Chapter Four

ConcepTests for Section 4.1

1. Concerning the graph of the function in Figure 4.1, which of the following statements is true?
   (a) The derivative is zero at two values of x, both being local maxima.
   (b) The derivative is zero at two values of x, one is a local maximum while the other is a local minimum.
   (c) The derivative is zero at two values of x, one is a local maximum on the interval while the other is neither a local maximum nor a minimum.
   (d) The derivative is zero at two values of x, one is a local minimum on the interval while the other is neither a local maximum nor a minimum.
   (e) The derivative is zero only at one value of x where it is a local minimum.

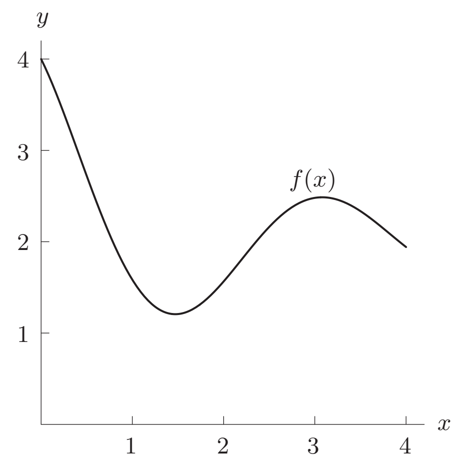

Figure 4.1

ANSWER:

(b). The derivative is zero where it has a horizontal tangent, having a local maximum if it is concave down there, a local minimum if it is concave up.

COMMENT:

You could sketch a different graph and ask the same question.

2. The table records the rate of change of air temperature, H, as a function of time, t, during one morning. When was the temperature a local minimum?

(a) 7:00 (b) 8:00 (c) 9:00 (d) 10:00 (e) 11:00

t (hours since midnight)	6	7	8	9	10	11	12
dH/dt (°F/hour)	1	2	0	-2	0	3	2

ANSWER:

(d)) At 6:00, 7:00, 11:00 and 12:00 the temperature is rising and at 9:00 the temperature is falling, so none of these times corresponds to a local minimum of temperature. Since the temperature is rising before 8:00 and falling just afterward, the temperature at 8:00 is a local maximum. The temperature is falling just before 10:00 and rising just afterward, thus the temperature is at a local minimum at 10:00.

COMMENT:

Students may think they need a second derivative to distinguish a local maximum from a local minimum. This problem encourages them to make the distinction using the first derivative.

3. Which of the following pieces of information from a daily weather report allow you to conclude with certainty that there was a local maximum of temperature at some time after 10:00 am and before 2:00 pm?
   (a) Temperature 50° at 10:00 am and 50° and falling at 2:00 pm.
   (b) Temperature 50° at 10:00 am and 40° at 2:00 pm.
   (c) Temperature rising at 10:00 am and falling at 2:00 pm.
   (d) Temperature 50° at 10:00 am and 2:00 pm, 60° at noon.
   (e) Temperature 50° at 10:00 am and 60° at 2:00 pm.

   ANSWER:
   (a), (c), and (d). There could be a local maximum with any of the five scenarios, but we can not be sure with (b) and (e). Scenario (b) is consistent with a steady decrease in temperature throughout the four hour time period, and (e) is consistent with a steady increase in temperature.
   COMMENT:
   Underlying this problem is the theorem that every continuous function on a closed interval attains a maximum, but you need not bring this theorem into the discussion. Student intuition about temperatures is sufficient to make this exercise meaningful for the class.

4. If the graph in Figure 4.2 is that of $f'(x)$, which of the following statements is true concerning the function f?
   (a) The derivative is zero at two values of x, both being local maxima.
   (b) The derivative is zero at two values of x, one is a local maximum while the other is a local minimum.
   (c) The derivative is zero at two values of x, one is a local maximum on the interval while the other is neither a local maximum nor a minimum.
   (d) The derivative is zero at two values of x, one is a local minimum on the interval while the other is neither a local maximum nor a minimum.
   (e) The derivative is zero only at one value of x where it is a local minimum.

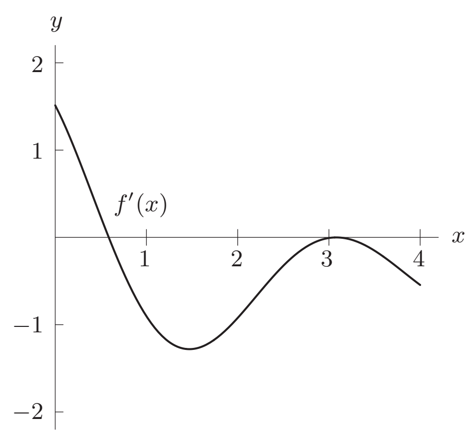

Figure 4.2

   ANSWER:
   (c). When $x \approx 0.6$, the derivative is positive to the left and negative to the right of that point. This gives a local maximum. When $x \approx 3.1$, the graph of the derivative is below the axis (negative) on both sides of this point. This is neither a local minimum nor a maximum.
   COMMENT:
   You could sketch a different graph and ask the same question.

5. Given that $f'(x)$ is continuous everywhere and changes from negative to positive at $x = a$, which of the following statements must be true?
   (a) a is a critical point of $f(x)$
   (b) $f(a)$ is a local maximum
   (c) $f(a)$ is a local minimum
   (d) $f'(a)$ is a local maximum
   (e) $f'(a)$ is a local minimum

ANSWER:
(a) and (c). a is a critical point and $f(a)$ is a local minimum.
COMMENT:
Follow-up Question. What additional information would you need to determine whether $f(a)$ is also a global minimum?

Answer. You need to know the values of the function at all local minima. In addition, you need to know the values at the endpoints (if any), or what happens to $f(x)$ as $x \to \pm\infty$.

For Problems 6–8, consider the graph of $f(x)$ in Figure 4.3.

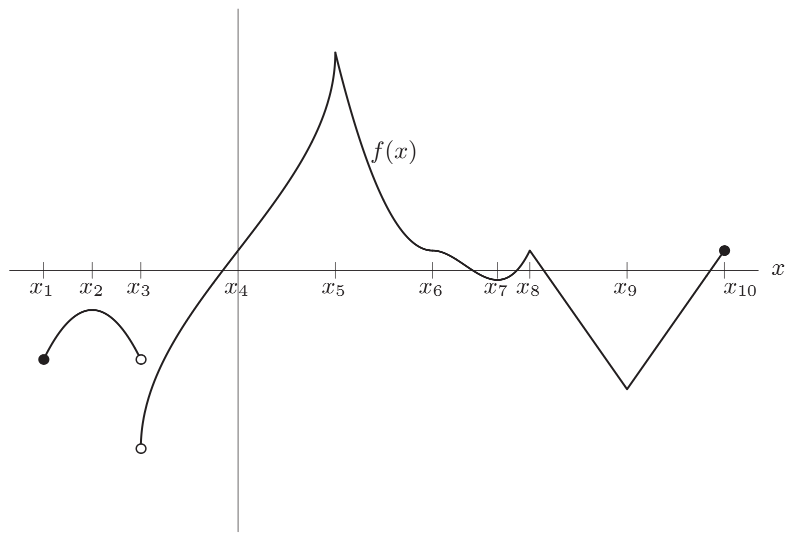

Figure 4.3

6. How many critical points does f have?
ANSWER:
There are 6 critical points. One occurring at x_2, x_5, x_6, x_7, x_8, and x_9.
COMMENT:
You might note that endpoints are not defined to be critical points and critical points can only occur where the function exists.

7. How many local minima does f have?
ANSWER:
There are two local minima. They occur at x_7 and x_9.
COMMENT:
Follow-up Question. What happens if one of the holes at x_3 is filled in? Does it matter which hole?

8. Consider the graph of $y = f(x)$ in Figure 4.3. How many local maxima does f have?
ANSWER:
There are three local maxima. They occur at x_2, x_5, and x_8.
COMMENT:
Follow-up Question. What happens if one of the holes at x_3 is filled in? Does it matter which hole?

9. Let $f(x) = ax + b/x$. What are the critical points of $f(x)$?

(a) $-b/a$	(b) 0	(c) $\pm\sqrt{b/a}$
(d) $\pm\sqrt{-b/a}$	(e) No critical points	

ANSWER:
(c) or (e). $f'(x) = a - b/x^2$. Therefore the critical points, if they exist, of $f(x)$ are $x = \pm\sqrt{b/a}$. Thus the answer is (c) when a, b are either both positive or both negative, (e) when a and b have opposite signs.
COMMENT:
Students have a hard time with parameters. To get them warmed up, you may ask the same question where you have assigned specific values to a and b.

ConcepTests for Section 4.2

1. The function $y = f(x)$ is shown in Figure 4.4. How many critical points does this function have on the interval shown?

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

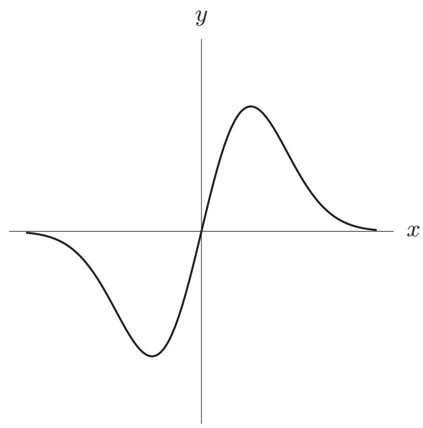

Figure 4.4

ANSWER:

(c) There are two critical points, one at the minimum to the left of the origin, and one at the maximum to the right of the origin.

2. The function $y = f(x)$ is shown in Figure 4.5. How many inflection points does this function have on the interval shown?

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

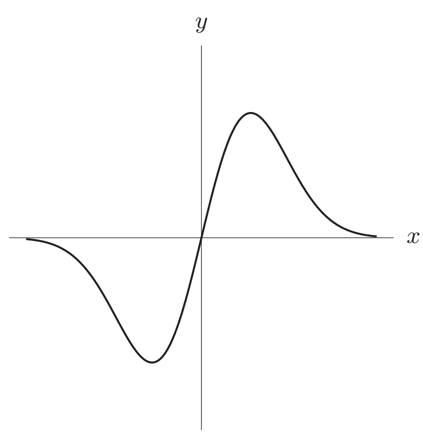

Figure 4.5

ANSWER:

(d) The graph is concave down on the far left, then concave up until the origin, then concave down again, then concave up on the far right. The graph changes concavity three times, so there are three inflection points.

3. The graph of the derivative $y = f'(x)$ is shown in Figure 4.6. The function $f(x)$ has an inflection point at approximately
   (a) $x = 0$
   (b) $x = 1$
   (c) $x = 3$
   (d) $x = 5$
   (e) None of the above

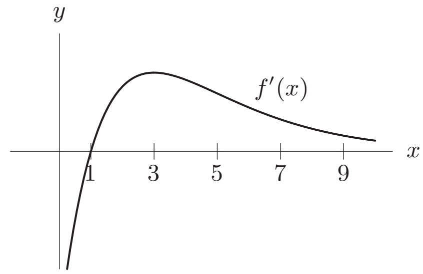

Figure 4.6

ANSWER:
(c) $x = 3$. The inflection points of $f(x)$ are the critical points of $f'(x)$ which are local extrema.

4. The graph of the derivative $y = f'(x)$ is shown in Figure 4.7. The derivative $f'(x)$ has an inflection point at approximately
   (a) $x = 0$
   (b) $x = 1$
   (c) $x = 3$
   (d) $x = 5$
   (e) None of the above

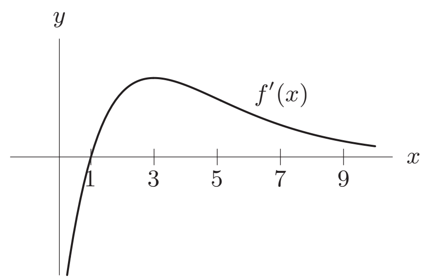

Figure 4.7

ANSWER:
(d) $x = 5$. The derivative function appears to change concavity from concave down to concave up at about $x = 5$.

5. The table records the rate of change of air temperature, H, as a function of time, t, as a warm front passes through one morning. Which of the following could be the rate at 11:00 if H has an inflection point at 10:00?
   (a) 3 °F/hour (b) 4 °F/hour (c) 5 °F/hour (d) 6 °F/hour

t (hours since midnight)	8	9	10	11
dH/dt (°F/hour)	2	3	4	?

ANSWER:
(a) At an inflection point of the temperature function H, the derivative dH/dt has a local maximum or a local minimum. Since dH/dt is increasing before $t = 10$, the derivative dH/dt can have a local maximum but not a local minimum at $t = 10$. Thus dH/dt decreases after $t = 10$.
COMMENT:
Students may think only of the connection between inflection points and second derivatives, but it is often the relation between inflection points and first derivatives that is physically easiest to understand. The temperature was increasing the fastest at the inflection point.

6. For the first three months of an exercise program, Joan's muscle mass increased, but at a slower and slower rate. Then there was an inflection point in her muscle mass, as a function of time. What happened after the first three months?
   (a) Her muscle mass began to decrease.
   (b) Her muscle mass reached its maximum and remained constant afterward.
   (c) Her muscle mass continued to increase, but now at a faster and faster rate.
   (d) The rate of change of her muscle mass changed from positive to negative.

   ANSWER:
   (c) For the first three months, the rate of change of muscle mass was positive and decreasing, so that the graph of muscle mass as a function of time was increasing and concave down. After the inflection point, the rate of change of muscle mass was increasing, so the rate never became negative. After the inflection point the graph of muscle mass was increasing and concave up.
   COMMENT:
   Ask the students to sketch a possible graph of Joan's muscle mass as a function of time.

ConcepTests for Section 4.3

1. The table records the rate of change of air temperature, H, as a function of time, t, one morning. At which of the following times is it coldest?
   (a) 7:00 (b) 8:00 (c) 9:00 (d) 10:00 (e) 11:00

t (hours since midnight)	7	8	9	10	11
dH/dt (°F/hour)	-5	-1	0	-2	-4

   ANSWER:
   (e) The temperature is decreasing all morning, except at 9:00 when it is neither increasing nor decreasing. Thus it is coldest at the latest time, 11:00.
   COMMENT:
   Ask the students to graph the temperature function.

2. Following a dose a of a drug, the average blood pressure of a patient is b. To compute the dose that maximizes the blood pressure, we
   (a) Find $\frac{da}{db}$ and substitute $b = 0$.
   (b) Find $\frac{db}{da}$ and substitute $b = 0$.
   (c) Solve for the values of a for which $\frac{da}{db} = 0$.
   (d) Solve for the values of a for which $\frac{db}{da} = 0$.

   ANSWER:
   (d) Since we want to maximize b, we are finding the critical points of b as a function of a. We find the critical points by setting the derivative db/da equal to zero, and solving for the variable a.

For Problems 3–4, consider the graph of $f(x)$ in Figure 4.8.

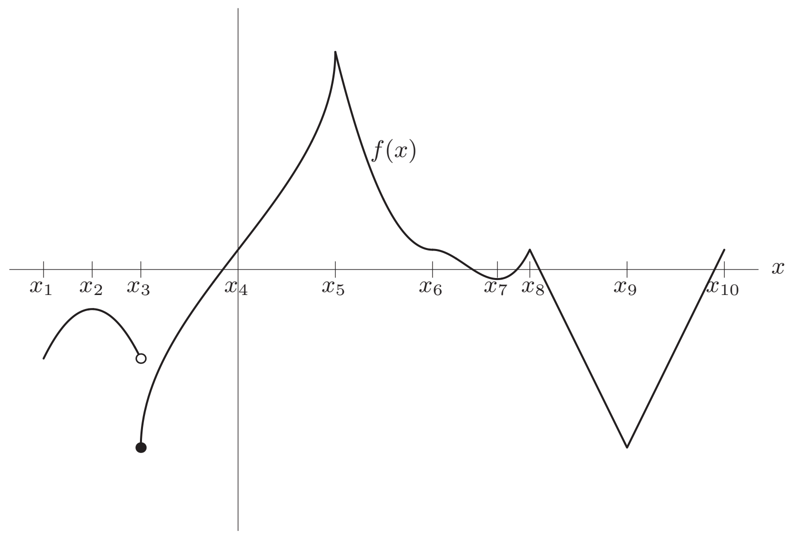

Figure 4.8

3. Where is the global maximum of $f(x)$?
   ANSWER:
   x_5
   COMMENT:
   Follow-up Question. What happens to the global maximum if f is defined only on the open interval from x_1 to x_4?
   Answer. There is no global maximum.

4. Where is the global minimum of $f(x)$?
   ANSWER:
   The global minimum is at x_3.
   COMMENT:

5. On the same side of a straight river are two towns, and the townspeople want to build a pumping station, S. See Figure 4.9. The pumping station is to be at the river's edge with pipes extending straight to the two towns. Which function must you minimize over the interval $0 \leq x \leq 4$ to find the location for the pumping station that minimizes the total length of the pipe?
   (a) $1 + |x| + |4 - x| + 4$
   (b) $\sqrt{x^2 + 1} + \sqrt{(4 - x)^2 + 16}$
   (c) $(1/2) \cdot x \cdot 1 + (1/2) \cdot (4 - x) \cdot 4$
   (d) $x(4 - x)$

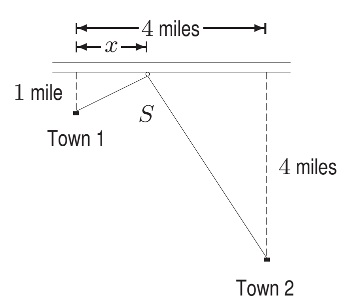

Figure 4.9

ANSWER:
(b). By the Pythagorean Theorem,

$$\text{Distance from Town 1 to } S = \sqrt{x^2 + 1^2}$$
$$\text{Distance from Town 2 to } S = \sqrt{(4-x)^2 + 4^2},$$

so the sum of these distances is given by (b).

COMMENT:

Follow-up Question. What function must be minimized if construction of the pipeline from Town 1 to the river is twice as expensive per foot as construction of the pipeline from Town 2 to the river and the goal is to minimize total construction cost?

Answer. $2\sqrt{x^2+1} + \sqrt{(4-x)^2+16}$.

ConcepTests for Section 4.4

1. Which is correct? A company generally wants to
   (a) Maximize revenue
   (b) Maximize marginal revenue
   (c) Minimize cost
   (d) Minimize marginal cost
   (e) None of the above

   ANSWER:
   (e). A company should maximize profit, that is, it should maximize revenue minus cost.
   COMMENT:
   Follow-up Question. Why are (a) and (c) incorrect?
   Answer. Both (a) and (c) are incorrect because the profit is generally not maximized at these points. Answer (a) generally does not give the maximum profit because a large revenue may be associated with a large cost and therefore give a small profit. Answer (c) is incorrect because the minimum cost may be associated with a low revenue and therefore give a small profit .

2. Which is correct? A company can increase its profit by increasing production if, at its current level of production,
   (a) Marginal revenue $-$ Marginal cost > 0
   (b) Marginal revenue $-$ Marginal cost $= 0$
   (c) Marginal revenue $-$ Marginal cost < 0
   (d) Marginal revenue $-$ Marginal cost is increasing

   ANSWER:
   (a) If $MR - MC > 0$, so $MR > MC$, then revenue is increasing faster than cost with production, so profit increases.
   COMMENT:
   Have the students discuss each of the scenarios (b)–(d) if the company increases production.

ConcepTests for Section 4.5

1. The average cost to produce 2000 items is \$5 per item. The marginal cost to produce the 2001^{st} item is \$4.35. We expect the average cost to produce 2001 items to be
   (a) Greater than \$5 per item.
   (b) Less than \$5 per item.
   (c) Equal to \$5 per item.
   (d) Impossible to tell without more information.

   ANSWER:
   (b) Less than \$5 per item, since we expect the marginal cost of \$4.35 to bring the average cost down slightly.

2. The marginal cost to produce the 100^{th} item is \$3. The fixed costs are \$1000. The average cost to produce 100 items is
   (a) \$13 per item.
   (b) \$3 per item.
   (c) \$1300 per item.
   (d) \$10.30 per item.
   (e) Impossible to determine without more information.

   ANSWER:
   (e) We are told the marginal cost for the 100^{th} item but not the marginal cost for all the other 99 items, so we don't have enough information to compute average cost.

ConcepTests for Section 4.6

1. An item currently sells for \$10 per item. A producer finds that increasing the price to \$11 per item has almost no effect on demand. Which of the following is most likely to be true?
   (a) Demand is inelastic at a price of \$10 and the producer will increase revenue by raising the price.
   (b) Demand is inelastic at a price of \$10 and the producer will increase revenue by lowering the price.
   (c) Demand is elastic at a price of \$10 and the producer will increase revenue by raising the price.
   (d) Demand is elastic at a price of \$10 and the producer will increase revenue by lowering the price.
   (e) We need more information to determine the elasticity.

   ANSWER:
   (a) Demand is inelastic since raising the price has almost no effect on the demand. Since the producer will sell almost as many items at \$11 as at \$10, the producer will increase revenue by raising the price.

2. Figure 4.10 shows a linear demand curve for a product. At which price, p_1 or p_2, is the elasticity of demand greater?
   (a) p_1
   (b) p_2

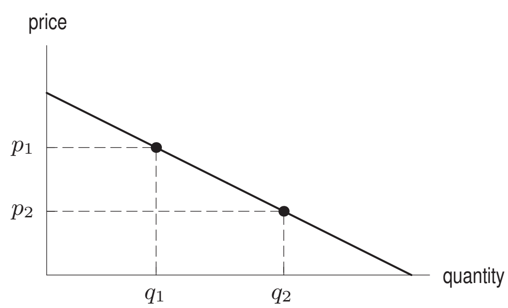

Figure 4.10

ANSWER:
(a) Elasticity is given by

$$E = \left| \frac{p}{q} \frac{dq}{dp} \right|.$$

Since the demand curve is linear, the derivative dq/dp is the same at both prices p_1 and p_2. Thus $E = kp/q$ for a positive constant k. Since $p_1 > p_2$ and $q_2 > q_1$, we have $p_1 q_2 > p_2 q_1$, and thus

$$\frac{p_1}{q_1} > \frac{p_2}{q_2}.$$

Elasticity is greater at price p_1.

COMMENT:
Students may prefer to justify their answers more intuitively. A change in price near p_1 is a smaller percentage of p_1 than the same change in price near p_2 is of p_2, because p_1 is greater than p_2, but they cause the same change in quantity. On the other hand, that change in quantity is a larger percentage of q_1 than it is of q_2 because q_1 is smaller than q_2.

ConcepTests for Section 4.7

1. A population follows a logistic growth pattern during a 24 month period, and essentially reaches its carrying capacity of 650 by the end of the 24 months. The population is growing fastest

(a) During the first month.
(b) During the 24^{th} month.
(c) When the population is about 325.
(d) During the 12^{th} month.
(e) At a time that is impossible to tell without more information.

ANSWER:

(c) The population is growing fastest at the inflection point, and the inflection point of a logistic growth curve occurs at a point where the population is one-half the carrying capacity.

2. Table 4.1 shows values of a quantity Q which grows logistically over time, t. Which of tables (a)–(e) could contain values of Q after $t = 10$?

Table 4.1

t	0	2	4	6	8	10
Q	10	30	70	160	300	420

(a)

t	10	12	14	16
Q	420	620	770	870

(b)

t	10	12	14	16
Q	420	520	620	720

(c)

t	10	12	14	16
Q	420	300	150	50

(d)

t	10	12	14	16
Q	420	500	540	560

(e)

t	10	12	14	16
Q	420	420	500	600

ANSWER:

(d). We see in Table 4.1 that the inflection point occurs at about $t = 8$ where the quantity Q is 300. The carrying capacity is approximately $2 \cdot 300 = 600$, and since the inflection point is somewhere between $t = 6$ and $t = 10$, the carrying capacity must be between $2 \cdot 160 = 320$ and $2 \cdot 420 = 840$. The table in (a) cannot represent logistic growth since the quantity 870 is too large. The table in (b) cannot represent logistic growth since the rate of growth in that table does not slow down after the inflection point (the growth appears to be linear). The table in (c) cannot represent logistic growth as the quantity decreases after the inflection point. The table in (d) could represent a function that increases at an increasing rate up to an inflection point, after which it continues to increase but at a decreasing rate toward a horizontal asymptote at twice the height of the inflection point. The table in (e) cannot represent logistic growth because the quantity takes the value of 420 for two different t values.

COMMENT:

Ask students to explain why table (d) could represent a logistic function.

ConcepTests for Section 4.8

1. Figure 4.11 shows the drug concentration curve for one dose of a drug, with the minimum effective concentration shown by the dotted line. When should the second dose be given if the doses are to be as far apart as possible while still maintaining effectiveness?
   (a) At $t = 10$
   (b) At $t = 30$
   (c) At $t = 60$
   (d) At $t = 70$
   (e) At $t = 80$

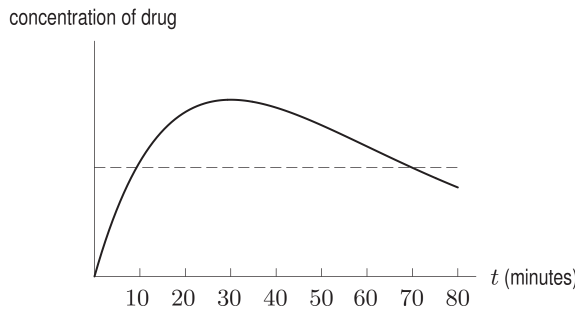

Figure 4.11

ANSWER:
(d) At $t = 70$. We want the drug concentration to remain above the minimum effective concentration. The second dose is needed just as the first becomes ineffective.

2. Figure 4.12 shows drug concentration curves for two different products, A and B, which contain the same drug, with the minimum effective concentration for the drug shown by the dotted line. Which of the following statements are true?
   (a) Product A has a higher peak concentration than Product B.
   (b) Product A is absorbed faster than Product B.
   (c) Product A is effective for a longer period of time than Product B.
   (d) None of the above.

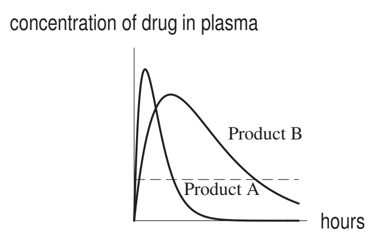

Figure 4.12

ANSWER:
(a) and (b). Product A is absorbed faster since the concentration increases faster at the start, and it reaches a higher peak concentration than Product B. However, Product B stays effective for a longer period of time.