, for example,
$$\frac{\sqrt{2x+1}}{\sqrt{5x-2}}$$
for the quotient rule, or
$$\sqrt{2x+1} \cdot (5x-2)^{-1/2}$$
for the product rule, or a variety of other possible ways.

3. To differentiate the function
$$x^2 \cdot \sqrt{5x+3},$$
begin by using
(a) The product rule
(b) The quotient rule
(c) The chain rule
(d) None of the above

ANSWER:
(a) This is the product of two functions, so the product rule makes the most sense.
COMMENT:
If your students can argue a way to rewrite the expression so that one of the other rules is more appropriate, that would be great! You might ask them if there is any way to rewrite the expression so that the quotient rule is more appropriate, for example, or the chain rule. All are possible, depending on whether we rewrite the expression as, for example,

$$\sqrt{x^4 \cdot (5x+3)}$$

for the chain rule, or (if the students really get creative!)

$$\frac{\sqrt{5x+3}}{x^{-2}}$$

for the quotient rule, or a variety of other possible ways.

4. To differentiate the function

$$(x^2+1)^3 \cdot \sqrt{\frac{5x}{x^3+4}},$$

begin by using

(a) The product rule
(b) The quotient rule
(c) The chain rule
(d) None of the above

ANSWER:
(a) This is the product of two functions, so the product rule makes the most sense.
COMMENT:
If your students can argue a way to rewrite the expression so that one of the other rules is more appropriate, that would be great! You might ask them if there is any way to rewrite the expression so that the quotient rule is more appropriate, for example, or the chain rule.

5. To differentiate the function

$$f(x) = \frac{1}{x+1},$$

it would be appropriate to use (check all that apply)

(a) The product rule
(b) The quotient rule
(c) The chain rule

ANSWER:
(b) and (c). If students recognize this function as $(x+1)^{-1}$, the chain rule is the best way to find this derivative.

ConcepTests for Section 3.5

1. For the function

$$f(x) = 3\sin(2x) + 5,$$

give the following four values:
Amplitude of $f(x)$;
Maximum value of $f(x)$;
Amplitude of $f'(x)$;
Maximum value of $f'(x)$.

(a) 3; 8; 6; 11
(b) 8; 8; 6; 6
(c) 3; 3; 3; 0
(d) 3; 8; 6; 6
(e) 6; 3; 3; 8

ANSWER:

(d). The amplitude of $f(x) = 3\sin(2x) + 5$ is 3 and the maximum value is $5 + 3 = 8$. The amplitude of $f'(x) = 6\cos(2x)$ is 6 and the maximum value is 6.

2. List in order (from smallest to largest) the following functions as regards to the maximum possible value of their slope.

(a) $\sin x$ (b) $\sin(2x)$ (c) $\sin(3x)$ (d) $\sin(x/2)$

ANSWER:

(d), (a), (b), (c). The derivative of $\sin(\alpha x)$ is $\alpha\cos(\alpha x)$. The largest value of $\cos(\alpha x)$ is 1, so the ranking is based only on the value of α.

COMMENT:

You could mimic this question replacing the sine function by cosine.

3. At which of the following values of x does $\sin(4\pi x^2)$ attain the largest slope?

(a) $x = 0$ (b) $x = 1/2$ (c) $x = 1$ (d) $x = 2$

ANSWER:

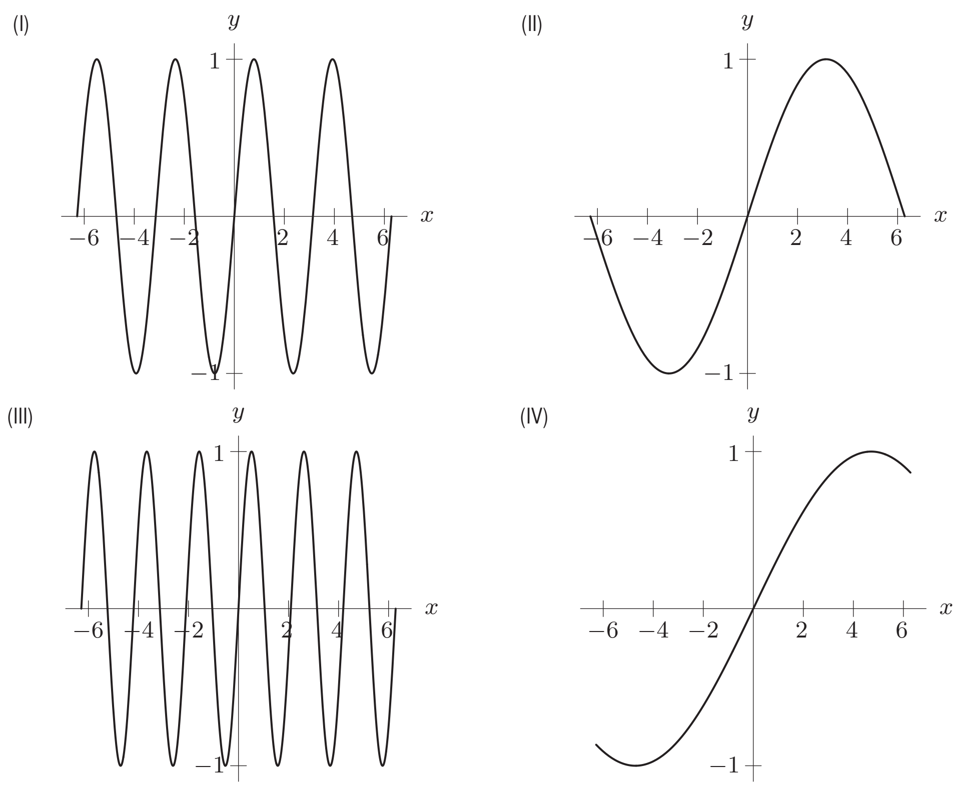
(d). The derivative of $\sin(4\pi x^2)$ is $8\pi x\cos(4\pi x^2)$. At these specific values of x, we have the values $0, -4\pi, 8\pi$, and 16π.

COMMENT:

You could mimic this question replacing the sine function by cosine.

4. Which of the graphs are those of $\sin(2x)$ and $\sin(3x)$?

(a) (I) $= \sin(2x)$, (II) $= \sin(3x)$ (b) (I) $= \sin(2x)$, (III) $= \sin(3x)$
(c) (II) $= \sin(2x)$, (III) $= \sin(3x)$ (d) (III) $= \sin(2x)$, (IV) $= \sin(3x)$



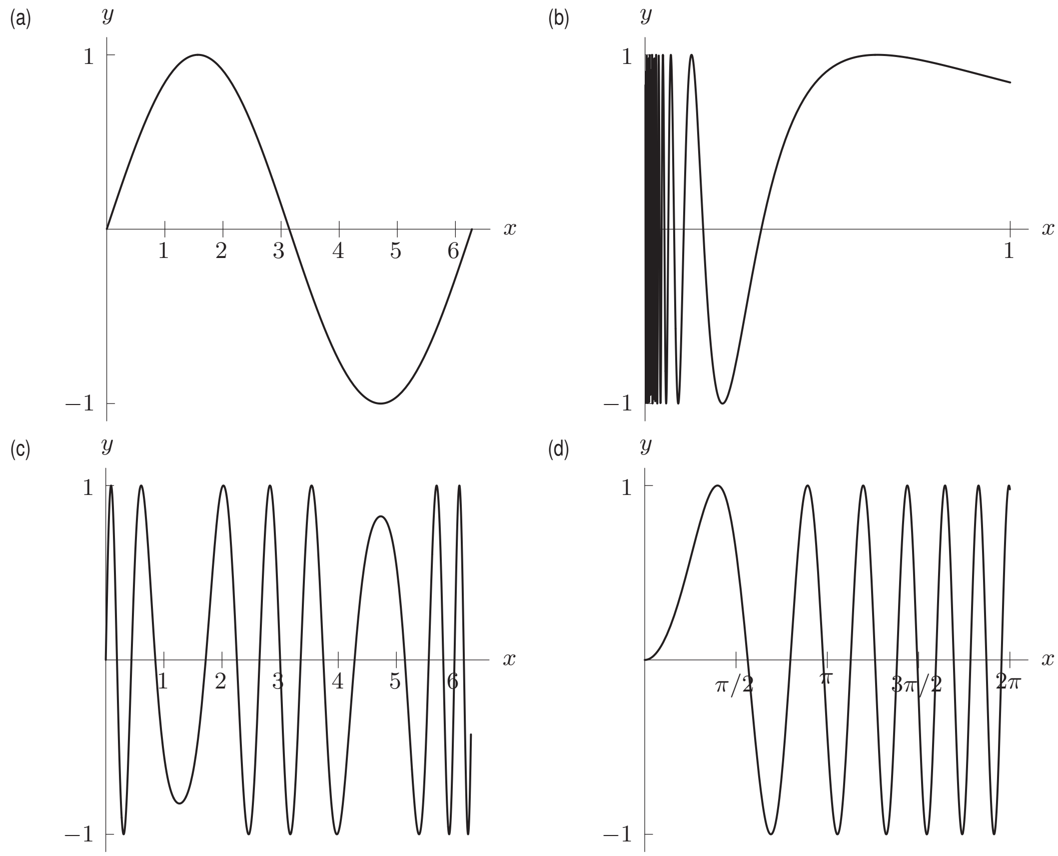
ANSWER:

(b). Because $\frac{d}{dx}\sin(\alpha x) = \alpha\cos(\alpha x)$, the graph of $\sin(3x)$ will have steeper slopes than $\sin(2x)$. The first positive zeros are at $x = \pi/2$ for $\sin(2x)$ and $x = \pi/3$ for $\sin(3x)$.

COMMENT:

You may want to emphasize that there are many ways of solving this problem, for example, finding the x-intercepts, remembering the properties of trigonometric functions, and using calculus to look at the slope of the function at various points. You could ask for equations for the graphs in choices (II) and (IV).

5. Which of the graphs is that of $\sin(x^2)$?



ANSWER:

(d). Since $\dfrac{d}{dx}\sin(x^2) = 2x\cos(x^2)$, the maximum value of the slope of the graph increases as x increases. Also, the zeros of $\sin(x^2)$ are $x = \sqrt{n\pi}$ for $n = 0, 1, 2, 3, \ldots$, which are closer together as x increases.

COMMENT:

You could have students give specific points on the graphs in the other choices which have properties not consistent with those of $y = \sin(x^2)$.